COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

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COWLES FOUNDATION DISCUSSION PAPER NO. 556

EXTENSION OF A DYNAMICAL MODEL OF POLITICAL EQUILIBRIUM

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July 21, 1980

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July 3, 1980

This paper extends an earlier analysis [4] of the dynamical behavior of a competitive two-party political process. The underlying framework is essentially that of the Hotelling-Downs model, generalized to a multi-dimensional Euclidean space of alternatives. In multidimensional spaces it is well known that the classical static or pure strategy Hotelling-Downs equilibrium will not exist, in general; the approach taken here is explicitly dynamical in character. The two political parties are assumed to compete repeatedly for votes, over an infinite series of elections, by advocating alternative policies or alternatives to the electorate. In each period, the party whose policy is preferred by a majority is elected; it is assumed to enact the policy it advocated, and to defend this same policy in the next election against the opposing party. The "out" party may adopt any policy it wishes, to maximize its prospects. In general, the incumbent's policy will always be defeated, and the two parties will alternate in office. As the process is repeated over time, a sequence of successively-enacted "winning" policies is thus generated. Our analysis focuses on the behavior of these sequences, or *trajectories*, of policies.

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*This work was supported by the National Science Foundation, under grant SOC-77-03278.

In [4] it is shown that when voter preferences are "Euclidean," i.e., have indifference surfaces which are concentric hyperspheres in R^K, the trajectories generated by this process will converge on a specific subset of the policy space, the minmax set. To give a rough idea of the nature of the minmax set, consider a collection of n voters, each with a preference ordering defined on a set A of alternatives. $v(x,y)$ the vote for x over y, is the number of voters who strictly prefer x to y. By varying x over the feasible set, we define $\bar{v}(y)$, the maximum vote against y : $\bar{v}(y) = \max_{x \in A} v(x,y)$. If $\bar{v}(y) \leqq n/2$, y is a Condorcet alternative or majority winner in the usual sense; typically, however, no such alternative exists when the alternatives are drawn from R^K, i.e., $\bar{v}(x) > n/2$ for all x. The function $\bar{v}$ nevertheless does have a minimum on R^K. The set of alternatives which minimize $\bar{v}$ constitute the minmax set; they are the alternatives which, in a sense, are "closest" to being Condorcet points. This minmax set is a typically small subset of the policy space, and moreover can be shown to get progressively smaller as the number of n of voters increases, becoming a single point in the limit as $n \to \infty$ [5].

Thus, at least for Euclidean preferences, these results suggest the minmax set as the natural dynamical generalization of the classical Hotelling-Downs equilibrium. As McKelvey has shown, however, convergence is not assured with arbitrary quasi-concave voter preferences [6]. Thus the robustness of the convergence result with respect to assumptions on voter preferences, and the extent to which it can be generalized beyond Euclidean preferences, is an important issue.

In this paper we extent the analysis to, essentially, the family of intermediate preferences introduced by Grandmont [2]. In particular, we consider a class of preferences which can be represented by utility functions of the form $u_a(x) = f_0(x) + a_1 f_1(x) + \dots + a_L f_L(x)$ linear in parameters $a_1, a_2, \dots, a_L$; this family is a large one, and contains close approximations to any smooth utility function on R^K . In Section 2 we show that if voter preferences are of this form, there exists an analog of the minmax set, denoted by $M[\hat{\lambda}]$, on which the vote maximizing trajectories tend to converge. $M[\hat{\lambda}]$ is thus the natural dynamical equilibrium for this class of processes.

In general the set $M[\hat{\lambda}]$, while typically a small compact subset of R^K , is nevertheless larger than the minmax set itself. Moreover, unlike the minmax set, $M[\hat{\lambda}]$ need not necessarily shrink to a point as $n \to \infty$. In Section 3 we explore the relation between the minmax set and $M[\hat{\lambda}]$, and show that the minmax set coincides with $M[\hat{\lambda}]$ if $L \leqq K$, where K is the dimension of the alternative space R^K , and L the dimension of the parameter space R^L which characterizes the voter utility functions. When this dimension condition is satisfied, competitive vote maximization then leads to convergence to the minmax set.

Some of the results obtained here also have a social choice-theoretic interpretation. In particular, for any $a \in [1/2, 1)$ we can define the α-majority preference relation P_α by $xP_\alpha y \iff v(x,y) > \alpha$. Within this family of possible social preference relations, particular interest focuses on the acyclic P_α , and on the P_α which have maximal elements. In Section 4 it is shown that the minmax and $\hat{\lambda}$ parameters characterize these sets.

1. Definition and Preliminary Results

1.1. We shall be concerned with societies composed of individuals whose preferences can be represented by utility functions of the form

$$u_a(x) = f_0(x) + a_1 f_1(x) + a_2 f_2(x) + \dots + a_L f_L(x) ,$$

where each $f_i : R^K \to R^1$ is a continuous function on R^K, and $a_1, \dots, a_L$ are parameters. In vector notation,

$$u_a(x) = f_0(x) + f(x)a ,$$

where $f(x) = (f_1(x), f_2(x), \dots, f_L(x))$ is $1 \times L$ and $a = (a_1, a_2, \dots, a_L)'$ is an $L \times 1$ column vector. (All subsequently introduced vectors are understood to be column vectors unless explicitly defined otherwise.)

The parameter vector $a \in R^L$ varies from voter to voter, but the functions $f_0, \dots, f_L$ are the same for all. R^K is the policy or alternative space; all points in R^K are assumed feasible. The parameter vector a is chosen from a convex body A in R^L, the parameter space. The functions $f_0, \dots, f_L$ and the set A are taken as given and fixed in all that follows. We denote by $\mathcal{U}$ the class of utility functions obtained as a ranges over A. $\mathcal{U}$ thus defines a family of intermediate preferences, in the sense of Grandmont [2]. (Various interpretations of this family can be suggested. For example, we can think of $f_0, \dots, f_L$ as alternative criteria for evaluation; all voters use the same criteria, but weight them differently in arriving at their final overall assessments. Alternatively, we can think of $x_1, \dots, x_K$ as a vector of policy instruments which lead to a set $z_1, \dots, z_L$ of possible outcomes or target variables, where $z = f(x)$. If voters have additively separable preferences with a common structure on the outcome space, their implied preferences for different policy vectors $x \in R^K$ will again be representable by utility functions belonging to the family $\mathcal{U}$.)

We note in passing that the Type I or Euclidean preferences belong to this family. Euclidean preferences are representable by a utility function of the form

$$u(x) = -\|x-s\|^2 = -\sum_{i=1}^{K} (x_i - s_i)^2 ,$$

where s is the point of satiation of the voter in question. This can be rewritten as

$$-\sum x_i^2 + \sum 2s_i x_i - \sum s_i^2 ,$$

and since the constant $\sum s^2$ has no effect on the location of the indifference surfaces, this is evidently equivalent to

$$- \sum_{i=1}^{K} x_i^2 + \sum_{i=1}^{K} s_i 2x_i$$

$$= f_0 + \sum_{i=1}^{K} a_i f_i(x) = u_a(x) ,$$

where the parameters $a_i = s_i$ are the components of the satiation point, and $f_i = 2x_i$ for $i = 1, 2, \ldots, L = K$. Clearly the usual "spatial modelling" utility function $u(x) = (x-s)'A(x-s)$, where A is a positive definite matrix of parameters, can also be rewritten as a second-order polynomial in the x_i , so also belongs to $\mathcal{U}$. Indeed, since any smooth function on R^K can be approximated by a polynomial of sufficiently high order, the family $\mathcal{U}$ (in the absence of further restriction on L or on the form of the f_i) contains approximate representations of almost every representable preference ordering.

We note the following useful property of $\mathcal{U}$:

Comment 1: For any $x, y \in R^K$, the set $\{a \in R^L : u_a(x) > u_a(y)\}$ is an open halfspace in R^L .

Proof: $u_a(x) > u_a(y)$ implies

$$0 < u_a(x) - u_a(y) = [f_0(x) + f(x)a] - [f_0(y) + f(y)a]$$

$$= f_0(x) - f_0(y) + [f(x) - f(y)]a ,$$

i.e., that $h'a > c$, where

$$h' = f(x) - f(y) \in R^L \quad \text{and} \quad c = f_0(y) - f_0(x) \in R .$$

Q.E.D.

As a matter of notation, we henceforth denote by H_{xy} the set

$$H_{xy} = \{a \in R^L : u_a(x) > u_a(y)\} ,$$

by $\mathcal{H}^0$ (resp. $\mathcal{H}^c$) the set of open (resp. closed) halfspaces of R^L , and by

$$\mathcal{H} = \mathcal{H}^0 \cup \mathcal{H}^c$$

the set of open or closed halfspaces of R^L . H, H', H", ... denote arbitrary (open or closed) halfspaces.

1.2. A society is a probability distribution μ on (the Borel sets of) A , which we assume to be a discrete distribution with finite support. We denote the support of μ by

$$\text{supp}\,[\mu] = \{a_j : j \in N\} \, ,$$

where N is the index set $N = \{1, 2, \ldots, n\}$. Thus there are n voters in the society, each voter i with parameter vector a_i.

Given μ, we now define $v(x,y)$, the <u>vote for</u> x <u>over</u> y as the proportion of the electorate who strictly prefer x to y, i.e. as $v(x,y) = \mu\{a \in A : u_a(x) > u_a(y)\} = \mu[H_{xy}]$ (from Comment 1). $\bar{v}(x)$, the <u>maximum vote</u> against x, is defined by $v(x) = \max_{y \in R^K} v(y,x)$ (since μ is a discrete distribution with finite support, the range of v is finite, so this maximum necessarily exists; this justification also applies to the other maxima and minima referred to below). We denote by $M(\cdot)$ the level sets of the maximum-vote function $\bar{v}$:

$$M(r) = \{x \in R^K : \bar{v}(x) \leqq r\} \, , \quad \text{for all} \quad r \in R^1 \, .$$

The <u>minmax number</u> v^* is the minimum value of $\bar{v}$,

$$v^* = \min_{x \in R^K} \bar{v}(x) \, ,$$

and $M(v^*)$ is the <u>minmax set</u>.

Recall that $\text{supp}\,\mu = \{a_i : i \in N\}$. For any coalition $C \subset N$ of voters, we define its Pareto set $P(C)$ by $P(C) = \{x \in R^K :$ for no y is $u_{a_i}(y) > u_{a_i}(x)$, all $i \in C\}$, and denote by $\mu[C]$ the quantity $\mu\{a_i : i \in C\}$. For future reference we note:

<u>Comment 2</u>: If $r < 1$,

$$M(r) = \bigcap_{\{C \subset N : \mu[C] > r\}} P(C) \, .$$

Proof. Obvious.

Turning now to the parameter space R^L , we define the function λ on A by

$$\lambda(a) = \max_{\{H \in H^0 : a \notin H\}} \mu[H] ,$$

and

$$\hat{\lambda} = \min_{a \in A} \lambda(a) ,$$

the minimum of λ . We denote by $A(r)$ the set

$$A(r) = \{a \in A : \lambda(a) \leqq r\} ,$$

and note the following:

Comment 3: $\hat{\lambda} < 1$ always. Moreover if $r < 1$, then

$$A(r) = \bigcap_{\{H \in H : \mu[H] > r\}} H .$$

Proof: If $a = a_j$ for some $a_j \in \text{supp}\,[\mu]$, clearly $a \notin H$ implies $\mu[H] < 1$, whence $\lambda(a) < 1$, whence $\hat{\lambda} \leqq \lambda(a) < 1$, giving the first assertion.

Suppose $a \notin H$ and $\mu[H] > r$. H open would imply

$$\lambda(a) \geqq \mu[H] > r , \quad \text{i.e.,} \quad a \notin A(r) ;$$

or, if H is closed, then there exists a plane P which separates a from the closed set H , which itself defines a new open halfspace, $H' \supset H$, such that

$$a \notin H' \quad \text{and} \quad \mu[H'] \geqq \mu[H] > r ,$$

again implying $a \notin A(r)$.

Conversely, since $\mathcal{H}^0 \subset \mathcal{H}$, $a \in \bigcap_{\{H \in \mathcal{H}:\mu[H]>r\}} H$ implies $a \in \bigcap_{\{H \in \mathcal{H}^0:\mu[H]>r\}} H$ whence $a \notin H \in \mathcal{H}^0$ implies $\mu[H] \leqq r$, whence $\lambda(a) \leqq r$.

Q.E.D.

1.3. There is clearly a close parallel between v^* , the minimum of the function $\bar{v}$ defined on the alternative space R^K , and $\hat{\lambda}$, the minimum of an analogous function λ defined on the parameter space R^L . But while the minmax set $M(v^*)$ is necessarily non-empty, some additional structure on $\mathcal{U}$ is required to ensure the nonemptiness of $M(\hat{\lambda})$. In particular, if all voters' utility functions are monotonically increasing in $x_1, \ldots, x_K$ and the entire space R^K is feasible, $M(r)$ will be empty for all $r < 1$; the minmax set $M(v^*)$ will consist of all of R^K , with $v^* = 1$. This will be true irrespective of the distribution of preferences; e.g. even if all preferences are identical, the majority preference relation is transitive, but there still exists no Condorcet alternative (and in fact no Pareto-optimal alternative) in the unbounded feasible set R^K .

These anomolies do not arise if the utility functions are satiated,*

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*Alternatively if the utility functions are monotonic but the set S of feasible points is compact, each u_a will have a maximum in the "frontier set" $B = \{x \in S : y > x \Longrightarrow y \notin S\}$. Any non-frontier point $z \in S-B$ will be Pareto-dominated, so the Pareto optimal sets $\mathcal{P}(C)$ and hence the sets $M(\cdot)$ (by Comment) all lie in B . Clearly both parties will wish to avoid non-boundary points, which make them vulnerable to defeat by a unanimous vote in the following election, so there is no serious loss of generality in confining attention to points lying in B . But B itself is homeomorphic to a lower-dimensional subspace R^{K^*}, $K^* \leqq K-1$, so we can take R^{K^*} itself as the alternative space with the utility functions in the following sense:

Satiation: For every $a \in A$ and $q \in R^1$, the set $\{x \in R^K : u_a(x) \geqq q\}$ is compact.

Clearly this condition implies u_a is satiated in the usual sense, i.e., has a maximum in R^K. We note the following consequences of the satiation assumption:

Comment 4: If u_{a^0} is satiated at x^0, then $\bar{v}(x^0) \leqq \lambda(a^0)$.

Proof: Let y be a point satisfying $\bar{v}(x^0) = v(y, x^0)$, i.e., $\mu[H_{yx^0}] = \bar{v}(x^0)$. Since u_{a^0} has a maximum at x^0, clearly $u_{a^0}(y) \leqq u_{a^0}(x^0)$, i.e., $a^0 \notin H_{yx^0}$. Hence $\lambda(a^0) = \max_{\{H \in \mathcal{H}^0 : a^0 \notin H\}} \mu[H]$ $\geqq \mu[H_{yx^0}] = \bar{v}(x^0)$.

Q.E.D.

More importantly, we also have the following:

Comment 5: Assume satiation. Then $v^* \leqq \hat{\lambda}$ always, and moreover $M(r)$ is compact for all $r < 1$.

Proof: Let $\hat{a}$ be an element which minimizes $\lambda(\cdot)$, i.e. $\lambda(\hat{a}) = \hat{\lambda}$, and let $\hat{x}$ be a point which maximizes $u_{\hat{a}}$. Then $\bar{v}(\hat{x}) \leqq \lambda(\hat{a})$ from

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$$u_a^*(y) = u_a(g^{-1}(y)) = f_0(g^{-1}(y)) + f(g^{-1}(y))a$$

(where $g : R^K \to R^{K*}$ is the homeomorphism in question) are now satiated, in the sense given above, in the unbounded set R^{K*}. Under this interpretation the dimension condition in Proposition 2 (Section 3 below) refers to K^*, the dimension of the boundary manifold itself, rather than that of the original space R^K.

Comment 4, so from the definition of v^* it follows that $v^* \leqq \overline{v}(\hat{x}) \leqq \hat{\lambda}(a) = \hat{\lambda}$, which proves the first assertion.

To show $M(r)$ compact, recall from Comment 3 that $M(r) = \bigcap_{\{C \subset N: \mu[C] > r\}} P(C)$. Since the number of coalitions C is finite, it suffices to show that each of the sets $P(C)$ is compact, i.e., bounded and closed.

Bounded: For each $i \in C$ let s_i be a point which maximizes $u_{a_i}(\cdot)$, and define

$$w_i = \min_{j \in C} u_{a_i}(s_j)$$

and

$$W_i = \{x \in R^K : u_{a_i}(x) \geqq w_i\} .$$

Each W_i is compact, from the satiation assumption, and C is finite, so the set $W = \bigcup_{i \in C} W_i$ is bounded, Let $x^* = s_{i*}$ for some arbitrary $i^* \in C$. For any $y \notin W$ it then follows from the definitions of w_i and W that

$$u_{a_i}(y) < w_i \leqq u_{a_i}(x^*) \quad \text{for all} \quad i \in C ,$$

and therefore that $y \notin P(C)$. Hence $P(C) \subset W$, so $P(C)$ is bounded.

Closed: Suppose the contrary, that $P(C)$ is not closed. Then there must exist a sequence of points $x^j \in P(C)$ converging to a limit $(x^j) \to \overline{x}$ which does not belong to $P(C)$, i.e. for which there exists a $\overline{y} \in R^K$ such that

$$u_{a_i}(\overline{y}) > u_{a_i}(\overline{x}) \quad \text{for every} \quad i \in C .$$

Let $y^j = x^j + (\overline{y}-\overline{x})$. Since $x^j \in P(C)$, for each j there must exist a voter $i_j \in C$ for whom

$$u_{a_{i_j}}(y^j) \leqq u_{a_{i_j}}(x^j) ,$$

and since C is finite there must be a subsequence (j') on which $i_{j'}$ is constant, i.e. $i_{j'} = i^* \in C$ for all j' . Clearly $(y^{j'}) \to \overline{y}$ and $(x^{j'}) \to \overline{x}$; hence, since $u_{a_{i^*}}(\cdot)$ is continuous and $u_{a_{i^*}}(y^{j'}) \leqq u_{a_{i^*}}(x^{j'})$ it follows that

$$0 \geqq \lim_{j'\to\infty} [u_{a_{i^*}}(y^{j'}) - u_{a_{i^*}}(x^{j'})] = u_{a_{i^*}}(\overline{y}) - u_{a_{i^*}}(\overline{x}) .$$

This is impossible, since by hypothesis $u_{a_i}(\overline{y}) > u_{a_i}(\overline{x})$ for every $i \in C$, a contradiction which proves the result.

Q.E.D.

1.5. With these preliminaries, we can now establish two fundamental results. First, we have

Theorem 1: If $\hat{a} \in A(r)$ then for any $x, y \in R^K$, $v(x,y) > r$ implies $u_{\hat{a}}(x) > u_{\hat{a}}(y)$.

Proof: From Comment 1, $v(x,y) = \mu[H_{xy}]$, where $H_{xy} \in \mathcal{H}^0 \subset \mathcal{H}$, and $\mu[H_{xy}] = v(x,y) > r$; hence, from Comment 3, $\hat{a} \in H_{xy}$, i.e. $u_{\hat{a}}(x) > u_{\hat{a}}(y)$ by the definition of H_{xy} .

Q.E.D.

Theorem 1 implies, in particular, that if (x^t) is a vote-maximizing sequence of alternatives (i.e. $v(x^{t+1}, x^t) = \overline{v}(x^t)$ for all

t) which lie outside the set $M(\hat{\lambda})$, then there exists a function $u_{\hat{a}}$, $\hat{a} \in A$, which is strictly increasing on the sequence; thus no such sequence can cycle. To show that such sequences in fact converge to $M(\hat{\lambda})$, we need a somewhat stronger version of Theorem 1. One additional rather weak assumption on $\mathcal{U}$ is required:

Weak Independence: There exists an $\alpha^* > 0$ such that for every $x, y \in R^k$,

$$\|f(x) - f(y)\| > \alpha^* \|x-y\| .$$

We now prove the following.

Theorem 2: Assume Weak Independence. Then, if $A(r)$ is a body, there exists an $a^0 \in A(r)$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $v(y,x) > r$ and $\|y-x\| > \varepsilon$ implies $u_{a^0}(y) - u_{a^0}(x) > \delta$.

Proof: If $A(r)$ is a body, there exists $a^0 \in A(r)$ and $\rho > 0$ such that $A(r) \supset \{a \in A : \|a - a^o\| \leqq \rho\} \equiv N_\rho(a^o)$. For any $a \in A$ and $z \in R^K$ we can write $u_a(z) = u_{a^o}(z) + f(z)(a - a^o)$, so in particular we have

$$u_a(y) - u_a(x) = u_{a^o}(y) - u_{a^o}(x) + [f(y) - f(x)](a - a^o) .$$

If $v(y,x) > r$ this quantity will be positive, implying

$$u_{a^0}(y) - u_{a^0}(x) > -[f(y) - f(x)](a - a^0) ,$$

for any $a \varepsilon N_\rho(a^0)$. If $||y-x|| > \varepsilon$, then from Weak Independence we have $||f(y) - f(x)|| > \alpha^*\varepsilon$. Hence define

$$a^* = a^0 - \rho \frac{[f(y) - f(x)]'}{||[f(y) - f(x)]||}$$

so that

$$-[f(y) - f(x)](a^* - a^0) = \rho \frac{[f(y) - f(x)](f(y)) - [f(x)]'}{||f(y) - f(x)||}$$

$$= \rho \frac{||f(y) - f(x)||^2}{||f(y) - f(x)||} > \rho\alpha^*\varepsilon .$$

Hence, taking $\delta = \rho\alpha^*\varepsilon > 0$, we have

$$u_{a^0}(y) - u_{a^0}(x) > -[f(y) - f(x)](a^* - a^0) > \delta ,$$

which proves the result.

Q.E.D.

2. Competitive Vote Maximization on R^K

2.1. We now return to the competitive political process described in the beginning. Two political parties are assumed to compete for votes by proposing alternative policy packages, or points in the alternative space R^K, in each of a series of electoral contests. Voters are assumed to have no loyalties to parties as such, and to vote solely on the basis of their preferences for the current policies offered by each. Since there are only two alternatives available in each election, there is no incentive for strategic voting, and the party whose policy vector is preferred by the greatest number of voters will win a majority and take office. Each party is assumed to be consistent, in the sense that upon winning office it proceeds to enact the policies it advocated in the election, and defends these policies in the next election. The "out" party, however, having lost the previous election, is free to adopt a new policy platform in attempting to regain office in the next election, and in making this choice is assumed to choose a point which will maximize its vote against the incumbent. Since the potential policies are drawn from a multidimensional space R^K, in general there will be no Condorcet alternative or majority-dominant policy, so the incumbent's policy can always be defeated, and the two parties will alternate in office. As they compete repeatedly over a series of elections a sequence of successively-enacted winning policies will be generated. We focus here on the behavior of these sequences, or trajectories.

To fix terminology, a vote-maximizing trajectory is a sequence (x^t) of points $x^t \in R^K$, such that $v(x^{t+1}, x^t) = \bar{v}(x^t)$ for all $t \geq 1$. A trajectory is non-degenerate if there exists an $\varepsilon > 0$ such that for every $T > 0$, there exists a $t' > T$ such that $\|x^{t'+1} - x^{t'}\| > \varepsilon$, i.e.

the parties occasionally differentiate themselves by at least ε . In all that follows, trajectories are understood to be vote-maximizing and non-degenerate.

There is considerable indeterminacy in the short-run behavior of these trajectories (for any x^t , the set of potential vote-maximizing x^{t+1} constitute an open set in R^K), so we shall concentrate on their long-run or asymptotic behavior. There are a variety of equilibrium concepts which can usefully characterize the long-run behavior of this type of dynamic process, but the following three seem to capture most of the relevant characteristics of interest:

First, we can look for the possible steady states, or rest points, of the process. In the usual definition, x^* would be said to be a rest point if there exists a trajectory (x^t) for which $x^t = x^*$ for all $t > T$, for some T . But such a trajectory would be neither non-degenerate nor vote-maximizing (since $v(x^*, x^*) = 0$), so clearly rest points in this exact sense cannot exist. However, if we relax the definition to admit trajectories which remain arbitrarily close to x^* , there will exist rest points in this approximate sense. Thus, we shall see x^* is a potential rest point, or <u>equilibrium</u>, if for every $\varepsilon > 0$ there exists a trajectory (x^t) and number T such that $x^{t'} \in N_\varepsilon(x^*)$ for all $t' > T$. We denote by E the set of such equilibrium points.

The definition of a rest point ensures the existence of some trajectories which remain in its vicinity indefinitely, but of course there also exist other trajectories which are not so well behaved. Thus to characterize the asymptotic behavior of arbitrary trajectories, we introduce the idea of a point of recurrence, or limit point. Thus we shall say a point x^* is a <u>limit point</u> if, for every $\varepsilon > 0$, there exists a trajectory (x^t) which contains a subsequence $(x^{t'})$ such that $x^{t'} \varepsilon N_\varepsilon(x^{t'})$ for all t. The set of limit points is denoted by $\mathcal{L}$. Clearly every equilibrium point is a limit point, so $\mathcal{E} \subset \mathcal{L}$.

Equilibrium and limit points are points which are potentially sustainable (in the sense of their respective definitions), if the proper trajectory is chosen, but no such point is necessarily sustained if other trajectories happen to be generated: there is no assurance that any particular trajectory will tend toward any particular limit or equilibrium point. Thus, to characterize a set of points which are necessarily sustained, by any trajectory, we introduce a third and final concept, that of a region of recurrence. A set $S \subset R^K$ is a region of recurrence if for every trajectory (x^t) and $\varepsilon > 0$, there exists a subsequence $(x^{t'})$ such that $x^{t'} \varepsilon N_\varepsilon(S)$ for all t'. A region of recurrence S is minimal if none of its proper subsets is also a region of recurrence. We denote by $\mathcal{R}$ the minimal region of recurrence. It is clear that the set $\mathcal{L}$ is itself a region of recurrence, and that any region of recurrence must contain every equilibrium point; hence $\mathcal{L} \supset \mathcal{R} \supset \mathcal{E}$. ($\mathcal{L}$ and $\mathcal{R}$ are nonempty and unique, from general considerations, and the set $\mathcal{E}$ can also be shown to be nonempty under the assumptions of Section 1.)

2.2. It will now be shown that under the structure described in Section 1, these equilibria are all located in or near the set $M(\hat{\lambda})$. Proposition 1 below is conditional upon one further premise--that the set $A(r)$ is a body in R^L --but for large n this condition is not restrictive and will be satisfied by almost every society μ. (In particular, we conjecture that $A(\hat{\lambda})$ is a body whenever $n > L+1$, if the a_i are in general position in R^L.) In any event Proposition 1 remains valid if $\hat{\lambda}$ is replaced by r throughout, with any $\hat{\lambda} \leqq r \leqq 1$ for which $A(r)$ is a body.

We now have:

Proposition 1: Assume satiation, Weak Independence, and that $A(\hat{\lambda})$ is a body. Then $M(\hat{\lambda}) \supset \mathcal{R} \supset \mathcal{E}$. Moreover there exists $\delta > 0$ such that $N_\delta(M(\hat{\lambda})) \supset \mathcal{L}$.

Proof: We first show that $M(\hat{\lambda})$ is a region of recurrence. Suppose the contrary, i.e. that there exists a trajectory (x^t) such that for some $\varepsilon^* > 0$, $T > 0$, $x^t \notin N_{\varepsilon^*}(M(\hat{\lambda}))$ for all $t > T$. Since (x^t) is non-degenerate, there must exist an $\varepsilon > 0$ and a subsequence (x^{t_j}) such that $t_j > T$ and $\|x^{t_j+1} - x^{t_j}\| > \varepsilon$ for all j ; hence, from Theorem 2, there exists $\hat{a} \in A(\hat{\lambda})$ and $\delta > 0$ such that $u_a(x^{t_j+1}) - u_a(x^{t_j}) > \delta$ for all j . For any j we can write

$$u_{\hat{a}}(x^{t_j+1}) = u_{\hat{a}}(x^{T+1}) + \sum_{\{t:T<t\leqq t_j\}} [u_{\hat{a}}(x^{t+1}) - u_{\hat{a}}(x^t)] .$$

Since by hypothesis $t > T$ implies $x^t \notin M(\hat{\lambda})$, the quantity in square brackets is always non-negative, from Theorem 1, so

$$\sum_{\{t:T<t\leqq t_j\}} [\cdot] \geqq \sum_{i=1}^{j} [u_{\hat{a}}(x^{t_i+1}) - u_{\hat{a}}(x^{t_i})] > j\delta ;$$

hence $u_{\hat{a}}(x^{t_j+1}) > u_{\hat{a}}(x^{T+1}) + j\delta$, implying that $u_{\hat{a}}(x^{t_j+1})$ increases without bound as $j \to \infty$, which is impossible since $u_{\hat{a}}$ is satiated. Hence no such trajectory can exist, i.e. $M(\hat{\lambda})$ is a region of recurrence. From the definitions, it clearly follows that $\mathcal{R} \subset M(\hat{\lambda})$ and that $\mathcal{E} \subset \mathcal{R}$, which prove the first part of the result.

To prove the remainder, let $S_x = \{y : v(y,x) = \bar{v}(x)\}$, for any x , and note that

$$S_x = \bigcup_{\{C \subset N : \mu[C] = \bar{v}(x)\}} \bigcap_{i \in C} \{z : u_{a_i}(z) \geqq u_{a_i}(x)\}$$

is a finite union of finite intersections of compact sets (from satiation), so is itself compact. Let $\hat{a} \in A(\hat{\lambda})$. Since $u_{\hat{a}}$ is continuous, $p(x) = \min_{y \in S_x} u_{\hat{a}}(y)$ is defined and continuous at all x , so has a minimum $\bar{p} = \min_{\{x \in M(\hat{\lambda})\}} p(x)$ on the compact (Comment 2) set $M(\hat{\lambda})$. The set $Q = \{x : u_{\hat{a}}(x) \geqq \bar{p}\}$ is also compact, and evidently contains $M(\hat{\lambda})$; i.e. $Q \supset M(\hat{\lambda})$. From the proof of the first part of the Proposition it follows that on any trajectory (x^t) , $x^T \in M(\hat{\lambda})$ for some T , whence $x^T \in Q$. If $x^t \in Q$ for any $t > T$, there are two possibilities:

(i) $x^t \in Q - M(\hat{\lambda})$. Then $\bar{v}(x^t) > \hat{\lambda}$, so $v(x^{t+1}, x^t) = \bar{v}(x^t)$ implies $u_a(x^{t+1}) > u_a(x^t) > \bar{p}$ from Theorem 1 and the fact that $x^t \in Q$; hence $x^{t+1} \in Q$ also. The only other possibility is

(ii) $x^t \in M(\hat{\lambda})$. Then $x^{t+1} \in S_{x^t}$, i.e. $u_{\hat{a}}(x^{t+1}) \geqq p(x^t)$, and since $x^t \in M(\hat{\lambda})$, $p(x^t) \geqq \bar{p}$. Hence $u_a(x^{t+1}) \geqq \bar{p}$, i.e. $x^{t+1} \in Q$ also. Thus by induction, $x^t \in Q$ for all $t > T$, so $L \subset Q$. Since Q is compact, there clearly exists δ s.t. $N_\delta(M(\hat{\lambda})) \supset Q \supset L$.

Q.E.D.

The second part of Proposition 1 is not very strong as stated, since δ may be quite large (even so, it still shows that the vote maximizing trajectories are much better-behaved than are the simple majority rule trajectories, for example, which in general are not confined to any proper subset of the alternative space [7]). It can be shown, however, that as the number of voters increases (in the proper, but rather general, way) this δ tends to zero [5]. Hence in large societies the all limit points will be found in or near $M(\hat{\lambda})$.

3. <u>The Relation between</u> $M(\hat{\lambda})$ <u>and</u> $M(v^*)$

Proposition 1 shows that the trajectories generated by competitive vote maximization will eventually converge on the set $M(\hat{\lambda})$, and remain in or near it indefinitely thereafter. $M(\hat{\lambda})$ is thus the natural dynamical generalization of the Hotelling-Downs equilibrium. But this set $M(\hat{\lambda})$, while typically a small compact subset of R^K , is nevertheless larger than the minmax set itself, in general (cf. Comment 5). The minmax set is of some intrinsic normative interest as a natural generalization of the Condorcet criterion; moreover as the number n of voters increases (in the proper fashion), the minmax set can be shown to become smaller, shrinking to a single point in the limit as $n \to \infty$ [ ]. This is not true of the $M(\hat{\lambda})$ set; hence it is of some interest to find the conditions which ensure convergence to the minmax set itself.

Clearly the relation between $M(\hat{\lambda})$ and $M(v^*)$ depends in part on the distribution μ ; e.g., if voter preferences are all identical, or single-peaked, these two sets will coincide. The condition given below is distribution-free, however, thus allowing for the possibility of evolution or change in voter preferences.

In developing this condition it will be convenient to assume the f_i functions are differentiable, i.e. C^1 . As a matter of notation, we denote by ∇_i the gradient vector of f_i ,

$$\nabla_i(x) = \frac{\partial}{\partial x}(f_i(x)) = \begin{bmatrix} \frac{\partial f_i(x)}{\partial x_1} \\ \vdots \\ \frac{\partial f_i(x)}{\partial x_K} \end{bmatrix}$$

and by $D(x)$ the derivative of f, i.e. the matrix

$$D(x) = [\nabla_1(x) \quad \nabla_2(x) \quad \ldots \quad \nabla_L(x)] \ .$$

Clearly the u_a are also C^1, with gradients ∇_a given by

$$\nabla_a(x) = \frac{\partial}{\partial x}(u_a(x)) = \nabla_0(x) + D(x)a \ , \quad \text{for all} \quad a \in A \ .$$

Voter preferences until now have not been assumed to be convex; henceforth, however, we shall assume

Pseudoconcavity: $u_a(x)$ pseudoconcave for all $x \in R^K$ and $a \in A$, i.e. $\nabla_a(x)(y-x) \leqq 0$ implies $u_a(y) \leqq u_a(x)$ for any $y \in R^K$.

We note without proof two obvious properties of pseudoconcavity: first, every critical point is a point of satiation, i.e. $\nabla_a(x^0) = 0$ implies $u_a(x^0) \geqq u_a(y)$, all y. Secondly, if $\nabla_a(x^0) > 0$ there exists an $\varepsilon > 0$ such that $\|y-x\| < \varepsilon$ and $\nabla_a(x^0)(y-x) > 0$ imply $u_a(y) > u_a(x^0)$. We also have:

Comment 6: Assume Satiation and Pseudoconcavity. Then for any $x^0 \in R^K$, there exists a society μ for which $\overline{v}_\mu(x^0) < 1$ if and only if there exists an $a^* \in A$ such that u_{a^*} is satiated at x^0.

Proof: If: Take μ such that supp $[\mu] = \{\hat{a}\}$. From pseudoconcavity all voters are satiated at x^0, so $\overline{v}_\mu(x^0) < 1$.

Only if: From Comment 2, $\overline{v}(x^0) < 1$ implies $x^0 \in P(N)$. From the standard first-order condition for a Pareto Optimum, there must exist non-negative multipliers $\alpha_i \geqq 0$, $i \in N$, not all zero, such that

$$\sum_{i\in N} \alpha_i \nabla_{a_i}(x^0) = 0 .$$

Let

$$a* = \sum_{i\in N} \frac{\alpha_i a_i}{\sum_{j\in N} \alpha_j} ;$$

then $a*$ is a convex combination of points $a_i \in A$, so $a* \in A$, and evidently the utility function u_{a*} satisfies

$$\begin{aligned}\nabla_{a*}(x^0) &= \nabla_0(x^0) + D(x^0)a* \\ &= \nabla_0(x^0) + D(x^0)\left[\sum_{i\in N} \frac{\alpha_i a_i}{\sum\alpha}\right] \\ &= \sum_{i\in N} \frac{\alpha_i}{\sum\alpha}[\nabla_0(x^0) + D(x^0)a_i] \\ &= \frac{1}{\sum\alpha}[\sum_{i\in N} \alpha_i \nabla_{a_i}(x^0)] = 0 .\end{aligned}$$

Q.E.D.

This result gives some insight into the structure of the function $\overline{v}$, and in particular shows that the nested level sets $M(r)$, $r < 1$, must be confined to S , the set of satiation points generated as a ranges over A . This set S is typically a K-dimensional subset of R^K when $L > K$. If $L \leqq K$, however (and if each u_a is uniquely satiated), S will be a subset of an L-dimensional manifold in R^K . For any society μ the $M(r)$ sets will all be confined to this manifold; moreover, as we will now show, the minmax and $M(\hat{\lambda})$ sets will coincide To establish this, we require one further condition on the component f_i

functions, a strengthening of the earlier Weak Independence condition. We now assume

Strong Independence: The matrix $D(x)$ is of full rank at all $x \in S$.

We now prove the following:

Proposition 2: Assume Satiation, Pseudoconcavity, and Strong Independence. Then $L \leqq K$ implies $\hat{\lambda} = v^*$.

Proof: Let x^* be a minmax point, i.e. $\bar{v}(x^*) = v^*$. From Comment 6, there exists $a^* \in A$ such that

$$0 = \nabla_{a^*}(x^*) = \nabla_0(x^*) + D(x^*)a^* ,$$

i.e.

$$\nabla_0(x^*) = -D(x^*)a^* . \qquad (1)$$

By definition of $\lambda(\cdot)$, there must exist an open halfspace $H = \{a \in R^L : (a,h) > (a^*,h)\}$ such that $\mu[H] = \lambda(a^*)$ (where $h \neq 0$ and $a^* \notin H$, clearly). Strong Independence and $L \leqq K$ imply that $D(x^*)$ is of rank L, and hence that the equation

$$d'D(x^*) = h' \qquad (2)$$

has a solution $d \in R^K$. For any $a \in A$, evidently

$$\begin{aligned}(\nabla_a(x^*), d) = d'\nabla_a(x^*) &= d'[\nabla_0(x^*) + D(x^*)a] \\ &= d'[-D(x^*)a^* + D(x^*)a] \quad \text{from (1)} \\ &= -h'a^* + h'a \quad \text{from (2).}\end{aligned}$$

Hence $(\nabla_a(x^*), d) > 0$ implies $h'a > h'a^*$, and therefore $a \in H$, for any $a \in A$. Since the u_a are pseudoconcave, for sufficiently small $\alpha > 0$ the point $y = x^* + \alpha d$ will satisfy $(\nabla_{a_i}(x^*), d) > 0 \Longrightarrow u_{a_i}(y) > u_{a_i}(x^*)$ for all $a_i \in \text{supp}\,[\mu]$, so $\overline{v}(y,x) \geqq \mu[H]$. Hence, from the definitions of $\overline{v}$, v^* and $\hat{\lambda}$ and choice of x^* and H we have

$$v^* = \overline{v}(x^*) \geqq \overline{v}(y, x^*) \geqq \mu[H] = \lambda(a^*) \geqq \hat{\lambda} ,$$

which from Comment 5 shows that $v^* = \hat{\lambda}$.

Q.E.D.

4. Consistent Majority Rules on R^K

4.1. We conclude with a brief voting-theoretic interpretation of Theorem 1. For any society we can define the family of social preference relations P_α for all $\alpha \in [1/2, 1)$, by $xP_\alpha y$ iff $v(x,y) > \alpha$. These relations are nested, in the sense that $\alpha' > \alpha$ implies $P_{\alpha'} \subset P_\alpha$, and range from the simple majority rule, $P_{1/2}$, to the unanimity rule or weak Pareto-dominance relation, $P_{1-\varepsilon}$ (for sufficiently small $\varepsilon > 0$). Within this family of relations, the P_α of potential interest as possible bases for social decision are the ones which are "consistent," in some appropriate sense.

In particular, one natural class to focus on are the P_α which possess maximal elements, i.e. those for which the choice set $C_\alpha = \{x : \text{for no } y \text{ is } yP_\alpha x\}$ is nonempty. Conditions for this type of consistency have been studied by several authors; i.e., for the case where the alternatives belong to a multidimensional space R^K , Slutsky [8] gives necessary and sufficient conditions for $x \in C_\alpha$ for given μ , and Greenberg [3] gives conditions on α which ensure $C_\alpha \neq \phi$ for all μ .

In the present context, clearly $C_\alpha \neq \phi$ if and only if $\alpha \geqq v^*$, so for a given society or profile the minmax number v^* characterizes the P_α which are "consistent" in this weak sense.

For some purposes however, we may be interested in finding the P_α which are consistent in the stronger sense of yielding a ranking or consistent ordering of the entire set of alternatives. The broadest family of such relations will be the P_α which are acyclic, in the usual sense that there exists no finite sequence $x^1, x^2, \ldots, x^r$, $r > 1$ of alternatives such that $x^{i+1} P_\alpha x^i$ for all $i = 1, 2, \ldots, r$, and $x^r = x^1$. It is readily seen that the $\hat{\lambda}$ parameter gives a characterization of the acyclic P_α.

Proposition 3: For a given society μ, P_α is acyclic if $\alpha \geqq \hat{\lambda}_\mu$. Moreover P_α is acyclic in every society μ if $\alpha \geqq L/L+1$. If $L \leqq K$ and the other conditions of Proposition 2 are satisfied, we can replace "if" by "if and only if" in both assertions. (α being understood as restricted to the range of v_μ in the first.)

Proof: Let $z^1, z^2, \ldots, z^r$ be any finite sequence of alternatives for which $z^{i+1} P_\alpha z^i$. If $\alpha \geqq \hat{\lambda}$, then by Theorem 1 there exists a function u_a which is strictly increasing on z^i, whence $u_a(z^r) > u_a(z^1)$, so $z^1 \neq z^r$, i.e. the sequence cannot be a cycle.

To prove the second assertion, consider a society μ whose members have utility functions u_{a_i}, $a_i \in \operatorname{supp}[\mu] \subset A \subset R^L$ defined on an alternative space R^K. We can always interpret these parameter vectors as also defining another society with Euclidean preferences on a different alternative space $R^{\tilde{K}}$, $\tilde{K} = L$, by taking $f_0(x) = x'x$ and $\tilde{f}_j(x) = x_j$, $j = 1, \ldots, L = \tilde{K}$, for all $x \in R^{\tilde{K}}$. The conditions of Proposition 2 are satisfied by these $\tilde{f}_j$, so the minmax number $\tilde{v}^*$ in this hypothetical alternative space $R^{\tilde{K}}$ must satisfy $\tilde{v}^* = \hat{\lambda}$. Moreover, from Theorem 2 of Greenberg [3] it must also be true that $\tilde{v}^* \leqq \tilde{K}/\tilde{K}+1$; hence, since $\hat{\lambda} = \tilde{v}^*$ and $\tilde{K} = L$, it follows that $\hat{\lambda} \leqq L/L+1$. This holds for any distribution μ defined on R^L ; hence, from the first sentence of Proposition 3, $\alpha \geqq L/L+1$ implies P_α acyclic for every μ .

Let X be a large compact set containing $P(N)$. For any x the set $\{y : yP_\alpha x\} = \bigcup_{\{C\subset N:\mu[C]>\alpha\}} \bigcap_{i\in C} \{z : u_{a_i}(z) > u_{a_i}(x)\}$ is clearly open, so from a standard result on acyclicity (e.g. [1], Theorem 7), if P_α is acyclic it has a maximal element on X ; i.e., if α is restricted to the range of v , P_α acyclic $\Longrightarrow \alpha \geqq v^*$. As shown above, $\alpha \geqq \hat{\lambda} \Longrightarrow P_\alpha$ acyclic, and if the conditions of Proposition 2 are satisfied $v^* = \hat{\lambda}$, implying P_α acyclic iff $\alpha \geqq \hat{\lambda} = v^*$.

Theorem 1 of Greenberg shows that the bound $L/L+1$ is sharp, i.e. for any $\beta < L/L+1$ there exist societies μ for which $\hat{\lambda}_\mu > \beta$ and hence in which P_α cycles for $\beta \leqq \alpha < \hat{\lambda}_\mu$.

Q.E.D.

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