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A THEORY OF MONEY AND FINANCIAL INSTITUTIONS

PART 35

BANKRUPTCY AND OPTIMALITY IN A CLOSED TRADING MASS ECONOMY MODELLED AS A NONCOOPERATIVE GAME

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by

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1. INTRODUCTION

In this paper we establish the existence of an optimal bankruptcy rule which enables us to describe the Walrasian trading economy as a game with trade in fiat money and noncooperative equilibrium points which (in the sense specified below) include the competitive equilibria of the trading economy.

The construction suggested here raises several questions concerning the uniqueness of competitive equilibria, the role of a money rate of interest in providing reserves against bankruptcy and the information needed by a banking system to influence a market economy. Hopefully an understanding of the answers to these questions may lead to the formulation of some fruitful dynamic models.

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In a previous paper[1] a simple model of a trading economy with two types of traders, two goods, an outside bank, borrowing and bankruptcy rules; has been studied exhaustively. In that paper a specific example was used to illustrate the effect of varying the bankruptcy conditions. It was shown that there were essentially four cases to be distinguished. They are when no firm goes bankrupt, some go bankrupt and all go bankrupt, and also when no one goes bankrupt but some or all hoard money.

The details of the case distinctions can best be appreciated by working through a specific example in full. However this is both time consuming and lengthy hence rather than provide a new example here, the reader who wants an example is referred to the other paper.[1]

In this paper we establish in general for a trading economy a series of results concerning games with penalties against those who fail to repay loans. In particular:

(1) We can define a class of games related to the original trading economy for which we can prove the existence of non-trivial noncooperative equilibria.

(2) These equilibria can be compared with and related to the competitive equilibria of the original trading economy.

(3) The severity of the penalty for each individual can be characterized by a single parameter λ^i for each trader i. If the penalties are sufficiently high then the game we can define has among its equilibria all of the competitive equilibria of the original trading economy, in the sense that the distribution of goods and prices associated with any competitive equilibrium (C.E.) are also associated with a noncooperative equilibrium (N.E.).

(4) With high penalties, the money rate of interest will be zero at equilibrium and hoarding may take place.* The money rate of interest in the context of these models is a loss reserve premium to recoup bankruptcy losses rather than a price for the financing of intertemporal trade.

(5) If the penalties λ^i are picked to be sufficiently small and if they are not in the ratio of some set of Lagrangian multipliers at a C.E. then there will be a positive rate of interest and the distribution of goods at an N.E. will not coincide with that at a C.E.

(6) If the original trading economy has K competitive equilibria, associated with each C.E. for a given utility function representation of preferences will be a set of Lagrangian multipliers. For the k^{th} C.E. we may have $({}^k\lambda^1, \ldots, {}^k\lambda^n)$. These multipliers will vary along with any order preserving transformation of the utility functions. If it is possible to find some utility function representation of the preferences of the traders such that there is no pair of C.E.'s for which ${}^j\lambda^i \geq {}^k\lambda^i$ for $i = 1, \ldots, n$ then we may construct K trading games such that each of these games has a specific C.E. as its unique N.E. with a zero interest rate.

In the previous paper of Shubik and Wilson a market with a finite number of traders was considered. It appears to be considerably easier to work with a nonatomic trading model to establish the general results noted here. The recent work of Dubey and Shapley[2] enables us to adopt this approach in Section 3 and subsequently.

These rather informally stated assertions are made precise and the appropriate theorems are proved in the subsequent sections.

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*This possibility is illustrated by Shubik and Wilson[1] but is ruled out in our analysis here by a simplification of the strategy spaces.

2. MODELS AND MODELLING PROBLEMS

In several previous papers[3, 4, 5] trade in an economy using a commodity money has been studied. The conditions under which trade is hampered by a shortage of money have been noted. A shortage of commodity money can be overcome by the introduction of credit. However if the economy is regarded as even implicitly dynamic we must take into account the possibility that individuals may not be able to pay back that which they have borrowed. When this happens, rules must be given concerning the treatment of default.

We may regard an economy which uses fiat money as one in which a form of credit system is used. An outside bank issues a supply of money to all traders in return for their promissory notes. After trade has taken place the traders are required to redeem their promissory notes. Thus we model the economy as a two stage game where stage one is used to get the fiat money into the system and stage two is used for trade.

2.1. The Extensive Form

In Figures 1a and 1b two alternative forms of the extensive form of the two stage game are suggested. In Figure 1a all traders bid simultaneously for fiat money, then all are completely informed. In the second stage they then bid in the market for goods. The trees have been drawn as though the set of moves is finite.

In Figure 1b individuals are not informed of the outcome of the first stage prior to having to make their plans for the second stage. These represent two extremes in information conditions and imply extremely different strategy sets. The strategy sets are discussed in detail in 2.4 below.

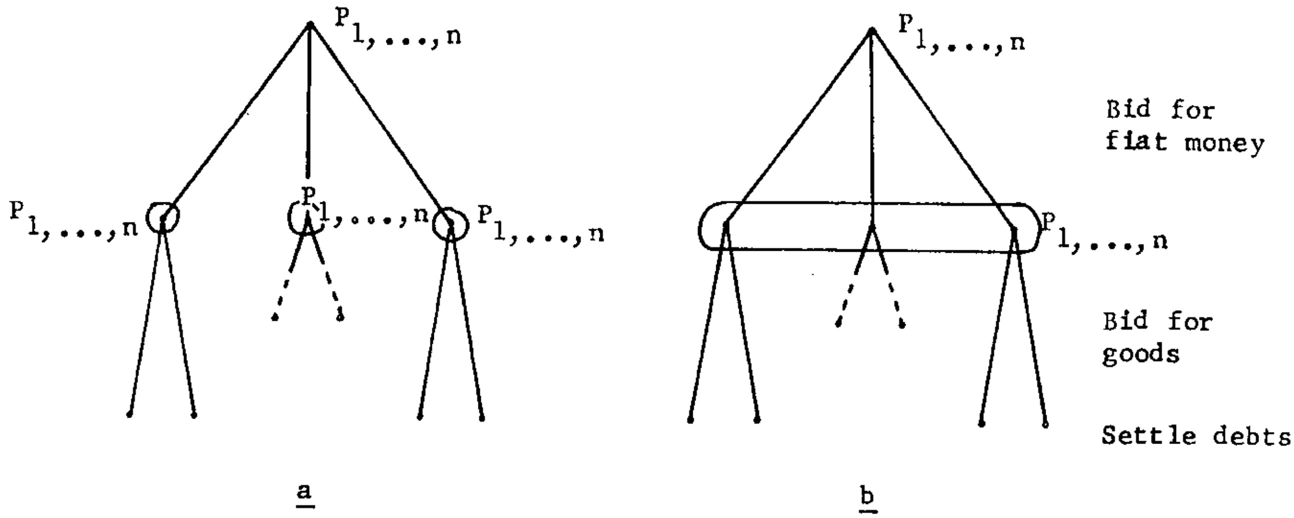

FIGURE 1

2.2. Endowments and Utility Functions

Although in Section 3 we restrict our interest to the nonatomic game; for convenience in discussion modeling problems prior to setting up this game we consider an economy with a finite number of traders.

Let there be n traders trading m commodities. Trader i has an initial endowment of $(a_1^i, \ldots, a_m^i)$ where $a_j^i \geq 0$ and $\bar{a}_j \ (= \sum_i a_j^i) > 0$.

The preferences of a trader i for any vector of resources $(x_1^i, x_2^i, \ldots, x_m^i)$ can be represented by a utility function $u^i(x_1^i, \ldots, x_m^i)$.

2.3. Moves, Outcomes and Modified Utility Functions

Let a move by trader i in the fiat money market be a bid (of a personal I.O.U. note) of size w^i.

Suppose* that the total fiat supply is M. He will obtain:

$$\frac{w^i}{\bar{w}} M .$$

*The case where there is no bound on the total fiat supply is analyzed in [2].

A move by a trader i in the second stage consists of a set of numbers $b_1^i, \ldots, b_m^i$ which in one model described in 2.4 can be interpreted as actual expenditures subject to the condition

$$\sum_{j=1}^{m} b_j^i \leq \frac{w^i}{\bar{w}} M$$

in another model a move will be a set of numbers $\eta_1^i, \ldots, \eta_m^i$, which will be interpreted as a percentage allocation of funds* thus actual expenditures will be a set of numbers $b_1^i, \ldots, b_m^i$ where $b_j^i = \eta_j^i (w^i/\bar{w})M$, $j = 1, \ldots, m$.

We define a set of penalties $\lambda^i > 0$, $i = 1, \ldots, n$ and a payoff function for each:**

$$\Pi^i(x_1^i, \ldots, x_m^i, x_{m+1}^i) = u^i(x_1^i, \ldots, x_m^i) \quad \text{for} \quad x_{m+1}^i \geq 0$$

$$= \hat{\Pi}^i(x_1^i, \ldots, x_m^i, x_{m+1}^i) \quad \text{for} \quad x_{m+1}^i < 0$$

where $\hat{\Pi}^i(x_1^i, \ldots, x_m^i, x_{m+1}^i) \leq u^i(x_1^i, \ldots, x_m^i) + \lambda^i x_{m+1}^i$;

where $x_{m+1}^i = \frac{w^i M}{\bar{w}} - \sum_{j=1}^{m} b_j^i + \sum_{j=1}^{m} p_j a_j^i - w^i$;

where $p_j = \bar{b}_j / \bar{a}_j$.

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*In this formulation hoarding is ruled out.

**For the second model the b_j^i are replaced by ${}^*b_j^i$

2.4. Strategies, Rationing Mechanisms and Payoffs

In a game where all players move without information, moves and strategies coincide. When information is available to the players in the process of making a sequence of moves then strategies can be clearly differentiated from moves.

Figures 1a and 1b show two extreme cases of full information or no information after the bids for money. In the Model 1 a strategy is described by a number and m functions of n variables $(w^i, f_1^i(w^1, w^2, \ldots, w^n), \ldots, f_m^i(w^1, w^2, \ldots, w^n))$. This means that because each individual knows precisely how much money all others have obtained he may base his selection of the b_j^i's upon every w^i. In the Model 2 a strategy and a move coincide. A move or strategy consists of $m+1$ numbers $(w^i, b_1^i, \ldots, b_m^i)$.

The most realistic model of the information structure is one in which each individual knows the amount of money he has obtained from the bids in the first stage and the aggregate amount for all others. In this game a strategy is described by a number and m functions of two variables $(w^i, f_1^i(w^1, \bar{w}^1), \ldots, f_m^i(w^m, \bar{w}^m))$ where $\bar{w}^i = \sum_{j \neq i}^{n} w^j$.

Model 2 has the least information, and hence the simplest strategies. The price paid for the simplicity of the strategies and the lack of information is the necessity to introduce an extra rationing device, described in 2.3 which guarantees that all bids are feasible. The device suggested here calls for a simple scaling up or down of bids. One could argue that this *ad hoc* rule is not "realistic," it is however simple and well defined and is probably one of a large class of rules which lead to models with the same limit noncooperative equilibria.

In attempting to define trade as a process, apparently arbitrary allocation rules are called for. They may be regarded as the elemental description of the way trading and financial institutions ration resources when systems are in disequilibrium.

2.5. Noncooperative Equilibria

The solution concept employed in this paper is that of the Nash noncooperative equilibrium (N.E.). Although this solution concept can be formally defined with ease for a multistage game, frequently even for games as specially structured as game models of economic exchange, the extreme multiplicity of equilibrium points makes the solution concept less attractive.

In spite of the aesthetic appeal of the competitive equilibrium theory and the formal extension of the equilibrium concept to a multiperiod economy, it provides neither an adequate description of the economy in all states of disequilibrium nor a sufficiently comprehensive description of strategy in a multiperiod model. In particular because the Walrasian analysis has concentrated primarily upon equilibrium it has been vague about the information conditions, the specifics of price generation and the details of individual selection of strategy. The reformulation of multistage economic trade as a noncooperative game forces a specificity in describing information conditions and the details of process. Although it may be possible to identify classes of noncooperative equilibria which appear to be naturally related to the competitive equilibria, it is also possible that other classes of noncooperative equilibria exist which reflect more complex strategies and communication than are implicitly or explicitly assumed in the study of competitive equilibria or related noncooperative equilibria.

Even though we regard the general equilibrium analysis to be vague in its treatment of information we may give an interpretation of the C.E.s in a class of multistage games in terms of N.E.s. Consider the class of multistage economies where individuals are equally informed. This class of games contains at one extreme the game where all know everything (as in Figure 1a) and at the other extreme the game where no one obtains further information during the play of the game (Figure 1b). As the amount of information is increased so is the size and complexity of the sets of strategies available to each trader. However the games with more information contain the strategy sets of the games with less information as subsets. In particular for large trading games which have the C.E.s as N.E.s (or limit N.E.s) given complete ignorance during trade, there will be related games with more information which also have the C.E.s as N.E.s. More specifically the C.E.s will be the only N.E.s belonging to all games which can be constructed from an original game with trade under no information, by the symmetric increase of information.

The replacement of the Walrasian model by the game formulation provides a far more "open or loosely coupled" model than previously. The use of noncooperative equilibrium solution provides a more general class of solutions which contains the competitive equilibria as a special case. It must be stressed that we regard the emphasis laid here upon the noncooperative equilibrium as merely a preliminary to the construction of a truly dynamic theory.

3. THE NONCOOPERATIVE MARKET GAMES

3.1. Notation and Definitions

For a positive integer k, Ω^k denotes the non-negative orthant of Euclidean space of dimension k. For any $x \in \Omega^k$, x_j is the j^{th} component of x, and $\bar{x}$ is $\sum_{j=1}^{k} x_j$. $\|\ \|$ stands for the maximum norm, i.e., $\|x\| = \max\{|x_j| : j = 1, \ldots, k\}$. For any x and y in Ω^k, $x > y$ $[\geq, =]$ means $x_j > y_j$ $[\geq, =]$ for $j = 1, \ldots, k$. 0 denotes both the number zero and the origin of Ω^k, but the meaning will be clear from the context. e_j is the vector whose j^{th} component is 1 and all other components are 0. H^k is the unit simplex in Ω^k, i.e., $H^k = \{x \in \Omega^k : \bar{x} = 1\}$.

Let $\{I, \mathcal{B}, \beta\}$ be a measure space. If f and g are measurable functions from I to Ω^k, we write $f > g$ $[\geq, =]$ if $f(x) > g(x)$ $[\geq, =]$ for all $x \in I$ except perhaps those x that belong to a null set i.e. a set $S \in \mathcal{B}$ for which $\beta(S) = 0$. Also we denote $f(x)$ often by f^x.

3.2. The Nonatomic Market

Let $\{T, \mathcal{C}, \mu\}$ be the measure space of traders, where T is the closed unit interval $[0,1]$ and corresponds to the set of traders, $\mathcal{C}$ is the Borel σ-algebra of coalitions, and μ is a finite, positive, non-atomic, complete measure on $\{T, \mathcal{C}\}$. W.l.o.g. we set $\mu(T) = 1$. The phrase "all traders" (or "each trader") will mean all traders (or each trader) except perhaps those that belong to a null set. For a measurable function $f : T \to R^k$; we abbreviate $\int_T f(t)d\mu(t)$ to $\int f$, and $\int_S f(t)d\mu(t)$ to $\int_S f$.

Trade occurs in m commodities. A commodity bundle is represented by a vector in Ω^m. The initial endowment of the traders is given by an integrable function $a : T \to \Omega^m$ where a^t is the endowment of trader t. $\int a_j$ is denoted by $\bar{a}_j$. $\hat{a}$ is the vector $(\bar{a}_1, \ldots, \bar{a}_m)$. We assume that $\hat{a} > 0$, i.e., there is a positive net amount of each commodity in the market.

To complete the data of the market we must also specify the preferences of the traders on commodity bundles. This is given (not uniquely) by a function $u : T \times \Omega^m \to \Omega^1$, where $u(t,\cdot)$ --also written $u^t(\cdot)$ --is the utility function of t. Our assumptions on u are as follows (compare with the assumptions in [6]):

(A1) u is measurable in the product space $T \times \Omega^m$ (where Ω^m is the standard Borel space).

(A2) u^t is continuous, concave, and nondecreasing for each t.

(A3) There is an integrable function $v : T \to \Omega^m$, $v > a$, such that:

(i) if $x_j \geq v_j^t$, then $u^t(x + \Delta e_j) = u^t(x)$ for all $\Delta > 0$;

(ii) if $x_j < v_j^t$, then u^t is strictly increasing in the j^{th} variable at x.

An allocation is a measurable function $x : T \to \Omega^m$ such that $\int x d\mu = \int a d\mu$. It describes a redistribution of the commodities among the traders. A competitive equilibrium (C.E.) of the market is a pair (p,x) where $p \in \Omega^m \setminus \{0\}$ is a price vector, and x is an allocation such that, for all t, x^t is optimal in t's budget set,

$$B^t(p) = \{y \in \Omega^m : p \cdot y \leq p \cdot a^t\},$$

i.e.,

$$u^t(x^t) = \max\{u^t(y) : y \in B^t(t)\}, \quad \text{and} \quad x^t \in B^t(p).$$

An allocation x [a price $p' \in \Omega^m \setminus \{0\}$] is called competitive if there is a $p \in \Omega^m \setminus \{0\}$ [an allocation x'] such that (p,x) $[(p',x')]$ is a C.E.

It is easy to check that, under our assumptions, if p is competitive then $p > 0$. For suppose $p_j = 0$ for some j. Let x^t be any optimal vector in $B^p(t)$. We must have $x^t_j = v^t_j$. So if x is any allocation such that (p,x) is a C.E., $\int x_j = \int v_j > \int a_j$, a contradiction.

Let us also recall the notion of "shadow prices of income"* at a C.E. Suppose (p,x) is a C.E. Then each trader t maximizes $u^t(y)$ subject to $y \in \Omega^m$, $p \cdot y - p \cdot a^t \leq 0$, and the maximum is obtained at $x^t \in B^t(p)$. By the Kuhn-Tucker theorem there exists a number $\lambda^t \geq 0$ such that x^t is also a solution of the following unconstrained problem:

$$\max[u^t(y) + \lambda^t(p \cdot a^t - p \cdot y)] , \quad \text{subject to } y \in \Omega^m .$$

The function $\lambda : T \to \Omega^1$ constitutes a choice of shadow prices at the C.E. (p,x). We show in Appendix A that for any C.E. a measurable choice of shadow prices exists, i.e. λ can be picked to be measurable. Note that if (p,x) is a C.E. with shadow prices λ, then for any $K > 0$, (Kp,x) is also a C.E. with shadow prices $\frac{1}{K}\lambda$. For any $B > 0$ we will call a C.E. (p,x) B-normalized if $p \cdot \hat{a} = B$.

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*Henceforth abbreviated "shadow prices."

3.3. Trade Using Fiat Money

To recast the market in the form of a noncooperative game we must describe the strategy sets and payoff functions of the traders. We will consider the continuum version of the finite game described earlier. Thus a strategy of trader t is to announce a bid ω^t of promissory notes, and a vector $\eta^t \in H^m$ which constitutes a decision as to how he will divide the fiat money he obtains from the bank into bids on the m trading-posts. Denoting t's strategy set by S^t, we then have

$$S^t = \{\{\omega^t, \eta^t\} : \omega^t \in \Omega^1, \eta^t \in H^m\} .$$

Given a choice of strategies by the traders, how are the trading-posts and the outside bank cleared? We are beset by a fundamental difficulty when we consider the mechanism of Section 2. This is because the mechanism calls for aggregating the bids in the bank and in each trading-post. These would be $\int \omega$ and $\int b_j$. But the integrals make sense only if the functions ω and b are assumed to be measurable. It is not clear how we would justify this assumption heuristically. Why should independent decision-makers behave in a jointly measurable way? We refer to Section 5 of [2], where a model of noncooperative behavior is suggested which leads to measurable strategies.

Assuming then that ω and η are measurable, and letting M stand for the fiat money in the bank, we define

$$1 + \sigma = \frac{\int \omega}{M} .$$

The amount of fiat money obtained by t is* $\omega^t/(1+\sigma)$, and thus his bid

*Division by 0 is defined to yield 0 throughout this paper.

b_j^t on the j^{th} trading-post is $\eta_j^t \omega^t/(1+\sigma)$. The price p_j is $\int b_j/\bar{a}_j$, and the final holding (of commodities) of t is: $x^t \in \Omega^m$, where $x_j^t = b_j^t/p_j$. His credit at the bank is given by

$$x_{m+1}^t = -\omega^t + \sum_{j=1}^m p_j a_j^t = -(1+\sigma)\bar{b}^t + \sum_{j=1}^m p_j a_j^t .$$

Hence, given a preassigned choice of bankruptcy penalties $\lambda : T \to \Omega^1$, the payoff to t is: $u^t(x^t) + \lambda^t \min[0, x_{m+1}^t]$.

We now have a game in strategic (or normal) form. With the rest of the data fixed, it depends on the choice of $\lambda : T \to \Omega^1$, and hence we will denote it by Γ_λ. A non-cooperative equilibrium (N.E.) of this game is a measurable $s_* : T \to \Omega^1 \times H^m$, $s_*^t \in S^t$, such that, for all t,

$$\Pi^t(s_*) = \max_{s^t \in S^t} \Pi\,(s_*|s^t) ,$$

where $(s_*|s^t)$ is the same as s_* except that s_*^t is replaced by s^t.

There exists a trivial N.E. of Γ_λ, namely the collection of strategies in which each trader bids nothing, i.e., $\omega = 0$. We will focus our attention on <u>active</u> N.E.'s, namely those which produce positive prices in each trading post. This implies that $1+\sigma > 0$, and in fact (as is easily checked) $\sigma \geq 0$.

We wish to investigate the N.E.'s of Γ_λ as λ varies. For this purpose it will be useful to demarcate certain regions in which λ may lie. Let

$$\Lambda = \{\lambda : T \to \Omega^1 : \lambda \text{ is measurable}\}$$

$$\Lambda_c = \{\lambda \in \Lambda : \lambda \text{ is a choice of shadow prices at some M-normalized C.E. of the market}\} .$$

For any ${}^*\lambda \in \Lambda$,

$$\Lambda_{*\lambda} = \{\lambda \in \Lambda : \lambda \geq {}^*\lambda\}$$

$$\Lambda_{(\lambda,\sigma)} = \{{}^*\lambda \in \Lambda : (1+\sigma){}^*\lambda = \lambda\}$$

$$\Lambda' = \Lambda\backslash[\Lambda_c \cup \{\Lambda_{*\lambda} : {}^*\lambda \in \Lambda_c\} \cup \{\Lambda_{(\lambda,\sigma)} : \lambda \in \Lambda_c,\ \sigma > 0\}] .$$

In the following when we say that an N.E. coincides with a C.E., we mean coincidence in prices and allocation.

<u>Proposition 1</u>. If $\lambda \in \Lambda_c$, there is an (active) N.E. of Γ_λ with $\sigma = 0$, which coincides with the C.E. associated with λ .

<u>Proof</u>. Let (p,x) be the C.E. associated with λ . Define $s : T \to \Omega^1 \times H^m$, $s^t \in S^t$, as follows:

$$\omega^t = p \cdot x^t$$

$$\eta_j^t = p_j x_j^t/\omega^t .$$

It is clear that, at s , $\sigma = 0$ and the prices and allocation produced are p and x . The proof will be finished by showing that s is an N.E. Denote, for any $b \in \Omega^m$, the vector $(b_1/p_1, \ldots, b_m/p_m)$ by $x(b)$. We must show that, for all t , $\max\{u^t(x(b)) + \lambda^t \min\{0, -\bar{b} + p\cdot a^t\}\}$ is achieved at $b = \omega^t\eta^t$. This follows by noting that, since λ is a choice of shadow prices at (p,x) , $\max\{u^t(x(b)) + \lambda^t(-\bar{b} + p\cdot a^t)\}$ is achieved at $b = \omega^t\eta^t = (p_1x_1^t, \ldots, p_nx_n^t)$, and that $u^t(x(b)) + \lambda^t(-b+p\cdot a^t) \geq u^t(x(b)) + \lambda^t \min[0, -\bar{b}+p\cdot a^t]\}$ for all $b \in \Omega^m$, with "=" when $b = \eta^t\omega^t$.

Q.E.D.

Proposition 2. If $^*\lambda \in \Lambda_\lambda$ for some $\lambda \in \Lambda_c$, there exists an N.E. of $\Gamma_{*\lambda}$ with $\sigma = 0$, which coincides with the C.E. associated with λ.

Proof. Let s be as in the previous proof, and observe that since $^*\lambda \geq \lambda$

$$u^t(x(b)) + {}^*\lambda^t \min\{0, -\bar{b} + p \cdot a^t\} \leq u^t(x(b)) + \lambda^t \min\{0, -\bar{b} + p \cdot a^t\}$$

with "=" when $b = \pi^t \omega^t$. Q.E.D.

Proposition 3. Suppose there is an (active) N.E. of $\Gamma_{*\lambda}$ with $\sigma = 0$. Then this N.E. coincides with some M-normalized C.E. (p,x). Moreover, there exists a set of shadow prices λ for (p,x) such that $^*\lambda \in \Lambda_\lambda$.

Proof. Let s be the N.E., (p,x) the prices and allocation produced at s, and x_{m+1} the credit. It is directly verified that $\int x = \int a$. Next note that $x^t_{m+1} \leq 0$ for all t. For suppose that $x^t_{m+1} > 0$ for t in a non-null set S. Then each trader $t \in S$ could improve his payoff by increasing his bids, contradicting that s is an N.E. But since $\sigma = 0$, $\int x_{m+1} = 0$. Hence $x^t_{m+1} = 0$ for all t. It follows immediately that (p,x) is a C.E. Let $\hat{\lambda} \in \Lambda$ be a choice of shadow prices associated with (p,x), i.e.,

$$\max_{b \in \Omega^m} \{u^t(x(b)) + \hat{\lambda}^t[-\bar{b} + p \cdot a^t]\}$$

occurs at $\hat{b}^t = (p_1 x_1^t, \ldots, p_m x_m^t)$. If $^*\lambda^t < \hat{\lambda}^t$, and $^*b^t$ solves

$$\max_{b \in \Omega^m} \{u^t(x(b)) + {}^*\lambda^t[-\bar{b} + p \cdot a^t]\}$$

it can be verified that we must have $-\overline{{}^*b}^t \leq -\bar{\hat{b}}^t$. Suppose $-\overline{{}^*b}^t < -\bar{\hat{b}}^t$ for t in a non-null set L. Then $-\overline{{}^*b}^t$ is also a solution of

$\max\{u^t(x(b)) + {}^*\lambda^t \min[0, -\bar{b} + p\cdot a^t]\}$ for $t \in L$, because $u^t(x(b)) + {}^*\lambda^t[-\bar{b} + p\cdot a^t] \geq u^t(x(b)) + {}^*\lambda^t \min[0, -\bar{b} + p\cdot a^t]$ with "=" when $b = {}^*b^t$ (since $-\overline{{}^*b}^t < -\bar{b}^t = p\cdot a^t$). But then $x^t_{m+1} < 0$ for $t \in L$, a contradiction. Hence $-\overline{{}^*b}^t = -\bar{\hat{b}}^t$ for all t such that ${}^*\lambda^t < \hat{\lambda}^t$. Let S' be the set $\{t \in T : {}^*\lambda^t < \hat{\lambda}^t\}$. Define λ as follows:

$$\lambda^t = \begin{cases} \hat{\lambda}^t & \text{if } t \notin S' \\ {}^*\lambda^t & \text{if } t \in S' . \end{cases}$$

Then λ is a choice of shadow prices for (p,x), and it is clear that ${}^*\lambda \geq \lambda$.

Q.E.D.

Proposition 4. Suppose there is an N.E. of $\Gamma_{{}^*\lambda}$ with $\sigma > 0$ which coincides with a C.E. Then ${}^*\lambda \in \Lambda_{(\lambda,\sigma)}$ where λ is a choice of shadow prices for the C.E.

Proof. Let (p,x) be the C.E. with shadow prices $\hat{\lambda}$, ${}^*s$ the N.E. of $\Gamma_{{}^*\lambda}$, ${}_*b$ the bids made in the second stage at ${}^*s$. Then

(1) $$\max_{b \in \Omega^m} \{u^t(x(b)) + \hat{\lambda}^t[-\bar{b} + p\cdot a^t]\} \text{ is attained at } {}_*b^t$$

(2) $$\max_{b \in \Omega^m} \{u^t(x(b)) + {}^*\lambda^t \min[0, -(1+\sigma)\bar{b} + p\cdot a^t]\} \text{ is attained at } {}_*b^t .$$

Denote the functions in (1) and (2) by $\hat{\xi}^t(b)$ and ${}^*\xi^t(b)$. We will find it useful to define, for $y \in \Omega^1$,

$$\hat{\phi}^t(y) = \max\{\hat{\xi}^t(b) : b \in \Omega^m, \bar{b} = y\}$$

$$^*\phi^t(y) = \max\{^*\xi^t(b) : b \in \Omega^m, \bar{b} = y\}$$

$$\phi^t(y) = {}^*\phi^t(y) + {}^*\lambda\sigma_*\bar{b}^t$$

$$\tilde{\phi}^t(y) = \max\{u^t(x(b)) + (1+\sigma)^*\lambda^t[-\bar{b} + p\cdot a^t] : b \in \Omega^m, \bar{b} = y\}$$

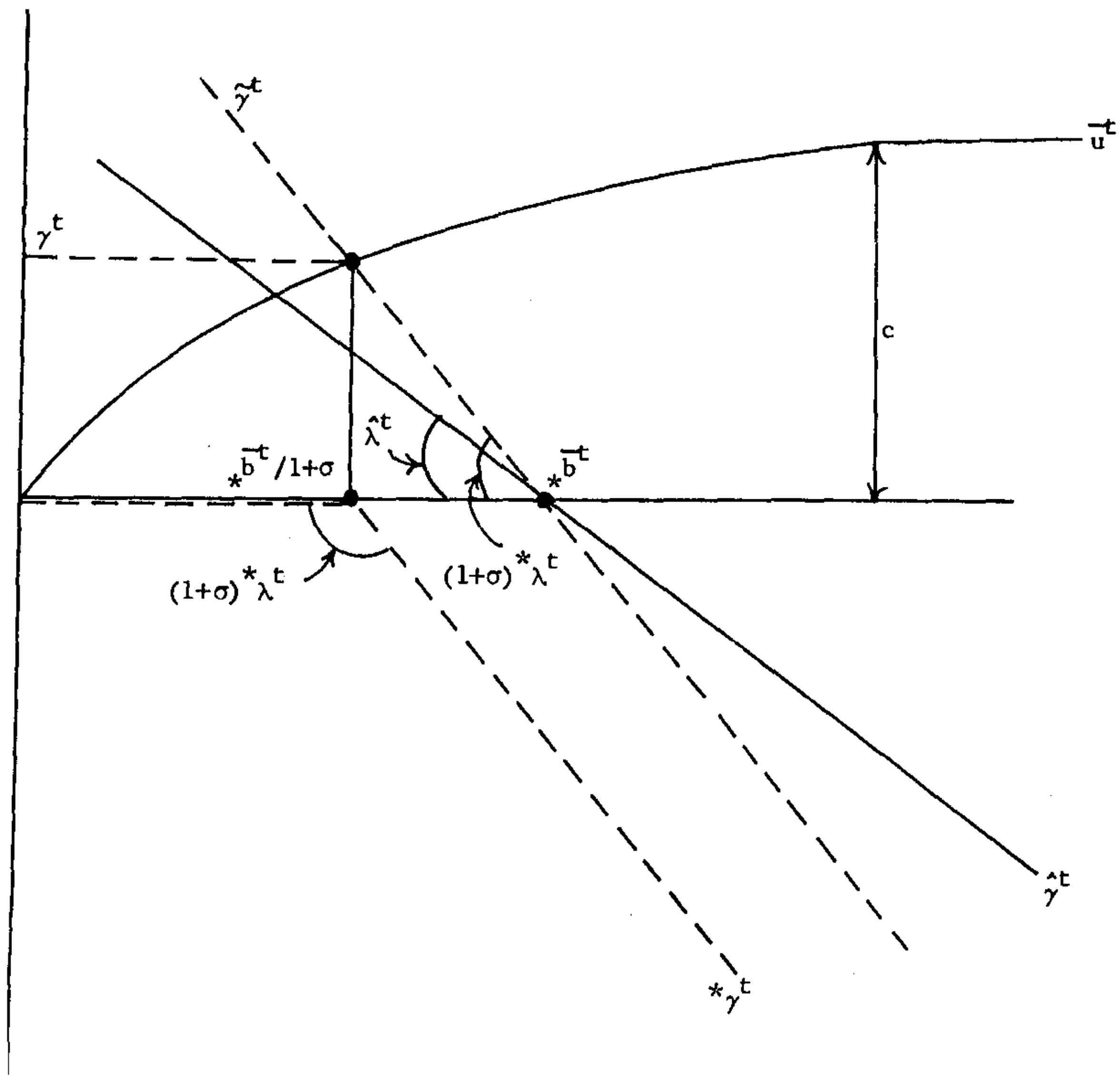


FIGURE 1

$$\bar{u}^t(y) = \max\{u^t(x(b)) : b \in \Omega^m, \bar{b} = y\} .$$

Note that, by (A3), $\bar{u}^t$ becomes constant (say c) for large enough y ; also note the relations (where $\hat{\gamma}^t$, ${}^*\gamma^t$, γ^t , $\tilde{\gamma}^t$ are as in Figure 1):

$$\hat{\phi}^t = \bar{u}^t + \hat{\gamma}^t$$

$${}^*\phi^t = \bar{u}^t + {}^*\gamma^t$$

$$\phi^t = \bar{u}^t + \gamma^t$$

$$\tilde{\phi}^t = \bar{u}^t + \tilde{\gamma}^t .$$

Let $\hat{G}^t$, ${}^*G^t$, $\tilde{G}^t$ and G^t denote the set of $y \in \Omega^1$ on which the max of $\hat{\phi}^t$, ${}^*\phi^t$, $\tilde{\phi}^t$ and ϕ^t are obtained. Note that these are closed and convex sets since the functions are continuous and concave. It is also easy to see that these sets must be compact as a consequence of the saturation Assumption A3 (see Figure 1). Hence we may represent them by closed intervals $[\hat{A}^t, \hat{B}^t]$, $[{}^*A^t, {}^*B^t]$, $[\tilde{A}^t, \tilde{B}^t]$, $[A^t, B^t]$. Moreover $[{}^*A^t, {}^*B^t] = [A^t, B^t]$ because ${}^*\phi^t$ differs from ϕ^t by a constant. Let $S_1 = \{t \in T : (1+\sigma){}^*\lambda^t > \hat{\lambda}^t\}$ and $S_2 = \{t \in S_1 : \tilde{B}^t < {}_*\bar{b}^t\}$. We will show that $\mu(S_2) = 0$. In the following let t be an element of S_2 .

If $\tilde{B}^t \geq {}_*\bar{b}^t/1+\sigma$, then $B^t = \tilde{B}^t$ because $\tilde{\phi}^t(y) \geq \phi^t(y)$ for all y , and $\tilde{\phi}^t(y) = \phi^t(y)$ if $y \geq {}_*\bar{b}^t/1+\sigma$. But, by (2), $B^t = {}^*B^t \geq {}_*\bar{b}^t$ for all t . This contradicts that $B^t = \tilde{B}^t < {}_*b^t$. Thus we see that for all $t \in S_2$, $\tilde{B}^t < {}_*\bar{b}^t/(1+\sigma)$. Take a $\delta > 0$ such that $\delta\tilde{B}^t + (1-\delta){}_*\bar{b}^t = y' \in [{}_*\bar{b}^t/1+\sigma, {}_*\bar{b}^t)$. Since $\tilde{\phi}^t$ is concave $\tilde{\phi}^t(y') \geq \delta\tilde{\phi}^t(\tilde{B}^t) + (1-\delta)\tilde{\phi}^t({}_*\bar{b}^t)$. But $\tilde{B}^t < {}_*b^t$ implies $\tilde{\phi}(\tilde{B}^t) > \tilde{\phi}^t({}_*\bar{b}^t)$. Hence $\tilde{\phi}^t(y') > \tilde{\phi}^t({}_*\bar{b}^t)$. This shows that $\phi^t(y') > \phi^t({}_*\bar{b}^t)$ because $\tilde{\phi}^t(y) = \phi^t(y)$ for $y \geq {}_*\bar{b}^t/(1+\sigma)$. Consequently $B^t < {}_*\bar{b}^t$. By (2) however $B^t = {}^*B^t \geq {}_*\bar{b}^t$. This proves that $\mu(S_2) = 0$.

Next let $S_3 = \{t \in T : (1+\sigma)^*\lambda^t < \hat{\lambda}^t\}$, and $S_4 = \{t \in S_3 : \tilde{A}^t > {}_*\bar{b}^t\}$. We show that $\mu(S_4) = 0$. Let t now refer to an element of S_4. Clearly $[\tilde{A}^t, \tilde{B}^t] = [A^t, B^t]$ because $\tilde{\phi}^t(y) \geq \phi^t(y)$ for all y, and $\tilde{\phi}^t(y) = \phi^t(y)$ if $y \geq {}_*\bar{b}^t/1+\sigma$. On the other hand, for all t, $A^t = {}^*A^t \leq {}_*\bar{b}^t$. This proves that $\mu(S_4) = 0$.

Define λ as follows:

$$\lambda^t = \begin{cases} \hat{\lambda}^t & \text{if } t \in T/(S_1 \cup S_3) \\ (1+\sigma)^*\lambda^t & \text{if } t \in S_1 \cup S_3 \end{cases}$$

Recalling that $\hat{\lambda}$ is a shadow price choice for (p,x) and that $\mu(S_2) = \mu(S_4) = 0$, we see that λ is also a shadow price choice for (p,x). But clearly ${}^*\lambda = (1+\sigma)\lambda$.

Q.E.D.

Proposition 5. Suppose $\lambda \in \Lambda'$. Then $\inf\{\sigma : \sigma$ occurs at an N.E. of $\Gamma_\lambda\} > 0$.

Proof. In the light of the previous propositions, it suffices to show that if $\inf\{\ldots\} = 0$, then there exists an N.E. of Γ_λ for which $\sigma = 0$.

Suppose ${}^ks$ is a sequence of N.E.'s of Γ_λ with rates of interest ${}^k\sigma$ and prices ${}^kp$, and suppose $\lim_{k\to\infty} {}^k\sigma = 0$.

Select a subsequence of ${}^ks$ such that $\lim_{k\to\infty} {}^kp$ exists and is, say, p. By Lemma 1 below $p > 0$. Also note that $\|{}^kp\| \leq F$ for all k, and hence $\|p\| \leq F$, where $F = \{\max M/\bar{a}_j : j = 1, \ldots, m\}$.

Let ${}^{k}b$ be the bids in fiat money made in stage 2 at ${}^{k}s$. Clearly, ${}^{k}b_j^t < Fv_j^t$. Therefore the functions ${}^{k}b$ are bounded from above (componentwise) by the integrable function Fv . For each t let $B(t)$ be the set of limit points of ${}^{k}b^t$ as $k \to \infty$. Let ${}^{k}B(t)$ be the set consisting of the single point ${}^{k}b^t$. Then $B(t) = \limsup {}^{k}B(t)$. Since all the ${}^{k}B$ are bounded above by the integrable function Fv , by Lemma 5.3 in [6], we have

$$\int p\cdot a = \lim \int {}^{k}p\cdot a = \lim \int {}^{k}b \in \limsup \int {}^{k}B$$

$$\subset \int \limsup {}^{k}B = \int B .$$

Let ${}^{*}b$ be such that ${}^{*}b^t \in B(t)$ for all t and $\int {}^{*}b = \int p\cdot a$.

First we show that ${}^{*}\bar{b}^t = p\cdot a^t$ for all t . Let ${}^{*}b^t = \lim_{r\to\infty} {}^{k_r}b^t$. Now ${}^{k_r}\bar{b}^t \geq {}^{k_r}p\cdot a^t/(1+\sigma^{k_r})$, hence by letting r tend to ∞ , we get ${}^{*}\bar{b}^t \geq p\cdot a^t$. But $\int {}^{*}\bar{b} = \int p\cdot a$, hence ${}^{*}\bar{b}^t = p\cdot a^t$ for all t .

Let ${}^{r}x(b)$ be the vector $(b_1/{}^{k_r}p_1, \ldots, b_m/{}^{k_r}p_m)$. Now ${}^{k_r}b^t$ maximizes $u^t(x(b)) + \lambda^t \min[0, (1+{}^{k_r}\sigma)\bar{b} - {}^{k_r}p\cdot a^t]$ at ${}^{k_r}b^t$. Since by Lemma 1 ${}^{k_r}p \xrightarrow[r\to\infty]{} p > 0$, ${}^{r}x(b) \to x(b) = (b_1/p_1, \ldots, b_m/p_m)$ for all b . Then recalling that ${}^{k_r}\sigma \xrightarrow[r\to\infty]{} 0$, and ${}^{k_r}b^t \to {}^{*}b^t$, we see that ${}^{*}b^t$ maximizes $u^t(x(b)) + \lambda^t \min[0, b - p\cdot a^t]$. Put $s^t = \{\omega^t, \eta^t\}$ where $\omega^t = {}^{*}\bar{b}^t$, and $\eta_j^t = {}^{*}b_j^t/\omega^t$. It is easy to verify that s is an N.E. of Γ_λ with $\sigma = 0$.

Q.E.D.

<u>Lemma 1</u>. Let ${}^{k}p$ be as in Proposition 5. Assume $\lim {}^{k}p \xrightarrow[k\to\infty]{} p$. Then $p > 0$.

Proof. Let α be the measure generated by the indefinite integral of v_1, i.e., $\alpha(S) = \int_S v_1$. α is absolutely continuous w.r.t. μ. Therefore for any $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$, such that $\mu(S) < \delta(\varepsilon)$ implies $\alpha(S) < \varepsilon$.

Let $A = \int v_1 - \int a_1 > 0$. Pick a K and an N^* small enough to ensure that $K + N^* < A$. For any allocation x, define

$$S_x = \{t \in T : x_1^t < v_1^t - N^*\} .$$

We assert that $\mu(S_x) \geq \delta(K)$, <u>for any allocation</u> x. Suppose not. Then $\int x_1 \geq \int_{T\setminus S_x} v_1 - N^*\mu(T\setminus S_x) \geq \alpha(T) - K - N^*$, therefore,

$$K + N^* \geq \int v_1 - \int x_1$$

$$= \int v_1 - \int a_1 > A ,$$

a contradiction.

From now on denote $\delta(K)$ by K^*.

Define $h : T \to R$ as follows:

$$h^t = \min\{u^t(x + e_1) - u^t(x) : x_1 < v_1^t - N^*\} .$$

By our assumptions on u^t, $h^t > 0$ for all t. Furthermore** h is measurable by Proposition 3 on page 60 in [7]. Let $Z_1 \subset Z_2 \subset \ldots$ be defined by:

$$Z_n = \{t \in T : h^t > 1/n\} .$$

**We are grateful to Gerard Debreu for this reference.

Since $h > 0$, $\bigcup_{n=1}^{\infty} Z_n = T$. Hence $\mu(Z_n) \xrightarrow[n\to\infty]{} 1$. Pick n_1 such that $\mu(Z_{n_1}) > 1 - K^*/2$. Then, <u>for any allocation</u> x ,

$$\mu(S_x \cap Z_{n_1}) > K^*/4 \; .$$

Finally define $V_1 \subset V_2 \subset \ldots$ by:

$$V_k = \{t \in T : 2\lambda^t < k\} \; .$$

Again $\bigcup_{k=1}^{\infty} V_k = T$, so $\mu(V_k) \xrightarrow[k\to\infty]{} \mu(T)$, and there exists a k_1 large enough to guarantee that $\mu(V_{k_1}) > 1 - K^*/8$. Put $W_x = S_x \cap Z_{n_1} \cap V_{k_1}$. Observe that, <u>for any allocation</u> x , $\mu(W_x) > K^*/8$.

We are prepared to prove that $p > 0$. Suppose not. W.l.o.g. let $p_1 = 0$ and assume** that ${}^{\ell}\sigma < 1$ for all ℓ . For large enough ℓ , ${}^{\ell}p_1 < 1/n_1 k_1$. Consider the set $W_{{}^{\ell}x}$. It is non-null (in fact its measure is at least $K^*/8$). If any $t \in W_x$ increases his bid on commodity 1 by the amount $1/n_1 k_1$ the increase in his final holding is given by:

$$\Delta^{\ell} x_j^t = 0 \quad \text{for} \quad j = 2, \ldots, m$$

$$\Delta^{\ell} x_1^t \geq 1/n_1 k_1 \, {}^{\ell}p_1 > 1 \; .$$

Hence the increase in the utility of the final bundle of t is greater than $1/n_1$. On the other hand the disutility for going bankrupt is at most $(1 + {}^{\ell}\sigma)\lambda^t / n_1 k_1 < k_1 / n_1 k_1 = 1/n_1$. Hence t can improve his payoff by increasing his bid on 1 in the amount $1/n_1 k_1$. Since each $t \in W_{{}^{\ell}x}$ can do this, ${}^{\ell}s$ cannot be an N.E. of Γ_λ , a contradiction. Q.E.D.

**Recall that we are assuming here that $\lim_{\ell\to\infty} {}^{\ell}\sigma = 0$.

Proposition 6. Let $\lambda \in \Lambda'$. Suppose (p,x) are the prices and allocation at an active N.E. of Γ_λ . Then (p,x) is not a C.E.

Proof. Obvious from the previous propositions.

Proposition 7. Let $^*\lambda = (1+\sigma)\lambda$, where $\lambda \in \Lambda_c$. Let (p,x) be the C.E. corresponding to λ . Then there exists an N.E. of $\Gamma_{*\lambda}$ coinciding with the C.E. and with rate of interest equal to σ .

Proof. The following s is easily verified to be the N.E.:

$$s^t = (\omega^t, \eta^t)$$

where

$$\omega^t = (1+\sigma) \sum_{j=1}^{m} p_j x_j^t$$

$$\eta_j^t = p_j x_j^t / \sum_{j=1}^{m} p_j x_j^t .$$

Q.E.D.

4. CONCLUDING REMARKS

4.1. On Uniqueness

Suppose that the penalties have been set at some function λ which corresponds specifically to the choice of shadow prices at some C.E. Can we deduce that the N.E. for this game is unique?

There appear to be two different questions which must be answered before such a conclusion could be drawn.

It is clear that for an economy with K C.E.'s if there is a utility function representation of preferences such that the shadow prices at any one C.E. are never $\geq$ (for each trader) the shadow prices at another C.E. then there will only be one C.E. with $\sigma = 0$ for any game with penalties set this way.

Question 1. We have not been able to establish that this will always be the case. Lloyd Shapley has provided an example for specific utility functions where the shadow prices at one C.E. strictly dominate those at another. But there are concavity preserving transformations of the utility functions for this case which get rid of the domination. We do not know, however, if this can be done in general. (Figure 2 provided by Shapley shows the example.)

Even supposing that we had an economy where the conditions for λ_c with a unique equilibrium with $\sigma = 0$ were met, another question remains.

Question 2. Given a game with a unique N.E. with $\sigma = 0$ is it possible that another N.E. with $\sigma > 0$ exists?

We have not been able to prove this but we suspect that the answer is no without too many qualifications beyond differentiability and perhaps strict concavity.

Figure 3 is drawn for a non-atomic economy with two types of traders. We consider only those λ which are type-symmetric, hence we may represent them by vectors in Ω^2. Figure 3 shows that λ-space. We assume that there are a finite number of C.E.'s, and for any C.E. there is a unique choice of (type-symmetric) shadow prices, represented by $\cdot$ in Figure 3.

From our propositions an intimate relationship emerges between the bankruptcy penalties and optimality. Ignoring those λ that lie in $\cup\{\Lambda_{(\lambda,\sigma)} : \lambda \in \Lambda_c, \sigma > 0\}$, which is in any case a "small" set (see Figure 3), the active N.E.'s of Γ_λ contain at least one C.E. if and only if $\lambda \in \Lambda_c \cup \{\Lambda_\lambda : \lambda \in \Lambda_c\}$. For such λ $\inf\{\sigma : \sigma$ occurs at an active N.E. of $\Gamma_\lambda\} = 0$; indeed, those N.E.'s with $\sigma = 0$ are precisely the ones that coincide with C.E.'s. For other λ, this $\inf > 0$, and the N.E.'s do not contain any C.E. We can think of this inf as a measure of the non-optimality (or deviation from a C.E.).

Another way of considering this is to conceive of all games defined by taking λ's associated with every point on the Pareto optimal surface of the trading economy. To each game there will be a smallest σ defined, where $\sigma = 0$ at the choice of λ's associated with C.E.'s thus we have a function of σ defined on the Pareto optimal surface.

Piecewise linear utility functions--can be made strictly concave and differentiable

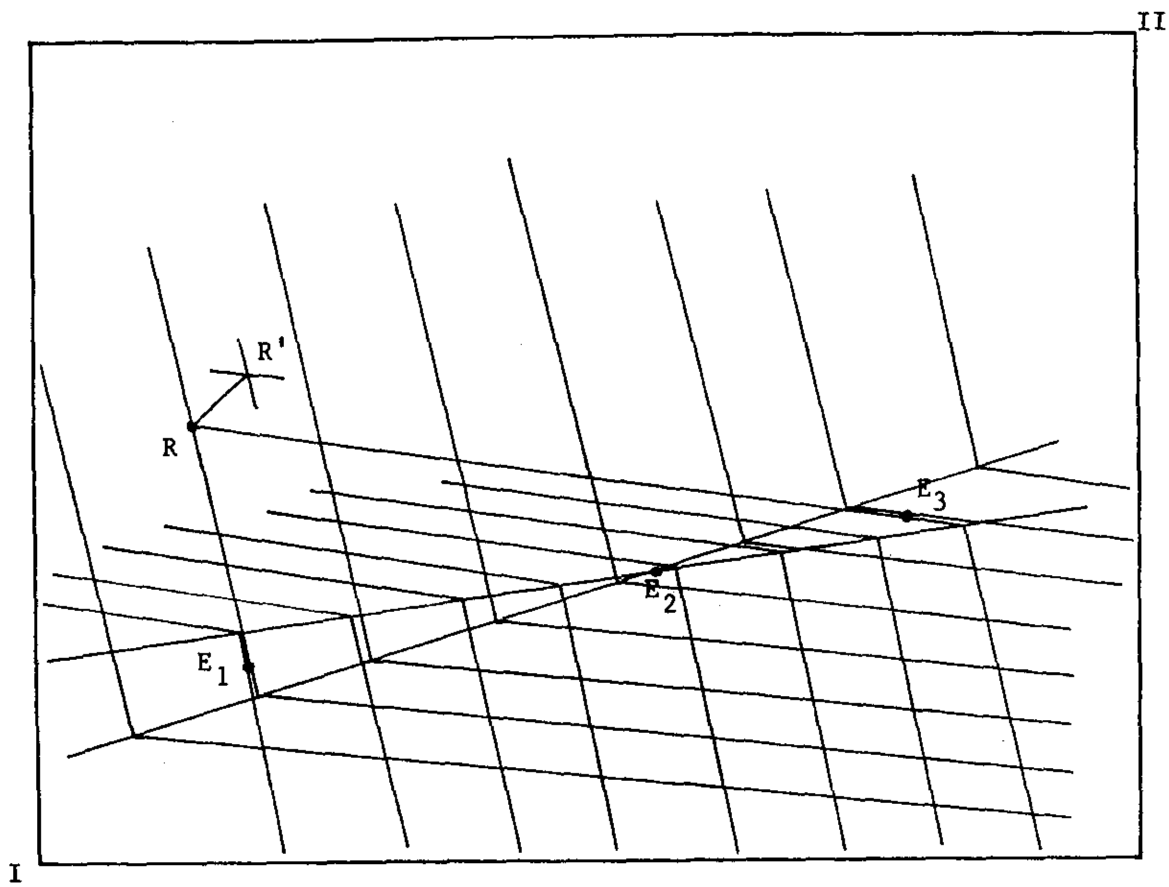


FIGURE 2

Prices at E_1 : (.8, .2)

Prices at E_3 : (.1, .9)

$$\frac{\text{I's marginal utility of income at } E_3}{\text{I's marginal utility of income at } E_1} \approx 3:1$$

$$\frac{\text{II's marginal utility of income at } E_3}{\text{II's marginal utility of income at } E_1} \approx 3:1$$

II is <u>worse</u> off at E_3 , but has <u>greater</u> <u>m.u.i.</u>

(I is better off at E_3 , and has greater m.u.i.)

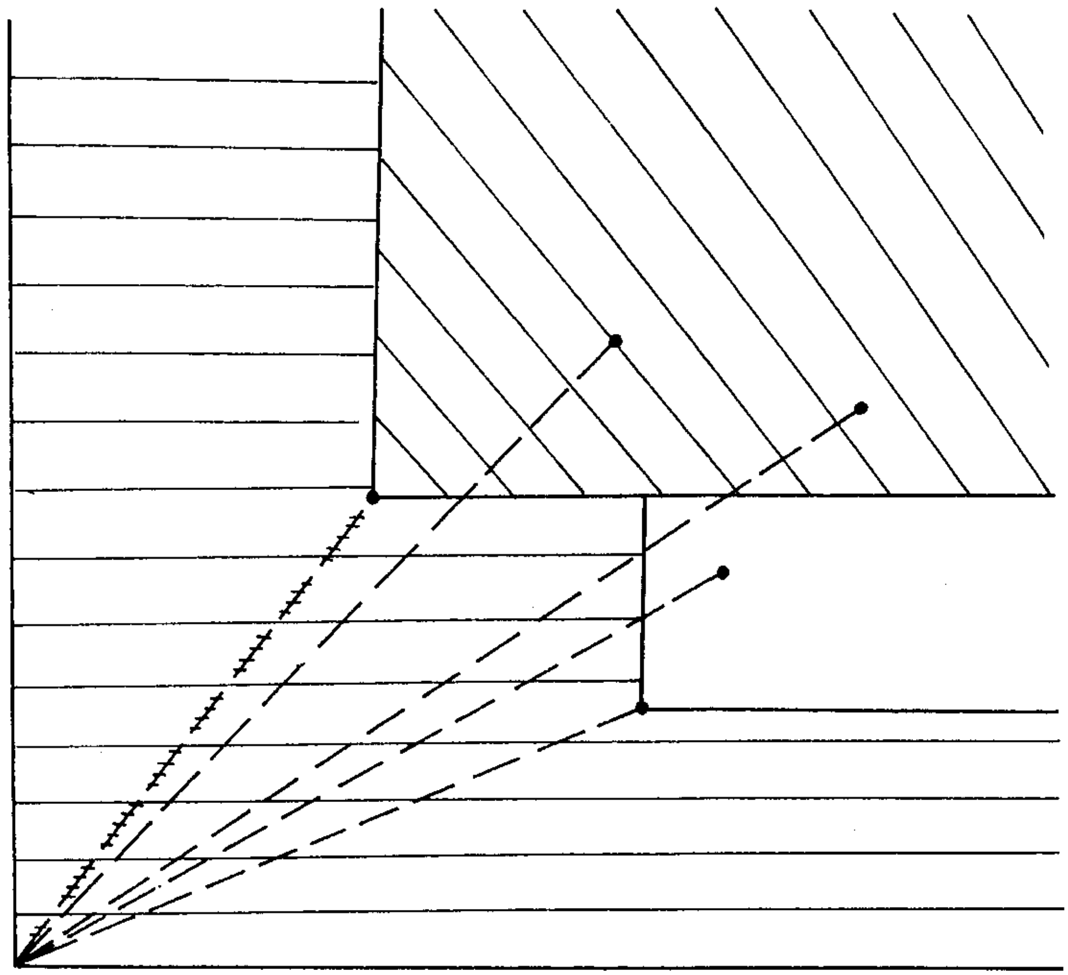
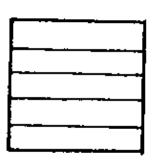
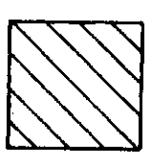


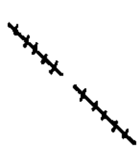
FIGURE 3



is Λ' , deleting the broken lines



is Λ_λ



is $\cup\{\Lambda_{\lambda,\sigma} : \sigma > 0\}$. σ increases as we move towards the origin.

APPENDIX A*

Consider the map $T \times \Omega^m \times \Omega^1 \xrightarrow{F} \Omega^1$, where

$$F(t, y, \mu) = u^t(y) + \mu(p \cdot y - p \cdot a^t) .$$

$\emptyset$ is measurable, and therefore, by in , so is $T \times \Omega^1 \xrightarrow{G} \Omega^1$, where

$$G(t,\mu) = \max_{y \in \Omega^m} \{F(t, y, \mu)\} .$$

Denoting the C.E. under consideration by (p,x), now define subsets L and R in $T \times \Omega^1 \times \Omega^1$ as follows:

$$L = \{[t, \mu, G(t,\mu)] : (t,\mu) \in T \times \Omega^1\}$$

$$R = \{[t, \mu, u^t(x^t)] : t \in T, \mu \in \Omega^1\} .$$

Clearly L and R are measurable, and so is $S = L \cap R$. Let P be the projection of S on $T \times \Omega^1$. By Proposition 3 in [7] (p. 60), P is measurable. Also for each $t \in T$, $(\{t\} \times \Omega^1) \cap P \neq \emptyset$ (by the Kuhn-Tucker theorem). Now we invoke the measurable choice theorem** to claim that there is a measurable function $\lambda : T \to \Omega^1$ such that $(t, \lambda^t) \in P$ for all t. This λ clearly constitutes a set of shadow prices for (p,x).

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*We are grateful to Gerard Debreu for suggesting the outline of this argument.

**See, e.g., Proposition 36.5 in [8].

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