COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 455

A THEORY OF MONEY AND FINANCIAL INSTITUTIONS

PART 22

A PRICE-QUANTITY BUY-SELL MARKET WITH AND WITHOUT CONTINGENT BIDS

Martin Shubik

May 16, 1977

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A PRICE-QUANTITY BUY-SELL MARKET WITH AND WITHOUT CONTINGENT BIDS*

by

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1. *INTRODUCTION*

In this paper two specific models of trade are described. These each may be used to define a game of strategy in strategic form. We may then examine each game utilizing the noncooperative equilibrium solution concept. Our specific concern is with the description of trade and the relationship between the noncooperative equilibria and the competitive equilibria of the economic trading system regarded as a simple Walrasian system.

In several previous papers different market mechanisms have been investigated. In particular a "sell-all" model has been considered by Shubik [1] and Shapley and Shubik [2]; Dubey and Shubik [3] have investigated a "bid-offer" model of trade and Shubik has considered a simple market with price-quantity strategies [4] and it has been suggested that

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*This work related to Department of the Navy Contract N00014-76-C-0085 issued by the Office of Naval Research under Contract Authority NR 047-006. However, the content does not necessarily reflect the position or the policy of the Department of the Navy or the Government, and no official endorsement should be inferred.

**The author is grateful to Pradeep Dubey for several valuable discussions concerning these models.

there are only a limited number of market mechanisms involving simultaneous bidding in terms of money, goods or prices [5].

2. A PRICE-QUANTITY, BUY-SELL MECHANISM

A mechanism is described for a single market and then it is generalized for m markets. Suppose that n traders have endowments of two commodities, one of which serves as a means of payment and a numeraire. We fix the price of a unit of numeraire at 1.

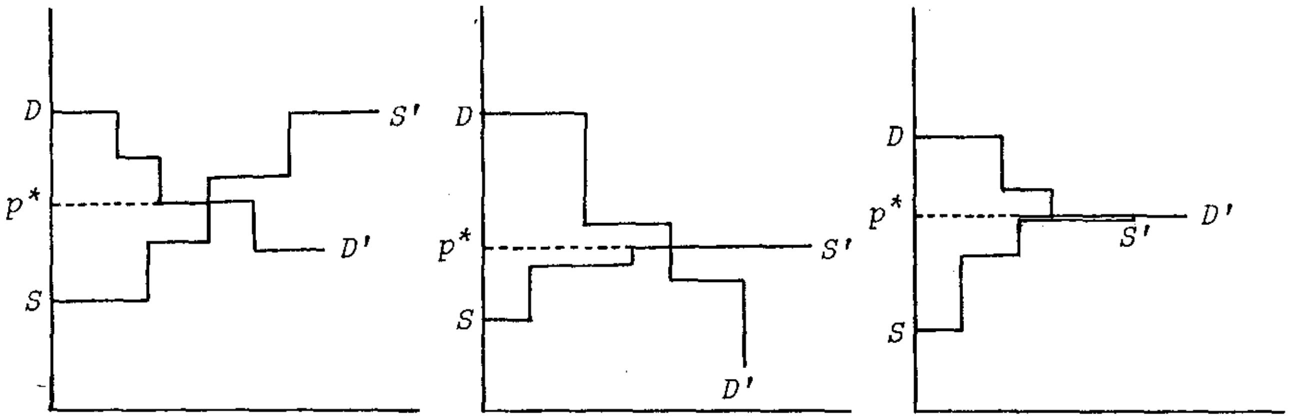
All traders are required to move simultaneously without knowledge of each other's actions by bidding and offering in a market. The endowment of trader i is given by (a_1^i, a_2^i) ; a move by trader i (which is also his strategy) is described by four numbers $(b^i, q^i; s^i, r^i)$ which are interpreted as follows:

b^i = the maximum price (in terms of the numeraire) that i will pay to buy q^i or fewer units of the first good. We require that $b^i q^i \leq a_2^i$, i.e., he cannot bid more money than he has on hand. We assume $q^i \leq 0$.

s^i = the minimum price that i will accept to sell r^i or fewer units of the first good. We require that $0 \leq r^i \leq a_1^i$, i.e., he cannot offer for sale that which he does not possess.

Although it may be unlikely that a trader will wish both to sell and buy the same item at the same time, there is no *a priori* reason to rule this behavior out, hence the strategies employed here permit an individual to be on both sides of the market if he so chooses.

Figures 1a, b, and c show possible configurations of supply and demand for the market in aggregate. The supply schedules are obtained by ranking the offer prices in ascending order and cumulating supply;

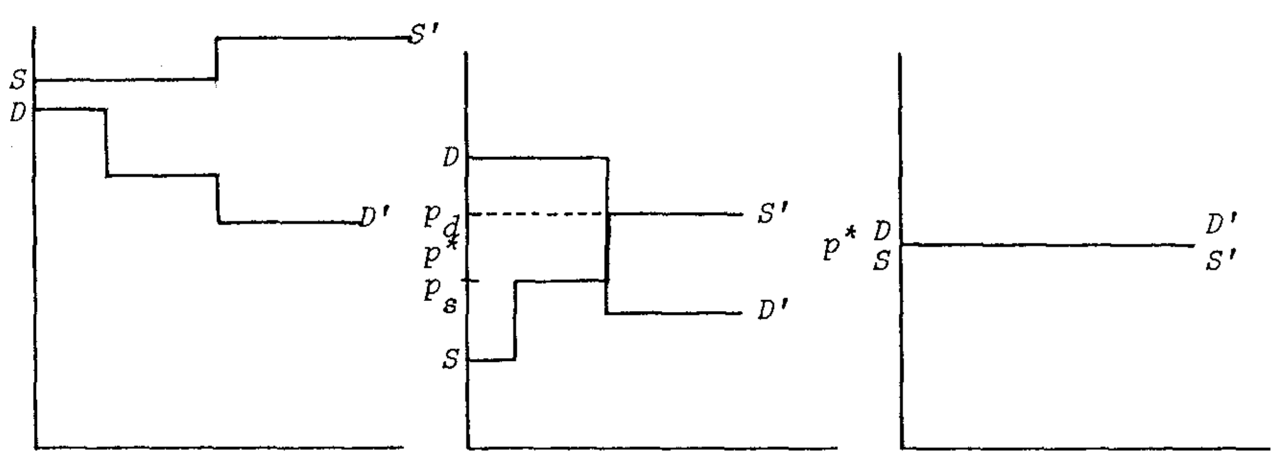


(a) (b) (c)

FIGURE 1

similarly demand prices are ranked, but in descending order.

The market mechanism works as follows. The aggregate supply and demand schedules are calculated and the market price is fixed at the price given by their intersection. All suppliers who require a higher price sell nothing; all buyers who require a lower price buy nothing. At the margin, as is shown in Figures 1a and c there may be some buyers willing to pay the market price of p^* for whom there is not enough supply. Similarly there may be sellers willing to supply more at p^* than the buyers wish to buy. This is shown in Figure 1b.



(a) (b) (c)

FIGURE 2

Figure 2 illustrates three further possibilities. In Figure 2a there will be no trade as there is no p^* for which both sides of the market will be active. Figure 2b illustrates an instance where there is an open range of prices at which effective demand will equal supply. Figure 2c shows an instance where all buyers and sellers have their demands and offers met.

When as in Figures 1a and 1c there is excess demand at p^* or, as in Figure 1b there is excess supply we adopt the convention that the marginal buyers or sellers are rationed in proportion to their demands or offers. In a market such as that shown in Figure 2a no trade will take place. In a market such as that shown in 2b we assume that a price interior to the range $p_s p_d$ is selected randomly.

3. *THE NONCOOPERATIVE EQUILIBRIA OF A PRICE-QUANTITY BID-OFFER GAME*

Let there be n individuals trading in m commodities using an $m+1^{\text{st}}$ commodity as a means of payment. Each trader i has an endowment of $(a_1^i, a_2^i, \ldots, a_{m+1}^i)$ where each is assumed to have enough money.* $a_j^i \geq 0$ for all $i = 1, \ldots, n$; $j = 1, \ldots, m+1$ and $a_j \ (= \sum_i a_j^i) > 0$.

Each individual i has a preference ordering which can be represented by a utility function $\phi_i(x_1^i, \ldots, x_{m+1}^i)$ which is assumed to be assumed to be concave increasing and differentiable. It is further assumed that for any distribution of resources there is always at least two traders who would desire more than they have of some particular re-

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*This has been defined elsewhere [6]. Intuitively all will have enough money if at any C.E. $\sum_{j=1}^{m} p_j w_j^i \leq a_{m+1}^i$, where w_j^i is the purchase of commodity j by trader i .

source. It is also assumed that in the initial distribution of resources at least two traders hold supplies of any particular resource.

A strategy by a trader i consists of a vector of $4m$ numbers $(b_1^i, q_1^i, s_1^i, r_1^i; \ldots; b_m^i, q_m^i, s_m^i, r_m^i)$ where:

$$\sum_{j=1}^{m} b_j^i q_j^i \leq a_{m+1}^i \quad \text{and} \quad 0 \leq r_j^i \leq a_j^i \quad \text{for} \quad j = 1, \ldots, m .$$

The market for each commodity j, $j = 1, \ldots, m$ operates as described in Section 2.

Theorem 1. The C.E.s of the trading market have associated* N.E.s in the price-quantity bid-offer game.

Proof. The C.E.s are obtained by solving the trading economy consisting of n traders trading in $m+1$ goods. At a C.E. aggregate supply will equal aggregate demand in each market. Consider all traders at a C.E. and allow some trader i to deviate. As can be seen from Figure 2c he can only influence price in a way he leaves his payoff reduced or as before. (For example he could indulge in a "wash sale" by offering k units of a commodity at a price lower than p^* and simultaneously bidding for k units at that price.)

Theorem 2. There will exist N.E.s with trade which differ from the C.E.s and hence will manifest excess supply or demand in various markets. For such N.E.s the excess supply or demand cannot be larger than the supply or demand of a single individual at the N.E. price.

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*An N.E. is associated with a C.E. if it has associated with it the same relative prices, distribution of goods and payoffs.

Proof. At any set of prices associated with an N.E. other than those also associated with a C.E. there must be excess supply or demand in some markets. Consider one such market. Suppose there is excess supply as is shown in Figure 1b and that there are two or more marginal suppliers (i.e. suppliers who are asking p^* or more for their supply). Each will be rationed and will sell a finite amount less than he offered. But this cannot be an N.E. because by an arbitrarily small price cut any marginal supplier can obtain a finite increase in sales and hence an increase in payoff. A similar argument holds for the cases shown in Figures 1a and 1c.

There remains the case shown in Figure 2b. Consider the inactive traders first. Either they are not in equilibrium, and hence this cannot be an equilibrium point, or they are. If they are, then the existence of an equilibrium depends upon the active traders. The moves of the active traders are such that over the range $p_s p_d$ supply equals demand, however excess supply or demand can be zero only at competitive equilibrium prices. Thus a case such as Figure 2b can be in equilibrium only if there is a continuous range of C.E.s from p_s to p_d.

Theorem 3. There exists a class of no-trade N.E.s.

Proof. This is obvious. Suppose all offer nothing for sale and all make no bids to buy. No single individual can improve. This equilibrium is clearly both trivial and highly unstable. It occurs in other models of trade as a noncooperative game and can be eliminated by introducing an "ε-offer and bid" much in the way a specialist is meant to make a market on the New York stock exchange. The introduction of an outside bid and offer has been discussed elsewhere [2, 3].

Theorem 4. Under replication the prices at any active N.E. differ by an amount $\varepsilon(k)$ from the prices at some C.E. where k is the replication number and $\varepsilon(k)$ can be made arbitrarily small as $k \to \infty$.

Proof. The same price rules for all in any market, at most excess supply or demand is determined by the excess supply or demand of a single individual of some type. Thus the failure to exactly satisfy the demands or utilize the supplies of others fails by some finite amount Δ , but $\Delta/k \to 0$ as $k \to \infty$ hence as the excess supply and demand functions are continuous functions of p in the neighborhood of a C.E. for a sufficiently large k , $|p^* - p| \leq \varepsilon(k)$.

4. *CONTINGENT BIDS*

In the general discussion on bidding and on tatonnement processes it has frequently been suggested that individuals could and should use contingent bids. For example they might submit a complete demand schedule dependent upon price. Hence an individual's strategy might be a set of m functions indicating how much he will buy or sell given any m market prices.

The device of having contingent bids does not enable us to avoid specifying a market mechanism. Price must still be formed. One simple way might be to aggregate all contingent bids and offers at the same price and calculate an excess supply or demand function. If points of zero excess were discovered the market manager would announce the price associated with one of them as the market price. Alternatively we might use the convention that whatever price minimizes excess supply or demand is selected as a market price and all individuals are rationed in proportion to their supply or demand at that price, if rationing is necessary.

The disadvantage with contingent bids as contrasted with the simpler bids suggested in Sections 2 and 3 are that they require a more elaborate or complicated market mechanism to determine price and furthermore contingent bids may permit the signalling of threats and cause a proliferation of equilibria. A simple two stage example shows this.

Consider two types of traders each trading twice in markets with perishable goods. Endowments for type 1 are (200, 0; 200, 0) and type 2 are (0, 2; 0, 2). All have the same utility functions

$$U^i = \log x_1^i + y_1^i + \log x_2^i + y_2^i \ .$$

We consider that no single trader influences price (i.e. we imagine a nonatomic market). It is easy to see that prices $p_{11} = p_{12} = 1$ and $p_{21} = p_{22} = 100$ with final endowments of (100, 1; 100, 1) for all, form the unique C.E. The payoff to all traders is:

$$\log 100 + 1 + \log 100 + 1 = 6 \ .$$

The following strategies however also constitute an N.E.

A Strategy for All Traders of Type 1

Offer 90 for sale in first period and a price of 1/90. If all traders of type 1 do likewise then offer 100 and a price of 1/100 in the second period. If any trader of type 1 does anything else offer 180 for sale in the second period and a price of 1/180. If any trader of type 2 does anything but offer to buy 90 at 1/90 offer 50 at 1/50 next period.

A Strategy for All Traders of Type 2

Offer to buy 90 at 1/90 or less in the first period and 100 at 1/100 or less in the second period if all traders of type 1 offered 90 at 1/90, otherwise offer to buy 180 at 1/180 in the second period.

These strategies yield

$$\Pi_1 = \log 110 + 1 + \log 100 + 1 = 6.0414$$

$$\Pi_2 = \log 90 + 1 + \log 100 + 1 = 5.954 \ .$$

An individual of type 1 can obtain at best, by using another strategy,

$$\Pi_1 = \log 90 + 11/9 + \log 20 + 1 = 3.176 + 2.301 = 5.477 \ .$$

The threat equilibrium may appear to be artificial, but so does the multistage C.E. Sometimes even masses of individuals "punish" or take action against a few who break a code of behavior even without formal communication.

At this time it appears that for multistage models the C.E. solution is unsatisfactory as it implicitly ignores all information whereas the N.E. models are unsatisfactory because as information is taken into account the strategy spaces become enormous and noncooperative equilibria proliferate and include many outcomes which do not appear to be reasonable.

The key element which appears to be central to resolving this unsatisfactory state is the coding of information and the specification of communication. Many of the *ad hoc* models of macroeconomics at least appear to offer a dynamics of behavior based on more information than is used in the Walrasian system, but less than that frequently called for by game theory models.

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