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MOST CONVEX FUNCTIONS ARE SMOOTH

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Most convex functions are smooth

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This note records one fairly simple observation, but it is somewhat surprising, and it also has some philosophical interest for, e.g., economics. It is well-known that a convex function is almost smooth, in the sense that it is differentiable almost everywhere. Indeed, considerably more precise facts are known. Furthermore, any convex function can be approximated arbitrary closely by a smooth (even analytic) convex function. The point being made here is that smoothness everywhere is a typical, that is, generic, property of convex functions.

Let us formulate this precisely in a technically simple but basic situation. Let X be a compact, convex body in R^n. Let $Cx(X)$ denote the set of convex continuous functions on X having values between 0 and 1. Then $Cx(X)$ is a convex, compact subset of the unit ball of $\mathbb{C}(X)$, the Banach space of continuous functions on X with the uniform norm. In particular $Cx(X)$ is a compact metric space.

Recall that a set Y in a metric space M is called of <u>first category</u> if it is contained in a countable union of closed sets without interiors. A set is a <u>residual</u> if it is the complement of a set of first category. The Baire Category Theorem implies that if M is complete, then any residual set is everywhere dense. A property of points of M is called <u>generic</u> if it holds for a residual set in M. The idea is that residual sets are in some topological sense large.

Theorem 1: The subset of $Cx(X)$ consisting of functions differentiable everywhere in the interior of X is a residual set in $Cx(X)$.

Remark: a) One reason why this result might be surprising is that in $C(X)$, the whole space of continuous functions, the differentiable functions form a subset of first category.

b) By duality, this result says

Theorem 2: The subset of $Cx(X)$ consisting of strictly convex functions is a residual set.

Since the intersection of residual sets is again residual, we see that the generic convex functions is both smooth and strictly convex. In mathematical economics, there is often discussion of the relative merits of assuming smoothness, or strict convexity, as opposed to plain convexity. Usually under smoothness or strict convexity arguments simplify, but there is a latent feeling that the simple convexity assumption is "more realistic". In accord with the general philosophy of genericity, as espoused notably by Smale and others in the study of differential topology and dynamical systems, the results here would indicate that, in the absence of strong reasons to the contrary, one should take a convex function to be smooth and strictly convex, since those properties are "typical" of convex functions as a family.

Proof: If f is a non-differentiable element of $Cx(X)$, then we can find a point $x \in X$, and two vectors v_1 and v_2 in R^n such that

$$f(y) - f(x) + v_1 (y-x) \geq 0 \quad \text{for all} \quad y \in X. \tag{1}$$

If X contains a ball of radius r around x, then putting $y = x - r(v_i/\|v_i\|)$, and using $0 \le f(x) \le 1$, we have

$$(2) \qquad r\,\|v_i\| \le 1$$

so that $\|v_i\|$ is bounded independently of f on any compact set in the interior of X.

Let $\{X_n\}$ be a sequence of compact convex subsets of the interior of X such that $X_n \subset X_{n+1}$ and $\bigcup_n X_n$ exhausts the interior of X. Consider the set B_n of functions f in $Cx(X)$ such that

(3) i) f is non-differentiable at a point $x \in X_n$, and

ii) the vectors v_i in (1) may be chosen so that the elements $(1,v_1)$ and $(1,v_2)$ of $\mathbb{R}^{n+1}$ make an angle $\ge 1/n$.

Let f_k be a sequence in B_n converging to g in $Cx(X)$. Let x_k be a point of X_n at which f_k is non-smooth, and let v_{1k} and v_{2k} be vectors satisfying (1) and (3) ii) above for f_k. By passing to a subsequence if necessary, we can assume that the x_k converge in X_n, and by (2) above we can assume that for a further subsequence v_{1k} and v_{2k} also converge. Then in the limit (3) and (1) clearly still hold, so $g \in B_n$. That is, B_n is closed. Since B_n is a subset of the non-smooth functions, and the smooth functions are dense in $Cx(X)$, B_n has no interior. Since the non-smooth functions are clearly just the union $\bigcup_n B_n$, they are a set of first category, so the smooth functions are residual. Theorem 1 is thus proved.