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MYOPIC ECONOMIC AGENTS

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Abstract

An economic agent is said to be weakly myopic if when he prefers a time-contingent consumption plan $\bar{x}$ to a time-contingent consumption plan $\bar{y}$, then he prefers $\bar{x}$ to $\bar{y}$ augmented by any stationary consumption plan which begins sufficiently far in the future.

An economic agent is said to be monotonically myopic if when he prefers a state-contingent consumption plan $\bar{x}$ to a state-contingent consumption plan $\bar{y}$, then he prefers any sufficiently large finite truncation of $\bar{x}$ to every sufficiently large finite truncation of $\bar{y}$.

A topology on the space of time (state)-contingent consumption plans is said to be weakly (monotonically) myopic if every complete preference relation which is continuous in this topology is weakly (monotonically) myopic.

A characterization of weakly (monotonically) myopic Hausdorff locally convex linear topologies and their dual spaces is given.

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I. Introduction

The standard treatment of general equilibrium theory in a Walrasian economy assumes consumption and production to occur in markets for each of a finite number of commodities. If we distinguish between commodities which are produced or consumed at different times, then the dimension of the commodity space may be quite large. In fact, the observation that the future has no natural termination date suggests an infinite dimensional space as the appropriate model for intertemporal economies.

Consequently, we consider a world with a denumerable number of time periods. In addition, we restrict our attention to bounded commodity bundles, hence our commodity space is $C_b(N)$, the space of bounded real-valued sequences. $C_b(N)$ is then the space of time-contingent consumption plans.

As in the finite dimensional setting, we characterize an economic agent by his initial endowment, a positive vector in $C_b(N)$, and his preference relation, a transitive binary relation on $C_b(N)$.

Given a topology, $\mathcal{T}$, on $C_b(N)$, we can make precise the notation that if an economic agent prefers a commodity bundle $\bar{x}$ to a commodity bundle $\bar{y}$, then he prefers bundles "close to $\bar{x}$" to bundles "close to $\bar{y}$". Preferences having this property are said to be continuous with respect to $\mathcal{T}$ or $\mathcal{T}$-continuous.

In the finite dimensional case, the assumption that an economic agent's preferences are continuous is made for technical reasons, e.g., to guarantee the existence of a preference maximizing commodity bundle in the agent's feasible choice set, if this set is compact. In this situation, where the commodity space is R_n, there are several equivalent "natural" topologies, e.g., the product or sup-norm topologies, denoted $\mathcal{T}_P$ and $\mathcal{T}_\infty$ respectively.

These topologies are equivalent in the sense that a complete preference relation on R_n is $\mathcal{T}_P$-continuous if and only if it is $\mathcal{T}_\infty$-continuous.

But these topologies are not equivalent on $C_b(N)$ and moreover there are other topologies on $C_b(N)$ which a priori seem as "natural" for economic analysis as the product or sup-norm topologies.

There is an economically interesting property shared by several of the topologies on $C_b(N)$, e.g., the product and strict (Mackey) topologies. The property of impatience or intertemporal myopia. Each of these topologies has the property that every complete continuous preference relation is impatient in the sense that present consumption is preferred to future consumption and the taste for future consumption diminishes as the time of consumption recedes into the future. Such topologies will be called myopic.

It is important to note that the sup-norm topology does not share this property, i.e., there exists a complete $\mathcal{T}_\infty$-continuous preference relation which does not "discount" the future.[(1)] In fact, if we relax the completeness requirement on preferences then the overtaking criterion induces a partially ordered $\mathcal{T}_\infty$-continuous preference relation on $C_b(N)$ which is not impatient, i.e., treats all "generations" equally.

In the capital theory literature, the behavioral assumption that economic agents are intertemporally myopic is central to the analysis and description of intertemporal economies. This suggests that the economically interesting intertemporal topologies on $C_b(N)$ are those which are myopic. Continuity is now a behavioral assumption rather than a technical requirement.

The product, strict (Mackey), and sup-norm topologies share the useful technical feature that they are Hausdorff locally convex (linear) topologies on $C_b(N)$. Since we wish to impose as few restrictions as possible on the myopic preferences of economic agents, we are led to consider the existence of the finest locally convex (linear) topology on $C_b(N)$ such that every complete continuous preference relation is (in some precise sense) impatient.

We shall investigate the following formal notions of intertemporal myopia:

A preference relation on $C_b(N)$ is said to be <u>weakly myopic</u> iff for all $\bar{x}, \bar{y}, \bar{c} \in C_b(N)$, where $\bar{c}$ is a constant vector; if $\bar{x}$ is preferred to $\bar{y}$ then $\bar{x}$ is also preferred to $\bar{y} + \hat{c}_n$ for all sufficiently large n, where $\hat{c}_n$ is a "tail" of $\bar{c}$, i.e., $\hat{c}_n(i) = 0$ for $1 \leq i \leq n$ and $\hat{c}_n(i) = \bar{c}(i) = c$ for $i > n$.

A topology, $\mathcal{T}$, on $C_b(N)$ is said to be weakly myopic iff every complete preference relation which is $\mathcal{T}$-continuous is weakly myopic. Bewley [3] attributes to Hildenbrand the observation that the strict (Mackey) topology is weakly myopic.(2)

A preference relation on $C_b(N)$ is said to be <u>strongly myopic</u> iff for all $\bar{x}, \bar{y}, \bar{z} \in C_b(N)$; if $\bar{x}$ is preferred to $\bar{y}$ then $\bar{x}$ is also preferred to $\bar{y} + \hat{z}_n$ for all sufficiently large n, wheren $\hat{z}_n$ is a "tail" of $\bar{z}$, i.e., $\hat{z}_n(i) = 0$ for $1 \leq i \leq n$ and $\hat{z}_n(i) = \bar{z}(i)$ for $i > n$. A topology, $\mathcal{T}$, on $C_b(N)$ is said to be strongly myopic iff every complete preference relation which is $\mathcal{T}$-continuous is strongly myopic. We note that the product topology is strongly myopic.

We shall show that the above model of intertemporal choice under certainty is formally equivalent to a model of atemporal choice under uncertainty if one adopts a state-preference model. Under this interpretation, we distinguish between commodities which are produced or consumed in different states of the world, and we posit a countable infinity of states.

We are now confronted with the task of defining linear topologies on this space which capture the essential aspects of economic decision making under uncertainty. The intuitive justification of the condition that we shall impose on linear topologies over the space of state-contingent consumption plans originates in Buffon's resolution of the St. Petersburg Paradox.

In the St. Petersburg game, a fair coin is tossed until heads appears, the individual is then paid 2^n dollars, where n is the number of tosses. What is the "fair price" for this gamble? It is generally argued that the "fair price" for a gamble is its mathematical expectation, but the St. Petersburg game has an infinite expectation.

As is well known, D. Bernouli suggested that is is the expected utility of money and not the expected money income that determines for an individual the "fair price" of a gamble. Moreover, if the marginal utility of money decreased at a sufficiently rapid rate, then the St.

Petersburg game has a finite expected utility which can be used as the "fair price".

Less well known, is another resolution of the St. Petersburg paradox, attributed by Arrow to Buffon, where it is suggested that individuals neglect improbable events. We quote from Arrow: "The principle of neglect of small probabilities was used by Buffon to resolve the St. Petersburg problem. The probability that a head will not appear until the n^{th} toss becomes very small for n sufficiently large; if the occurrence of that event is regarded as impossible for all n beyond a certain value, then the mathematical expectation of return becomes finite, and the paradox is resolved", [2].

Arrow goes on to give a cogent argument that this principle of decision making is inconsistent with the laws of probability and therefore is inadequate for a probabilistic theory of choice under risk.

We wish to argue that a consistent state-preference model of uncertain choice can be based on Buffon's principle of neglect of small probabilities. Moreover, this principle is simply a form of probabilistic myopia.

If we have a probability distribution over the states of the world, e.g., $\{P_t\}_1^\infty$, where $P_t > 0$ and $\sum_1^\infty P_t = 1$, then we can order the states of the world with respect to their probability of occurrence. We shall assume that $P_1 > P_2 > \cdots > P_n > \cdots$. More generally, we shall only assume that the states of the world are ordered with respect to their likelihood of occurrence and not assume, necessarily, that this ordering was derived from a probability distribution. Hence the state labeled 1 is the "most likely" to occur, the state labeled 2 is the "next most likely" to occur, etc.

$C_b(N)$ is now the space of state-contingent consumption plans. A topology, $\mathcal{T}$, on $C_b(N)$ is probabilisticly myopic if every complete $\mathcal{T}$-continuous preference relation has the property that if a state-contingent consumption plan $\bar{x}$ is preferred to another state-contingent consumption plan $\bar{y}$, then every sufficiently large finite truncation of $\bar{x}$ is preferred to all sufficiently large finite truncations of $\bar{y}$, i.e., in expressing a preference between $\bar{x}$ and $\bar{y}$, the economic agent "neglects" all states less likely than some sufficiently unlikely state.

As in the intertemporal setting, we restrict our attention to Hausdorff locally convex (linear) topologies on $C_b(N)$, for technical convience, and consider the existence of the finest Hausdorff locally convex (linear) topology on $C_b(N)$ such that every complete continuous preference relation is (in some precise sense) probabilisticly myopic.

We shall investigate the following formal notation of probabilistic myopia:

A preference relation on $C_b(N)$ is said to be <u>monotonicly</u> <u>myopic</u> if for all $\bar{x}, \bar{y} \in C_b(N)$; if $\bar{x}$ is preferred to $\bar{y}$ then for all sufficiently large n, $\check{x}_n$ is preferred to $\check{y}_n$, where $\check{x}_n$ is an "initial segment" of $\bar{x}$, i.e., $\check{x}_n(i) = \bar{x}(i)$ for $1 \leq i \leq n$ and $\check{x}_n(i) = 0$ for $i > n$.[3]

A topology, $\mathcal{T}$, on $C_b(N)$ is said to be monotonicly myopic iff every complete preference relation which is $\mathcal{T}$-continuous is monotonicly myopic. We show in part (a) of Theorem (5) that every strongly myopic, Hausdorff locally convex topology on $C_b(N)$ is monotonicly myopic.

If $\mathcal{T}$ is a Hausdorff locally convex (linear) topology on $C_b(N)$ which is weakly, strongly, or monotonicly myopic, we are naturally led to ask, what are some of its basic properties. That is, is it metrisable; what conditions characterize the compact sets; under what conditions do sequences converge; and most importantly for economic analysis, what is the nature of the dual space of continuous linear functionals. For it is the dual space that contains the family of permissible price systems.

An unpleasant fact about $C_b(N)$ is that it admits topologies, e.g., the sup-norm topology, such that some continuous linear functionals cannot be represented as a summable (ℓ_1) sequence $\{q_t\}_1^\infty$ where $\sum_1^\infty q_t < \infty$ and $q_t \geq 0$ if the linear functional is positive. It is difficult to imagine an economist taking seriously a notion of prices which did not allow him to talk about the social (or individual) marginal rate of substitution between two goods as the ratio of their social (or individual) prices. Hence the necessity of requiring the positive continuous linear functionals to have ℓ_1 representations.

In fact, the primary justification in the mathematical economics literature for considering the Mackey topology for the duality $\langle C_b(N), \ell_1 \rangle$ is that it is the finest Hausdorff locally convex (linear) topology on $C_b(N)$ such that all the continuous linear functionals on $C_b(N)$ can be represented as ℓ_1 sequences.

We would question this justification since what is important for economic analysis is only that the positive continuous linear functionals be ℓ_1 summable. As we show in part (b) of Theorem (4), the Mackey topology is not the finest Hausdorff locally convex (linear) topology on $C_b(N)$ such that all of the positive continuous linear functionals have ℓ_1 representations. In fact, $\mathcal{T}_{WM}$, the finest Hausdorff locally convex (linear) topology on $C_b(N)$ which is weakly myopic, is finer than the Mackey topology and every positive continuous linear functional in the dual of $\mathcal{T}_{WM}$ has a ℓ_1 representation -- see part (a) of Theorem (2). The intuition that the Mackey topology is an appropriate intertemporal topology is made precise by part (a) of Theorem (4), where it is shown that the Mackey topology is the finest Hausdorff locally convex (linear) topology on $C_b(N)$ which is strongly myopic.

In part (a) of Theorem (6), we give sufficient conditions for a monotonicly myopic, Hausdorff locally convex (linear) topology on $C_b(N)$ to be strongly myopic, where the crucial assumption is that the given topology be metrisable and complete. Hence these monotonicly myopic topologies are coarser than the Mackey topology.

$\mathcal{T}_{WM}$ is (as far as we know) a new topology on $C_b(N)$ and consequently its basic properties are unknown. In particular, we do not know if it is metrisable and complete. This is an important question. Since if $\mathcal{T}_{WM}$ is metrisable and complete and if V is the linear vector space consisting of linear functionals, L, on $C_b(N)$ where $L(\hat{u}_n) \to 0$, then $\mathcal{T}_{WM}$ is the finest Hausdorff locally convex (linear) topology, $\mathcal{T}_0$, on $C_b(N)$ such that V is the space of $\mathcal{T}$-continuous linear functionals.

To better understand the properties of $\mathcal{T}_{WM}$, we have also investigated $\mathcal{T}_M = \mathcal{T}_{WM} \cap \mathcal{T}_\infty$. Here we can give a complete characterization of the dual space -- part (b) of Theorem (3) -- and give sufficient conditions for a sequence to converge or a set to be compact. $\mathcal{T}_M$ is interesting in its own right and also appears to be a new topology on $C_b(N)$. The essential myopic feature of $\mathcal{T}_M$ is given in part (a) of Theorem (3).

II. Notation and Terminology

$C_b(N)$ is the family of bounded real-valued sequences.

$C_b^+(N)$ is the family of positive bounded real-valued sequences.

A preference relation on $C_b(N)$ is a transitive binary relation on $C_b(N)$. A preference relation $\succsim$ is said to be complete if for all $\bar{x}, \bar{y} \in C_b(N)$, either $\bar{x} \succsim \bar{y}$ or $\bar{y} \succsim \bar{x}$. $\bar{x} \succsim \bar{y}$ is to be read "$\bar{x}$ is preferred or indifferent to $\bar{y}$." $\bar{x} \succ \bar{y}$ is to be read "$\bar{x}$ is preferred to $\bar{y}$." $\bar{x} \sim \bar{y}$ is to be read "$\bar{x}$ is indifferent to $\bar{y}$."

If $C_b(N)$ is given a topology, $\mathcal{T}$, then a preference relation $\succsim$ is said to be $\mathcal{T}$-continuous if for all $\bar{x} \in C_b(N)$ the sets $\{\bar{y} \in C_b(N) | \bar{y} \succsim \bar{x}\}$ and $\{\bar{y} \in C_b(N) | \bar{x} \succsim \bar{y}\}$ are closed in the topology $\mathcal{T}$.

A preference relation is said to be monotonic if for all $\bar{x}, \bar{y} \in C_b(N)$ such that $\bar{x} > \bar{y}$, i.e., $\bar{x}(i) \geq \bar{y}(i)$ for all i and $\bar{x}(j) > \bar{y}(j)$ for some j, $\bar{x}$ is preferred to $\bar{y}$.

$\|\cdot\|$ is a seminorm on $C_b(N)$, if for all $\bar{x}, \bar{y} \in C_b(N)$ and every real number α:

(i) $\|\bar{x}\| \geq 0$,

(ii) $\|\bar{x} + \bar{y}\| \leq \|\bar{x}\| + \|\bar{y}\|$,

(iii) $\|\alpha\bar{x}\| = |\alpha| \|\bar{x}\|$.

If $\{\|\cdot\|_i\}$, $i \in \mathcal{J}$, is a family of seminorms on $C_b(N)$, then the topology they define on $C_b(N)$ is that generated by the following family of subsets of $C_b(N)$: $\{\bar{x} \in C_b(N) | \|\bar{x}-\bar{y}\|_i < \epsilon\}$ where $\bar{y}$ ranges over $C_b(N)$, $\epsilon > 0$, and i ranges over $\mathcal{J}$.

A family of seminorms $\{\|\cdot\|i\}$, $i \in \mathcal{J}$ is said to be saturated if the sup of any finite set of seminorms in the family is again a member of the family.

A linear functional, L, on $C_b(N)$ is said to be a countably additive integral if there exists a summable sequence $\{a_i\}_1^\infty$ such that for all $\bar{x} \in C_b(N)$, $L(\bar{x}) = \sum_1^\infty \bar{x}(i)\ a_i$.

A linear functional L on $C_b(N)$ is said to be a purely finitely additive linear functional if for every $\bar{x} \in C_b(N)$ such that $\bar{x}$ has at most a finite number of non-zero values, $L(\bar{x}) = 0$.

If $\bar{x} \in C_b(N)$, then $\check{x}_n$ is that vector in $C_b(N)$ such that $\check{x}_n$ $\check{x}_n(i) = \bar{x}(i)$ for all $i \leq n$ and $\check{x}_n(i) = 0$ for $i \geq n+1$. We define $\hat{x}_n$ as $\bar{x} - \check{x}_n$.

$\bar{u}$ is defined as that vector in $C_b(N)$ such that $\bar{u}(i) = 1$ for all i.

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on $C_b(N)$, then $\mathcal{T}_2$ is said to be finer than $\mathcal{T}_1$ if $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

$\mathcal{T}_P$ is the product topology on $C_b(N)$.

$\mathcal{T}_S$ is the strict topology on $C_b(N)$.

$\mathcal{T}_{SM}$ is the strong myopic topology on $C_b(N)$.

$\mathcal{T}_{WM}$ is the weak myopic topology on $C_b(N)$.

$\mathcal{T}_\infty$ is the sup-norm toplogy on $C_b(N)$.

$\mathcal{T}_M = \mathcal{T}_{WM} \cap \mathcal{T}_\infty$ is the myopic topology on $C_b(N)$.

The strict topology, $\mathcal{T}_S$, is generated by the following family of seminorms: Let $\{a_n\}_1^\infty$ be any sequence of real numbers converging to 0, then for any $\bar{x} \in C_b(N)$ define $\|\bar{x}\|$ as $\sup_{1 \leq n \leq \infty} |a_n\ \bar{x}(n)|$.

If E is a topological vector space, then F and G are supplementary subspaces of E iff $E = F \oplus G$, the direct sum of F and G. If the algebraic isomorphism of $F \oplus G$ onto E is a TVS isomorphism, then we say that F and G are topological supplements.

III. Statement of Theorems

Lemma (1). (a) If $\mathcal{T}$ is a Hausdorff locally convex topology on $C_b(N)$, then $\mathcal{T}$ is weakly myopic iff $\hat{u}_n \to \bar{0}$.

(b) If $\mathcal{T}$ is a Hausdorff locally convex topology on $C_b(N)$, then $\mathcal{T}$ is strongly myopic iff for all $\bar{x} \in C_b(N)$, $\hat{x}_n \to \bar{0}$.

Theorem (1). (a) There exists on $C_b(N)$ a finest Hausdorff locally convex topology, $\mathcal{T}_{WM}$, which is weakly myopic.

(b) There exists on $C_b(N)$, a finest Hausdorff locally convex topology, $\mathcal{T}_{SM}$, which is strongly myopic.

Lemma (2). (a) A purely finitely additive linear functional on $C_b(N)$ is $\mathcal{T}_{WM}$-continuous iff its null space contains $\bar{u}$.

(b) A purely finitely additive linear functional on $C_b(N)$ is $\mathcal{T}_{SM}$-continuous iff it is the zero functional.

Theorem (2). (a) A linear functional on $C_b(N)$, L, is $\mathcal{T}_{WM}$-continuous iff $L(\hat{u}_n) \to 0$. Moreover, a positive linear functional on $C_b(N)$ is $\mathcal{T}_{WM}$-continuous iff it is a positive countably additive integral.

(b) A linear functional on $C_b(N)$ is $\mathcal{T}_{SM}$-continuous iff it is a countably additive integral.

Theorem (3). (a) Let $\succsim$ be a complete, monotonic $\mathcal{T}_M$-continuous preference relation. For all $\bar{x}, \bar{y} \in C_b(N)$ if $\bar{x} > \bar{y}$ then there exists a constant vector $\bar{c}$ such that for all sufficiently large n, $\check{c}_n > \bar{y}$.

(b) A linear functional on $C_b(N)$ is $\mathcal{T}_M$-continuous iff it is the sum of a countably additive integral and a purely finitely additive linear functional whose null space contains $\bar{u}$.

Theorem (4). (a) $\mathcal{T}_S = \mathcal{T}_{SM}$.

(b) $\mathcal{T}_{SM} \subsetneqq \mathcal{T}_M \subsetneqq \mathcal{T}_{WM}$.

Theorem (5). (a) $\mathcal{T}_{SM}$ is monotonicly myopic.

(b) The $\mathcal{T}_{WM}$-interior of $C_b^+(N)$ is empty.

Theorem (6). (a) If $\mathcal{T}$ is a monotonicly myopic, complete metrisable Hausdorff locally convex topology on $C_b(N)$, such that for all $\bar{x} \in C_b(N)$; if $\lim\inf |\bar{x}(i)| = 0$, then $\hat{x}_n \to \bar{o}$. Then $\mathcal{T}$ is strongly myopic.

(b) If $\mathcal{T}$ is a complete metrisable Hausdorff locally convex topology on $C_b(N)$ such that every positive $\mathcal{T}$-continuous linear functional is a positive countably additive integral. Then $\mathcal{T}$ is weakly myopic.

IV. Proofs

Lemma (1). (a) Suppose $\hat{u}_n \to \bar{0}$ and $\bar{x} > \bar{y}$ for some complete $\mathcal{T}$-continuous preference relation $\succsim$. If $\bar{x} \not\succ \bar{y} + \hat{u}_n$ for all sufficiently large n, then there exists a subsequence $\hat{u}_{n_j}$ such that $\bar{y} + \hat{u}_{n_j} \succsim \bar{x}$. But $\hat{u}_{n_j} \to \bar{0}$ and therefore $\bar{y} \succsim \bar{x}$, a contradiction.

Suppose $\mathcal{T}$ is defined by a family of seminorns $\{p_\alpha | \alpha \in A\}$ and that every complete $\mathcal{T}$-continuous preference relation on $C_b(N)$ is weakly myopic. For every p_α there exists $\bar{x}_\alpha \in C_b(N)$ such that $p_\alpha(\bar{x}_\alpha) \neq 0$. $p_\alpha(\cdot)$ defines a complete continuous preference relation $\underset{\tilde{\alpha}}{>}$ on $C_b(N)$ where for all $\bar{x}, \bar{y} \in C_b(N)$, $\bar{x} \underset{\tilde{\alpha}}{>} \bar{y}$ iff $p_\alpha(\bar{x}) \geq p_\alpha(\bar{y})$. For any $\epsilon > 0$ and $\alpha \in A$, let $\beta = \epsilon / p_\alpha(\bar{x}_\alpha)$. Since $\epsilon = p_\alpha(\beta\bar{x}_\alpha)$, $\beta\bar{x}_\alpha \underset{\tilde{\alpha}}{>} \bar{0}$. Hence for sufficiently large n, $\beta\bar{x}_\alpha >_\alpha \hat{u}_n$, i.e., $\epsilon = p_\alpha(\beta\bar{x}_\alpha) > p_\alpha(\hat{u}_n)$. Therefore $\hat{u}_n \to \hat{0}$.(4)

(b) The argument is the same as that in part (a).

Theorem (1). (a) Let $\{P_\alpha | \alpha \in A\}$ be the family of seminorms on $C_b(N)$ such that for every $\alpha \in A$, $P_\alpha(\hat{u}_n) \to 0$. Then $\{P_\alpha | \alpha \in A\}$ is a saturated family of seminorms which contains the family of strict seminorms. Hence $\{P_\alpha | \alpha \in A\}$ generates a Hausdorff locally convex topology on $C_b(N)$, $\mathcal{T}_{WM}$. By part (a) of Lemma (1), $\mathcal{T}_{WM}$ is weakly myopic and is as fine as any other Hausdorff locally convex topology on $C_b(N)$, which is weakly myopic.

(b) Let $\{P_\beta | \beta \in B\}$ be the family of seminorms on $C_b(N)$ such that for every $\beta \in B$, and for all $\bar{x} \in C_b(N)$, $P_\beta(\hat{x}_n) \to 0$. Let $\mathcal{T}_{SM}$ be the topology generated by this family of seminorms. The argument that $\mathcal{T}_{SM}$ has the desired properties is the same as in part (a).

Lemma (2). (a) If L is a purely finitely additive linear functional on $\mathcal{C}_b(N)$, then for all n, $L(\check{u}_n) = 0$. Hence $L(\bar{u}) = L(\hat{u}_n) + L(\check{u}_n) = L(\hat{u}_n)$. Therefore, if L is $\mathcal{T}_{WM}$-continuous then $L(\hat{u}_n) \to 0$, i.e., $L(\bar{u}) = 0$.

If L is a purely finitely additive linear functional on $\mathcal{C}_b(N)$ and $L(\bar{u}) = 0$, then $L(\hat{u}_n) \to 0$. L defines a seminorm $\|\cdot\|$ on $\mathcal{C}_b(N)$, where for all $\bar{x} \in \mathcal{C}_b(N)$; $\|\bar{x}\| = |L(\bar{x})|$. This seminorm is $\mathcal{T}_{WM}$-continuous and for all $\bar{x} \in \mathcal{C}_b(N)$, $L(\bar{x}) \leq \|\bar{x}\|$. Hence L is $\mathcal{T}_{WM}$-continuous.

(b) If L is purely finitely additive linear functional, then for all $\bar{x} \in \mathcal{C}_b(N)$, $L(\check{x}_n) = 0$. Hence if L is $\mathcal{T}_{SM}$-continuous then $L(\bar{x}) = L(\hat{x}_n) + L(\check{x}_n) = L(\hat{x}_n) \to 0$. That is, $L(\bar{x}) = 0$, for all $\bar{x} \in \mathcal{C}_b(N)$.

Theorem (2). (a) If L is a linear functional on $\mathcal{C}_b(N)$ and $L(\hat{u}_n) \to 0$, then the seminorm $\|\cdot\|$, where $\|\bar{x}\| = |L(\bar{x})|$ for all $\bar{x} \in \mathcal{C}_b(N)$, is $\mathcal{T}_{WM}$-continuous. Hence L is $\mathcal{T}_{WM}$-continuous, since $L(\bar{x}) \leq \|\bar{x}\|$ for all $\bar{x} \in \mathcal{C}_b(N)$. If L is $\mathcal{T}_{WM}$-continuous then $L(\hat{u}_n) \to 0$, since $\hat{u}_n \to \bar{0}$ in the $\mathcal{T}_{WM}$-topology. The proof that a positive countably additive integral is a $\mathcal{T}_{WM}$-continuous linear functional is immediate. Suppose L is a positive $\mathcal{T}_{WM}$-continuous linear functional. We shall show that L is a $\mathcal{T}_{\infty}$-continuous linear functional. Let $\bar{y}_n$ be a sequence converging to $\bar{0}$ in the sup-norm topology, $\mathcal{T}_{\infty}$. Let $\alpha_n = \|\bar{y}_n\|_{\infty}$ and $\bar{z}_n = \alpha_n \bar{u}$, then $\bar{y}_n \leq \bar{z}_n$. Since $L(\bar{z}_n) = \alpha_n L(\bar{u}) \to 0$, $L(\bar{y}_n) \to 0$. Hence by the Hewit-Yosida respresentation Theorem, $L = L_c + L_p$ where L_c is a positive countably additive integral and L_p is a positive purely finitely additive linear functional. Since L_c is $\mathcal{T}_{WM}$-continuous, $L_p = L - L_c$ is $\mathcal{T}_{WM}$-

continuous. Therefore $L_p(\bar{u}) = 0$, by part (a) of Lemma (2). Hence if $\bar{x} \geq \bar{0}$, then $L_p(\bar{x}) = 0$. Since every $\bar{x} \in C_b(N)$ can be expressed as the difference of two positive vectors $\bar{x}^{(+)}$ and $\bar{x}^{(-)}$, $L_p(\bar{x}) = 0$, for all $\bar{x} \in C_b(N)$.

(b) Let L be a $\mathcal{T}_{SM}$-continuous linear functional and let $\alpha_i = L(\bar{e}_i)$. For every $\bar{x} \in C_b(N)$, $L(\bar{x}) = \lim_{n \to \infty} L(\check{x}_n)$. But $L(\check{x}_n) = \sum_1^n \bar{x}(i)\alpha_i$. Hence $L(\bar{x}) = \lim_{n \to \infty} \sum_1^n \bar{x}(i)\alpha_i$. Let $\bar{z}$ be the vector in $C_b(N)$ where $\bar{z}(i) = 1$ if $\alpha_i \geq 0$ and -1 otherwise. Then $L(\bar{z}) = \sum_1^\infty |\alpha_i|$. If L_c is the countably additive integral defined by the α_i, then $L = L_c$.

The converse is obvious.

Theorem (3). (a) Let A be the set of all constant vectors in $C_b(N)$, then A is connected in $\mathcal{T}_\infty$, hence connected in $\mathcal{T}_M = \mathcal{T}_{WM} \cap \mathcal{T}_\infty$. Suppose for some $\bar{x} \in C_b(N)$, either $\bar{x} > \bar{c}$ or $\bar{c} > \bar{x}$ for every $\bar{c} \in A$. Let $B = \{\bar{c} \in A | \bar{c} > \bar{x}\}$ and $D = \{\bar{c} \in A | \bar{x} > \bar{c}\}$, then B and D are $\mathcal{T}_M$-open and by monotonicity they are both nonempty. Therefore, $A = B \cup D$ and $B \cap D = \emptyset$, a contradiction. Hence if $\bar{x} > \bar{y}$ there exists a $\bar{c} > \bar{y}$, i.e., $\bar{c} \sim \bar{x}$. But $\check{c}_n \to \bar{c}$ with respect to $\mathcal{T}_{WM}$, hence $\check{c}_n \to \bar{c}$ with respect to $\mathcal{T}_M$. Therefore, $\check{c}_n > \bar{y}$ for all sufficiently large n.

(b) If L is a $\mathcal{T}_M$-continuous linear functional then L is $\mathcal{T}_\infty$-continuous, since $\mathcal{T}_M = \mathcal{T}_\infty \cap \mathcal{T}_{WM}$. Therefore by the Hewitt-Yosida representation theorem, $L = L_c + L_p$ where L_c is a countably additive integral and L_p is a purely finitely additive linear functional. It follows from Theorem (2) that L_c is $\mathcal{T}_{WM}$-continuous. Hence $L_p = L - L_c$ is $\mathcal{T}_{WM}$-continuous. Therefore by part (a) of Lemma (2), $\bar{u}$ is in the null space of L_p.

The converse follows immediately from Theorem (2) and part (a) of Lemma (2).

<u>Theorem 4</u>. (a) $\mathcal{T}_S \subseteq \mathcal{T}_{SM}$, since the family of strictly continuous seminorms is contained in the family of seminorms which generate $\mathcal{T}_{SM}$. In (4), Conway has shown that the finest Hausdorff locally convex topology on $C_b(N)$ such that every continuous linear functional is a countably additive integral is the strict topology, $\mathcal{T}_S$. Hence by part (b) of Theorem (2), $\mathcal{T}_{SM} \subseteq \mathcal{T}_S$. Therefore, $\mathcal{T}_S = \mathcal{T}_{SM}$.

(b) Every strongly myopic seminorm is clearly weakly myopic, hence $\mathcal{T}_{SM} \subset \mathcal{T}_{WM}$. Also $\mathcal{T}_S \subset \mathcal{T}_\infty$. Therefore $\mathcal{T}_{SM} \subset \mathcal{T}_M = \mathcal{T}_{WM} \cap \mathcal{T}_\infty$. It follows from Lemma (2) that there exists a non-zero $\mathcal{T}_M$-continuous linear functional which is not $\mathcal{T}_{SM}$-continuous. Hence $\mathcal{T}_{SM} \neq \mathcal{T}_M$. To complete the proof, we must show that $\mathcal{T}_M \neq \mathcal{T}_{WM}$.

The set consisting of $\{\bar{u}, \check{u}_1, \ldots, \check{u}_n, \ldots\}$ is a linearly independent subset of $C_b(N)$ and therefore may be extended to a Hamel basis of $C_b(N)$, which we shall denote as $\{\bar{z}_i\}_{i \in \mathcal{I}}$. Let $A = \{\check{u}_j\}_0^\infty$ where $\check{u}_0 \equiv \bar{u}$, then A is a proper subset of $\{\bar{z}_i\}_{i \in \mathcal{I}}$, e.g., $(1, 0, 1, 0, \ldots, 1, 0, \ldots)$ can not be expressed as a finite linear combination of vectors in A. In fact, there are at least a countable infinity of vectors B such that $A \cap B = \emptyset$ and $A \cup B$ a subset of $\{\bar{z}_i\}_{i \in \mathscr{I}}$. Let $\alpha_i : \mathscr{I} \to R$ such that $\alpha(i) = 1$ if $\bar{z}_i \in A$ and $\alpha(i) = n$ if $\bar{z}_i$ is the n^{th} element in some fixed enumeration of the vectors in B, otherwise $\alpha(i) = 0$. The function α determines a linear functional on $C_b(N)$, denoted T, where for each $\bar{x} \in C_b(N)$, $T(\bar{x}) = \sum_{i \in \mathscr{I}} \alpha(i) \beta_{\bar{x}}(i)$ and $\beta_{\bar{x}}(i)$ is the ith coordinate of $\bar{x}$ in the representation of $\bar{x}$ with respect to the Hamel basis $\{\bar{z}_i\}_{i \in \mathscr{I}}$. Now $T(\check{u}_j) = 1$ for $j \geq 0$, hence $T(\hat{u}_j) = T(\check{u}_0 - \check{u}_j) = T(\check{u}_0) - T(\check{u}_j) = 1-1 = 0$, for $j \geq 1$, T defines a seminorm $\|\cdot\|$, where for all $\hat{x} \in C_b(N)$, $\|\bar{x}\| = |T(\bar{x})|$. $\|\cdot\|$ is weakly myopic, since for all $j \geq 1$, $\|\hat{u}_j\| = |T(\hat{u}_j)| = 0$. Hence T is a $\mathcal{T}_{WM}$-continuous linear functional. Moreover, the vectors in B can be choosen to have sup-norm 1. Therefor T is not bounded with respect to $\mathcal{T}_\infty$ and hence not $\mathcal{T}_\infty$-continuous.

<u>Theorem (5)</u>. (a) Suppose $\succsim$ is a complete $\mathcal{T}_{SM}$-continuous preference relation on $C_b(N)$, and for some $\bar{x}, \bar{y} \in C_b(N)$; $\bar{x} > \bar{y}$. Since $\hat{x}_n \to \bar{o}$ and $\hat{y}_n \to \bar{o}$, it follows that $\check{x}_n \to \bar{x}$ and $\check{y}_n \to \bar{y}$. Hence there exists n_0 such that for all $n \geq n_0$, $\check{x}_n > \check{y}_n$.

(b). If we give $C_b(N)$ the natural ordering where $C_b^+(N)$ is the positive cone and assume that the $\mathcal{T}_{WM}$-interior of $C_b^+(N)$ is nonempty. Then by the following proposition, proven in (9), every positive linear functional on $C_b(N)$ is $\mathcal{T}_{WM}$-continuous. But there exists a positive purely finitely additive linear functional on $C_b(N)$, e.g., a linear extension of the first order Cesaro sum. This contradicts part (a) of Theorem (2).

<u>Proposition (1)</u>. If E is an ordered locally convex vector space and the positive cone of E has nonempty interior, then every positive linear function is continuous.

<u>Theorem (6)</u>. (a) Suppose $\mathcal{T}$ is defined by a family of seminorms $\{p_\alpha | \alpha \in A\}$ and that every complete $\mathcal{T}$-continuous preference relation on $C_b(N)$ is monotonicly myopic. If for some $\bar{x} \in C_b(N)$, $\hat{x}_n \not\to \bar{0}$, and $\liminf |\bar{x}(i)| > 0$, then there exists a $\mathcal{T}$-continuous seminorm $\|\cdot\|$ such that for some $\epsilon > 0$ and subsequence $\hat{x}_{n_j}$, $\|\hat{x}_{n_j}\| \geq \epsilon$ and $\inf |\hat{x}_{n_j}(j+1)| \geq$.

Let $\{\bar{z}_i\}_{i \in \mathcal{I}}$ be a maximal linearly independent subset of the $\{x_{n_j}\}_1^\infty$. Suppose $\mathcal{I}$ is finite, say of cardinality m. Let $\ell(i)$ be the index of the first non-zero component of $\bar{z}_i$, then $\ell(i) \neq \ell(j)$

for $i \neq j$. Let $\ell = \max_{1 \le i \le m} \ell(i)$ and consider some $\hat{x}_{n_{j_0}}$ where the index of the first non-zero component of $\hat{x}_{n_j}$ exceeds ℓ. The existence of $\hat{x}_{n_{j_0}}$ is guaranteed by our assumption that $\|\hat{x}_{n_j}\| \geq \epsilon$, for all j. But $\hat{x}_{n_{j_0}}$ cannot be expressed as a linear combination of the $\bar{z}_i$. Hence $\mathcal{I}$ is infinite and we can assume $\{\bar{z}_i\}_{i \in \mathcal{I}} = \{\hat{x}_{n_j}\}_1^{\infty}$. Moreover, we can also assume that the first component of $\bar{x}$ is not zero. Hence if $\hat{x}_{n_0} \equiv \hat{x}$, then $\{\hat{x}_{n_j}\}_0^{\infty}$ is also linearly independent. For any $\bar{y} \in C_b(N)$, let $V(\bar{y})$ be the linear subspace of $C_b(N)$ spanned by $\bar{y}$. For all $j \geq 1$, let $V_j = V(\hat{x}_{n_j}) \oplus \sum_{i=0}^{i<j} \oplus V(\hat{x}_{n_i} - \hat{x}_{n_j})$. The proofs of the following two propositions can be found in (5).

<u>Proposition (2)</u>. Let E be a Hausdorff locally convex space. Every finite dimensional vector subspace of E admits a topological supplement.

<u>Proposition (3)</u>. Let E be a topological vector space, F a closed vector subspace of finite codimension. Then every linear mapping of E into a topological vector space G which vanishes on F is continuous.

By Proposition (2), each V_j has a topological supplement W_j such that $C_b(N) \simeq U_j \oplus W_j$. Let $P_j : C_b(N) \to V_j$ where for all $\bar{y} \in C_b(N)$, $P_j(\bar{y}) = P_j(\bar{v} + \bar{w}) = \bar{v}$. Then P_j is continuous by Proposition (3). Let $T_j : V_j \to R$ where for all $\bar{q} \in V_j$, $T_j(\bar{q}) = T_j(\gamma\hat{x}_{n_j} + \sum_{i=0}^{i<j} \theta_i(\hat{x}_{n_i} - \hat{x}_{n_j}))$ $= \gamma\|\hat{x}_{n_j}\|$, then T_j is continuous. If $L_j = T_j \circ P_j$, then L_j is a continuous linear functional on $C_b(N)$.

If $\bar{y} \in C_b(N)$, then $\tilde{L}_j(\bar{y}) = \bar{y}(j+1)/\hat{x}_{n_j}(j+1)$. Hence $|\tilde{L}_j(\bar{y})| = |\bar{y}(j+1)|/|\hat{x}_{n_j}(j+1)| \leq \|\bar{y}\|_\infty/\inf|\hat{x}_{n_j}(j+1)| \leq \|\bar{y}\|_\infty/\varepsilon$, for all j.

That is, the L_j are pointwise bounded. Let $\delta_K = \sum_1^K \|\hat{x}_{n_j}\|$ and note that $\delta_K \geq K\varepsilon$. If $S_K = \frac{1}{\delta_K}\sum_1^K L_j$, then the S_K are pointwise bounded continuous linear functionals on $C_b(N)$. Since $\mathcal{T}$ is assumed to be complete and metrisable, $\{S_K\}_1^\infty$ is an equicontinuous family by the Banach-Steinhaus Theorem. We shall need the following proposition.

<u>Proposition (4)</u>. If $B = \{\bar{y} \in C_b(N) | \lim_{K\to\infty} S_K(\bar{y}) \text{ exists}\}$, then B is a closed subspace of $C_b(N)$.

<u>Proof</u>: Let $T : \langle C_b(N), \mathcal{T}\rangle \to \langle C_b(N), \mathcal{T}_\infty\rangle$ where for all $\bar{y} \in C_b(N)$, $T(\bar{y}) = \{S_K(\bar{y})\}^\infty$. Then T is obviously linear. Moreover, T is bounded and therefore continuous. Since C_{On}, the family of convergent real-valued sequences, is a closed subspace of $\langle C_b(N), \mathcal{T}\rangle$; we see that $B = T^{-1}(C_{On})$ is a closed subspace of $\langle C_b(N), \mathcal{T}\rangle$.(5)

Let $B = \{\bar{y} \in C_b(N) | \lim_{K\to\infty} S_K(\bar{y}) \text{ exists}\}$ and define $S : B \to R$ where for all $\bar{y} \in B$, $S(\bar{y}) = \lim_{K\to\infty} S_K(\bar{y})$. Then S is a continuous linear functional on B, by Banach-Steinhaus Theorem. Hence by the Hahn-Banach Theorem S can be extended to a $\mathcal{T}$-continuous linear functional $\tilde{S}$, defined on all of $C_b(N)$. $\tilde{S}$ defines a complete $\mathcal{T}$-continuous preference relation $\succsim$, where for all $\bar{x}, \bar{y} \in C_b(N)$; $\bar{x} \succsim \bar{y}$ iff $\tilde{S}(\bar{x}) \geq \tilde{S}(\bar{y})$. We shall show that $\succsim$ is not strongly myopic. Recall that for all j,

$\bar{x} = \hat{x}_{n_j} + \check{x}_{n_j}$. Hence for every $j \geq 1$, $\bar{x} = \hat{x}_{n_j} + (\hat{x}_{n_0} - \hat{x}_{n_j})$ $+ \sum_{i=1}^{i<j} O(\hat{x}_{n_i} - \hat{x}_{n_j}) + O\bar{W}_j$. Therefore, $L_j(\bar{x}) = \|\hat{x}_{n_j}\|$ and $S_K(\bar{x}) = \frac{1}{\delta_K} \sum_1^K \|\hat{x}_{n_j}\| = 1$. $S(\bar{x}) = \lim_{K \to \infty} S_K(\bar{x}) = 1$, i.e., $\bar{x} \in B$. Hence $\tilde{S}(\bar{x}) = S(\bar{x}) = 1$. But for all $j > \ell$, $\hat{x}_{n_\ell} = \hat{x}_{n_j} + (\hat{x}_{n_\ell} - \hat{x}_{n_j})$ $+ \sum_{i/\ell}^{i<j} O(\hat{x}_{n_i} - \hat{x}_{n_j}) + O\bar{W}_j$. Therefore, if $j > \ell$, then $L_j(\hat{x}_{n_\ell}) = \|\hat{x}_{n_j}\|$ and

$$S_K(\hat{x}_{n_\ell}) = \frac{1}{\delta_K} \sum_1^K L_j(\hat{x}_{n_\ell}) = \frac{1}{\delta_K} \left(\sum_1^\ell L_j(\hat{x}_{n_\ell}) + \sum_{\ell+1}^K L_j(\hat{x}_{n_\ell}) \right)$$

$$= \frac{1}{\delta_K} \left[\sum_1^\ell (L_j(\hat{x}_{n_\ell}) - \|\hat{x}_{n_j}\|) \right] + 1 , \qquad \text{for all } K > \ell .$$

Since $\delta_K \to \infty$ as $K \to \infty$, $\lim_{K \to \infty} S_K(\hat{x}_{n_\ell}) = 1$. Therefore $\{\hat{x}_{n_j}\}_0^\infty \subset B$ and $\tilde{S}(\hat{x}_{n_j}) = S(\hat{x}_{n_j}) = 1$. But $\tilde{S}(\bar{x}) = \tilde{S}(\hat{x}_{n_j} + \check{x}_{n_j}) = \tilde{S}(\hat{x}_{n_j})$ and therefore $\tilde{S}(\check{x}_{n_j}) = 0$. Since $\bar{x} > \bar{0}$ and for all j, $\check{x}_{n_j} \sim \bar{0}$; $\gtrsim$ is not monotonicly myopic.

(b) Suppose $\mathcal{T}$ is not weakly myopic, then we proceed as in the proof of part (a) to construct a $\mathcal{T}$-continuous linear functional $\tilde{S}$. Recall that $\tilde{S}$ is the extension of a linear functional S, whose domain is a subspace B. $\bar{u}$ and the $\{u_n\}_1^\infty$ are members of B. Also $\tilde{S}(\bar{u}) = \tilde{S}(\hat{u}_n) = 1$, for all n. Hence $\tilde{S}$ is not weakly myopic, i.e., the preference relation defined by $\tilde{S}$ is not weakly myopic. If $\tilde{S}$ is positive, then the proof is complete, since $\tilde{S}$ cannot be a countably additive integral by part (a) of Theorem (2). The positivity of $\tilde{S}$ follows from the following proposition proven in [9].

Proposition (5). If M is a linear subspace of a vector space E ordered by a cone K and if for each $\bar{x} \in K$, there is a $\bar{y} \in M$ such that $\bar{x} \leq \bar{y}$ then every positive linear functional on M can be extended to a positive linear functional on E.

We give $C_b(N)$ the natural ordering where the positive cone is $C_b^+(N)$ and will show that S is positive on B. If $\bar{x} \in B$ and $\bar{x} \geq \bar{0}$, then $S(\bar{x}) = \lim_{K \to \infty} S_K(\bar{x}) = \lim_{K \to \infty} \frac{1}{\delta_K} \sum_1^K L_j(\bar{x})$. To evaluate $L_j(\bar{x})$ we must express $\bar{x}$ in terms of $V_j \oplus W_j$. Recall that $V(\hat{u}_j) \oplus \sum_{i=0}^{i<j} \oplus V(\hat{u}_i - \hat{u}_j) = V_j$. (We have assumed that the subsequence $\hat{u}_{n_j}$ such that $\|\hat{u}_{n_j}\| \geq \epsilon > 0$ is, in fact, the original sequence $\hat{u}_n$.) If $\bar{x} \in C_b(N)$, then the coefficient of $\hat{u}_j$ in the representation of $\bar{x}$ is $\bar{x}(j+1)$. Therefore, $L_j(\bar{x}) = \bar{x}(j+1) \geq 0$ and $S_K(\bar{x}) \geq 0$, for all K. That is, $S(\bar{x}) \geq 0$.

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V. Footnotes

(1) The most comprehensive study of impatience with respect to the sup-norm topology is found in a series of papers by Koopmans and his collaborators. We refer the interested reader to his two most recent articles on intertemporal choice [6], [7].

(2) Hildenbrand's remark concerned the Mackey topology on $C_b(N)$, i.e., the finest locally convex topology on $C_b(N)$ such that all the continuous linear functionals can be represented as summable (ℓ_1) sequences. But Conway [4] has shown that on $C_b(N)$, the Mackey and strict topology are the same. The importance of the strict topology is that it is generated by a family of seminorms which can be interpreted as generalized discounted values. See Lewis [8] for a discussion of the strict topology as an intertemporal topology.

(3) M. Yaari suggested this notion of myopia in a personal communication, but it first appeared in the uncertainty literature, where Arrow [1] refers to it as monotone continuity.

(4) The idea for this proof orginated in a remark made by Roger Myerson.

(5) This is a minor modification of the proof of a similar result found in [10].

VI <u>References</u>

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