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PROBABILISTIC GENERALIZATIONS OF THE SHAPLEY VALUE

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by

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1. Introduction

The purpose of this note is to consider the set of "probability-values" for games in characteristic function form--defined by Blair in [1]--and to isolate a subset of these which coincide with the competitive payoffs of nonatomic economies. This leads us to the class of "diagonal $\mathcal{P}$-values." In the final section, we discuss some open questions connected with giving an axiomatic support to this class.

2. "$\mathcal{P}$-Values"

Let $v : 2^N \to R$, $v(\emptyset) = 0$, be a characteristic function game on the player-set $N = \{1, \ldots, n\}$; and let ${}^i\mathcal{P}$ be a probability distribution on the subsets of $N\setminus\{i\}$, $i \in N$. Denote the set $\{{}^1\mathcal{P}, \ldots, {}^n\mathcal{P}\}$ by $\mathcal{P}$. Blair** [1] defined a "$\mathcal{P}$-value" for v, $\mathcal{P}_\xi(v) \in R^n$, as follows:

$$\mathcal{P}_{\xi_i}(v) = \sum_{S: i \in S \subset N} {}^i\mathcal{P}(S\setminus\{i\})[v(S) - v(S\setminus\{i\})] .$$

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**Blair's is the first general definition, though he confined himself to simple games. Special cases of $\mathcal{P}$-values have been considered in, e.g., [2], [3].

This can be interpreted as the expected marginal worth of i when coalitions of the other players form according to ${}^{i}\mathcal{P}$.

The familiar Shapley and Banzhaf values can be obtained by certain choices of $\mathcal{P}$. For the Shapley value [4] define ${}^{i}\mathcal{P}$ on $N\setminus\{i\}$ by

$$\mathcal{P}_i(S) = \frac{1}{n}\cdot\frac{1}{{}^{n-1}C_s}\,, \quad \text{where} \quad s = |S|\ .$$

For the Banzhaf value* [5] let ${}^{i}\mathcal{P}$ assign equal probability to all subsets of $N\setminus\{i\}$, namely $1/2^{n-1}$.

Our interest is in selecting a certain set of $\mathcal{P}$-values and in examining their behavior when N has a "large" number of individually "insignificant" players in it.

In order to do so it will be helpful to recapitulate the notion of "the multilinear extension" of a game due to Owen [3].

3. $\mathcal{P}$-Values and Multilinear Extensions

Consider the unit cube U^n in R^n (whose axes are labelled by the players $1, \ldots, n$). Vertices of the cube correspond in a natural way to subsets of N. Indeed, if the vertex is given by the vector u (whose components are 0 or 1) it will correspond to the subset S defined as follows: $i \in S$ iff $u_i = 1$. Thus a game v can be thought of as a function from the vertices of U^n (denoted u_S for $S \subset N$ henceforth) to the reals. Owen extends this to a function on U^n as follows:

$$f(x_1, \ldots, x_n) = \sum_{S\subset N} \{ \prod_{i\in S} x_i \prod_{i\in N\setminus S} (1 - x_i) \} v(u_S)$$

*This definition is due to L. S. Shapley [2]. Banzhaf himself normalized the value, i.e., set $\sum_{i\in N} \xi_i = 1$.

He showed in [3] that the Shapley value ϕ is obtained by integrating the partial derivatives of f along the diagonal of U^n, i.e.,

$$\phi_i(v) = \int_0^1 f^i(t, t, \ldots, t)dt$$

where f^i is the i^{th} partial derivative of f. We can think of this integral as $\int_{U^n} f^i d\lambda'$ where λ' is the probability measure on U^n which is distributed uniformly on its diagonal. Similarly Owen's result [5] on the Banzhaf value (β) :

$$\beta_i(v) = f^i(1/2, \ldots, 1/2)$$

can be written $\beta_i(v) = \int_{U^n} f^i d\lambda''$ where λ'' is the probability measure concentrated at the midpoint $(1/2, \ldots, 1/2)$ of the diagonal.

We are tempted to integrate with respect to an arbitrary probability measure λ on the diagonal. [The reason why we restrict ourselves to the diagonal will be discussed later on.] If we do so, we get $\eta^\lambda \in R^n$,

$$\eta_i^\lambda = \int_U f^i d\lambda .$$

Is there a $\mathcal{P}$ for which η^λ is a $\mathcal{P}$-value? We wish to show that there is. To this end, let $(t, \ldots, t)$ be a point on the diagonal, $0 \leq t \leq 1$, and associate with it, for each $i \in N$, a probability distribution ${}^i\mathcal{P}^t$ on subsets of $N \setminus \{i\}$ given by

$${}^i\mathcal{P}^t(S) = t^s(1-t)^{n-1-s}$$

where $s = |S|$. That is to say* ${}^i\mathcal{P}^t$ is obtained by picking elements

*This interpretation is from Owen [3].

independently and without replacement from $N\setminus\{i\}$ with probability t. Put $\mathcal{P}^t = ({}^1\mathcal{P}^t, \ldots, {}^n\mathcal{P}^t)$. Since each ${}^i\mathcal{P}^t(S)$ depends only on s, we can represent $\mathcal{P}^t$ by an n-dimensional vector $\psi(\mathcal{P}^t)$, where $\psi_j(\mathcal{P}^t) = {}^{n-1}C_{j-1}t^{j-1}(1-t)^{n-j}$, for $j = 0, 1, \ldots, n-1$. For all $i \in N$, $\psi_j(\mathcal{P}^t)$ is the probability of getting a subset of size $j-1$ from $N\setminus\{i\}$ under the distribution ${}^i\mathcal{P}^t$. Thus $\psi(\mathcal{P}^t)$ lies in the unit simplex S^n of R^n. On the other hand, for any $e \in S^n$, we can construct a set of probability distributions ${}^i\hat{\psi}(e)$ on $N\setminus\{i\}$ as follows:

$$[{}^i\hat{\psi}(e)](S) = e_{s+1}/{}^{n-1}C_s \ .$$

Denote $\{{}^1\hat{\psi}(e), \ldots, {}^n\hat{\psi}(e)\}$ by $\hat{\psi}(e)$. Note that $\hat{\psi}(\psi(\mathcal{P}^t)) = \mathcal{P}^t$.

Let $C \subset S^n$ be the set $\{\psi(\mathcal{P}^t) : 0 \leq t \leq 1\}$, and let $\bar{C}$ be its convex hull, i.e.,

$$\bar{C} = \{\sum_{k=1}^{m} \alpha_k c_k : c_k \in C, \ \alpha_k \geq 0, \ \sum_{k=1}^{m} \alpha_k = 1\} \ .$$

Since $\psi(\mathcal{P}^t)$ is continuous in t, the set $\{\psi(\mathcal{P}^t) : 0 \leq t \leq 1\}$ is compact. Then it is clear that, for any probability measure λ on the diagonal,

$$\int_{U^n} \psi(\mathcal{P}^t)d\lambda(t) \in \bar{C} \ .$$

Let Λ be the set of all probability measures on U^n that are concentrated on its diagonal. For any $\lambda \in \Lambda$, let $\mathcal{P}^\lambda = \{{}^1\mathcal{P}^\lambda, \ldots, {}^n\mathcal{P}^\lambda\}$ stand for $\hat{\psi}[\int_{U^n} \psi(\mathcal{P}^t)d\lambda(t)]$. When λ concentrates with probability 1 on $(t, \ldots, t)$, we will--in accordance with our earlier notation--write $\mathcal{P}^t$ for $\mathcal{P}^\lambda$.

We are now able to assert: η^{λ} is a $\mathcal{P}^{\lambda}$-value. This will follow from

$$\mathcal{P}^{\lambda}_{\xi_i} = \int_{U^n}^{\mathcal{P}^t} \xi_i d\lambda(t) = \int_{U^n} f^i(t, \ldots, t) d\lambda(t) \;, \quad i \in N \;, \quad \text{for any}$$

$\lambda \in \Lambda$.

The second equality is immediate if we observe that--see [3]--

$$f^i(t, \ldots, t) = \sum_{S: i \notin S \subset N} t^s (1-t)^{n-1-s} [v(S) - v(S \setminus \{i\})]$$

$$= \sum_{S: i \notin S \subset N} {}^i\mathcal{P}^t(S)[v(S) - v(S \setminus \{i\}) \;.$$

For the first, note that

$$\int_{U^n}^{\mathcal{P}^t} \xi_i d\lambda(t) = \int_{U^n} \Big(\sum_{S: i \notin S} ({}^{n-1}C_s)^{-1} \psi_{s+1}(\mathcal{P}^t)[\ \ldots\ ] \Big) d\lambda$$

$$= \sum_{S: i \notin S} ({}^{n-1}C_s)^{-1} \Big[\int_{U^n} \psi_{s+1}(\mathcal{P}^t) d\lambda(t) \Big] [\ \ldots\ ]$$

$$= \sum_{S: i \notin S} ({}^{n-1}C_s)^{-1} \psi_{s+1}(\mathcal{P}^{\lambda})[\ \ldots\ ]$$

$$= \sum_{S: i \notin S} {}^i\mathcal{P}^{\lambda}(S)[\ \ldots\ ]$$

$$= \mathcal{P}^{\lambda}_{\xi_i} \;.$$

Q.E.D.

We know, by Caratheodory's theorem [7], that any $e \in \overline{C}$ can be expressed as a convex combination of at most $n+1$ points $e_1, \ldots, e_{n+1}$ picked from C . Hence, for any $\lambda \in \Lambda$, there is a $\lambda' \in \Lambda$ such that

$\psi(\mathcal{P}^{\lambda}) = \int_{U^n} \psi(\mathcal{P}^{t}) d\lambda'(t)$, where λ' is concentrated at most n+1 points on the diagonal. From Lemma 1, we then have

$$\mathcal{P}^{\lambda}\xi_i = \int_{U^n} f^i(t, \ldots, t) d\lambda'(t)$$

for all $i \in N$. In words: <u>to evaluate any $\mathcal{P}^{\lambda}$-value we need to take a convex combination of the partial derivatives of</u> f <u>computed at most</u> n+1 <u>points on the diagonal</u>. When λ is the uniform distribution, the $\mathcal{P}^{\lambda}$-value is the Shapley value, and we get a variation of Owen's result. Owen showed in [3], that when $|N|$ is large f^i can be approximated at points on the diagonal using the normal distribution. Our result would facilitate the computation (Owen approximates the integral along the diagonal by picking many points on it) if we could describe the n+1 points and their weights. We have not attended to this problem as yet. However see the Appendix.

4. <u>Asymptotic $\mathcal{P}$-Values for Non-Atomic Games</u>

The class of probability values that we shall be concerned with consist of the $\mathcal{P}^{\lambda}$-values, $\lambda \in \Lambda$.

To examine the behavior of $\mathcal{P}^{\lambda}$-values for large games, we start with (see [8]) a non-atomic game $v : \mathcal{B} \to R$, where $\mathcal{B}$ consists of the Borel sets of $I = [0,1]$. Then, following Kannai [9], we consider a sequence of finite games which "approaches" v , and ask if the $\mathcal{P}^{\lambda}$-values of these finite games converge to give a $\mathcal{P}^{\lambda}$-value of v . For the formal treatment, we quote* directly from Chapter III in [8].

*We have replaced "$\mathcal{P}^{\lambda}$-value" for "Shapley value."

A partition Π of the underlying space I is a finite family of disjoint subsets whose union is the whole space; a partition Π_2 is a refinement of another partition Π_1 if each member of Π_1 is a union of members of Π_2. A partition is called measurable if each of its members is measurable. A sequence $(\Pi_1, \Pi_2, \ldots)$ of partitions is said to be decreasing if Π_{m+1} is a refinement of Π_m for each m; and separating if for all $s, t \in I$ with $s \neq t$, there is an m such that s and t are in different members of Π_m. A decreasing separating sequence of measurable partitions is called an admissible sequence.

If u is a game with finitely many players we will denote by ${}^{\lambda}\xi(u)$ the $\mathcal{P}^{\lambda}$-value of u (simplifying the earlier notation $\mathcal{P}^{\lambda}_{\xi}$ to ${}^{\lambda}\xi$), considered as a measure on the set of players of u.

If v is a set function and Π a measurable partition of the underlying space, let v_{Π} be the finite game whose players are the members of Π, given by

$$v_{\Pi}(\Xi) = v\Big(\bigcup_{j\in\Xi} j\Big)$$

for all $\Xi \subset \Pi$. Now let T be a measurable subset of I, and let $\mathbf{\Pi} = (\Pi_1, \Pi_2, \ldots)$ be a decreasing sequence of measurable partitions whose first term Π_1 is the partition $\{T, I\setminus T\}$. For each ν let

$$T_{\nu} = \{j \in \Pi_{\nu} : j \subset T\} ;$$

T is a coalition in the original infinite game v, and T_{ν} is the corresponding coalition in the corresponding finite game $v_{\Pi_{\nu}} = v_{\nu}$, say. If the numbers $[{}^{\lambda}\xi(v_{\nu})](T_{\nu})$ approach a limit as $\nu \to \infty$, then this limit will be denoted $[{}^{\lambda}\xi_{\mathbf{\Pi}}(v)](T)$. If the limit exists for all admissible $\mathbf{\Pi}$ starting with $\{T, I\setminus T\}$ and is independent of the choice of such $\mathbf{\Pi}$, then we will denote it by $[{}^{\lambda}\xi(v)]$. If that is the case for all measurable T , then we will call the set function ${}^{\lambda}\xi(v)$ the <u>asymptotic $\mathcal{P}^{\lambda}$-value</u> of v .

We are now in a position to state our asymptotic result.

<u>Theorem 1</u>. Assume that $v = f \circ \mu$, where μ is a non-atomic vector measure, and f is a C^1 function defined on the range of μ (which is convex by Lyapunov's theorem). Then for any $\lambda \in \Lambda$, ${}^{\lambda}\xi(v)$ exists, and

$$[{}^{\lambda}\xi(v)](T) = \int_0^1 f_{\mu(T)}(t\mu(I))d\lambda(t) ,$$

where $f_{\mu(T)}$ is the derivative of f in the direction $\mu(T)$.

For the proof we first establish a lemma. Denote the members of Π_ν by $\{A_1^\nu, \ldots, A_{m^\nu}^\nu\}$, and set $\mu(A_i^\nu) = \alpha_i^\nu$, where μ is a non-atomic scalar measure. For any $S^\nu \subset \Pi_\nu$, let $\alpha^\nu(S^\nu)$ stand for $\sum_{i \in S^\nu} \alpha_i^\nu$. Finally let $\alpha_{max}^\nu = \max_{1 \le j \le m^\nu} \{\alpha_j^\nu\}$, and $\alpha = \mu(I)$.

<u>Lemma 1</u>.* Fix any $t \in [0,1]$, and choose $S \subset \Pi_\nu$ "at random" according to $\mathcal{P}^t$, i.e., with probability $t^s(1-t)^{m^\nu - s}$. Then, for p and q in $[0,\alpha]$,

$$\lim_{\nu\to\infty} \text{Prob}[p < \alpha^\nu(S) < q] = \begin{cases} 1 & \text{if } t\alpha \in (p,q) \\ 0 & \text{if } t\alpha < p \text{ or } q < t\alpha \end{cases} .$$

*This is a simple variation of Lemma 3 in [2].

<u>Proof</u>. Omit the superscript " ν " until needed. Let H_k denote the random variable $(\alpha(S) : S = k)$, obtained by summing exactly k of the numbers $\alpha_1, \ldots, \alpha_m$, chosen at random. Its mean is of course $\mu_k = k\alpha/m$, and as with any random variable* we have for any $d > 0$

$$\text{Prob}\{H_k \leqq \mu_k - d\} \leqq \frac{\sigma_k^2}{\sigma_k^2 + d^2}$$

where σ_k^2 denotes the variance of H_k. Also, we have

$$\sigma_k^2 \leqq k\sigma_1^2 \, .$$

since a "sample without replacement" has lower variance than the corresponding "sample with replacement."** Since

$$\sigma_1^2 \leqq \frac{1}{m} \Sigma \alpha_j^2 \leqq \frac{1}{m} \alpha\alpha_{max} ,$$

we conclude that

$$\text{Prob}\{H_k \leqq \mu_k - d\} \leqq \alpha\alpha_{max}/d^2 , \quad \text{for } d > 0 \, .$$

Now let $q < t\alpha$, and consider the sum

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*The worst case is a distribution concentrated at two points, namely $\mu_k - d$ and the unique point on the other side of μ_k that yields the required mean μ_k and variance σ_k^2. For such a distribution it is easy to calculate that $\sigma^2 = d^2 P_d/(1 - P_d)$, where P_d is the probability of $\mu_k - d$; from this the inequality follows.

**See Kemperman [10].

$$\text{Prob}\{\alpha(S) < q\} = \sum_{k=0}^{m} \text{Prob}\{|S| = k\} \cdot \text{Prob}\{H_k < q\} .$$

Choose d so that $q < q+d < t\alpha$, define $\delta = (q+d)/\alpha$, and split the sum into two parts:

$$\sum_{k=0}^{m} = \sum_{k=0}^{<\delta m>} + \sum_{k=<\delta m>+1}^{m} .$$

For the second part, we observe that $\mu_k > \delta\alpha = q+d$, so

$$\text{Prob}\{H_k < q\} \leqq \text{Prob}\{H_k < \mu_k - d\} \leqq \alpha\alpha_{max}/d^2 ,$$

and we have

$$\sum_{k=<\delta m>+1}^{<\delta m>} \leqq \frac{\alpha\alpha_{max}}{d^2} \sum_{k=<\delta m>+1}^{m} \text{Prob}\{|S| = k\} \leqq \frac{\alpha\alpha_{max}}{d^2} ,$$

this goes to zero since (pp. 130-31 in [5]) α_{max} goes to zero. For the first part, we have, trivially,

$$\sum_{k=0}^{<\delta m>} \leqq \text{Prob}\{|S| \leqq <\delta m>\} ,$$

which goes to zero with increasing m, by the Central Limit Theorem, since δ is a fixed number less than t. Thus, we have established that

$$\lim_{\nu\to\infty} \text{Prob}\{\alpha^{\nu}(S) < q\} = 0 \quad \text{if} \quad q < t\alpha .$$

The proof for $t\alpha > p$ is similar. Combining these two cases the third limit easily follows.

Q.E.D.

Lemma 2. Let μ be a non-atomic vector measure on I. Let U be a neighborhood of the point $t\mu(I)$ on the "diagonal" in the range of μ, i.e., the line segment with end points 0 and $\mu(I)$. Let $= \{\Pi_1, \Pi_2, \ldots\}$ be an admissible sequence. Pick subsets from Π_ν as in Lemma 2. Then for every $\epsilon > 0$, there is a ν_0, such that for $\nu > \nu_0$,

$$\text{Prob}\{\mu(S^\nu) \in U\} \geq 1-\epsilon .$$

Proof. This is obvious from Lemma 2.

Q.E.D.

Proof of Theorem 1.* First consider the case where λ concentrates on the point $(t, \ldots, t)$. Let $H^\nu = (h_1^\nu, \ldots, h_{x(\nu)}^\nu)$ be such that $A_i^\nu \in T_\nu$ iff $i \in H^\nu$. Then

$$\begin{aligned}[{}^t\xi(v_\nu)](T_\nu) &= \sum_{i\in H^\nu} \sum_{S\subset\Pi_\nu} {}^i\rho^t(S\setminus i)[v(S) - v(S\setminus i)] \\ &= \sum_{i\in H^\nu} \sum_{S\subset\Pi_\nu} {}^i\rho^t(S\setminus i)[f(\mu(S)) - f(\mu(S\setminus i))] \\ &= \sum_{i\in H^\nu} \sum_{S\subset\Pi_\nu} {}^i\rho^t(S\setminus i)(\nabla f)(x_S)\cdot\mu(A_i^\nu) .\end{aligned}$$

The last equality comes from the mean value theorem. ∇ stands for gradient, $\cdot$ for dot product, and x_S is a point on the line joining $\mu(S)$ and $\mu(S\setminus i)$. By Lemma 3 $\sum_{S\subset\Pi_\nu} {}^i\rho^t(S\setminus i)(\nabla f)(x_S)$ tends to $(\nabla f)(t\mu(I))$. Clearly $\sum_{i\in H^\nu} \mu(A_i^\nu)$ tends to $\mu(T)$. Thus we have proved that $\lim[{}^t\xi(v_\nu)](T_\nu)$ exists and is equal to $f_{\mu(T)}(t\mu(I))$.

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*We sometimes use i to stand for the player A_i^ν in Π_i^ν. No confusion results.

Now take any $\lambda \in \Lambda$. Note that

(a) $$^{\lambda}\xi(v_\nu)(T_\nu) = \int [^t\xi(v_\nu)](T_\nu) d\lambda(t)$$

(b) $$\lim [^t\xi(v_\nu)](T_\nu) = f_{\mu(T)}(t\mu(I)) \quad \text{for} \quad t \in [0,1] \ .$$

We could conclude, by the Lebesque Dominated Convergence Theorem, that

$$\lim [^{\lambda}\xi(v_\nu)](T_\nu) = \int f_{\mu(T)}(t\mu(I)) d\lambda(t)$$

if there existed a λ-integrable function $H : [0,1] \to R$ such that $[^t\xi(v_\nu)](T_\nu) \leq H(t)$ for all $t \in [0,1]$ and for all ν . We construct such an H . Let $M = \max |\nabla f|$, where the max is taken on the range of μ , and $| \ |$ denotes the max norm. Now, for all t , and for all ν ,

$$\begin{aligned} [^t\xi(v_\nu)](T_\nu) &= \sum_{i \in H_\nu} \sum_{i \notin S \subset \Pi_\nu} {}^i\mathcal{P}^t(S)[\nabla f(x_S)] \cdot \mu(A_i^\nu) \\ &\leq \sum_{i \in H_\nu} (Me) \cdot \mu(A_i^\nu) \sum_{i \notin S \subset \Pi_\nu} {}^i\mathcal{P}^t(S) \\ &\leq (Me) \cdot \mu(T) \end{aligned}$$

(e is the unit vector in R^m , where Range $\mu \subset R^m$.) Put $H(t) = (Me) \cdot \mu(T)$.

Q.E.D.

<u>Remark</u>. Note that Theorem 1 generalizes the result in [8], with a somewhat less cumbersome proof.

We will use the term* "<u>diagonal</u> $\mathcal{P}$-<u>values</u>" for the $\mathcal{P}^{\lambda}$-values, $\lambda \in \Lambda$.

Staying on the diagonal provided us with a simple way of defining a sequence of $\mathcal{P}$-values for the games $\{v_{\Pi_\nu}\}_{\nu=1}^{\infty}$. Here we develop a definition of a more general asymptotic value for v , by considering off-diagonal** $\mathcal{P}$-values of the games $\{v_{\Pi_\nu}\}$. Let $p^\nu = (p_1^\nu, \ldots, p_{m^\nu}^\nu)$ where p_i^ν is the probability of picking A_i^ν from Π_ν (and $1 - p_i^\nu$ is the probability of not picking it). Now p^ν gives a probability distribution ${}^{i}\mathcal{P}(p^\nu)$ on the subsets of $\Pi_\nu \backslash \{A_i^\nu\}$ if we put $[{}^{i}\mathcal{P}(p^\nu)](S) = \prod_{i \in S} p_i^\nu \prod_{i \notin S} (1 - p_i^\nu)$, for $S \subset \Pi_\nu \backslash \{A_i^\nu\}$. Let us denote $\{{}^{1}\mathcal{P}(p^\nu), \ldots, {}^{m^\nu}\mathcal{P}(p^\nu)\}$ by $\mathcal{P}(p^\nu)$. We have a $\mathcal{P}(p^\nu)$-value, ${}^{\mathcal{P}(p^\nu)}\xi$, v_{Π_ν} for each ν . As before we attempt to define a limiting value ξ of the ${}^{\mathcal{P}(p^\nu)}\xi$-values by

$$\xi(T) = \lim_{\nu \to \infty} {}^{\mathcal{P}(p^\nu)}\xi(T_\nu) ,$$

provided the limit exists for all $T \in \mathcal{C}$, and is the same for any admissible sequence of partitions. This will obviously require some constraints on the sequence $\{p^\nu\}_{\nu=1}^{\infty}$.

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*Strictly speaking, the word "value" has been misused by us, because (see [8]) it connotes a function on the linear space of games that satisfies certain axioms. Our $\mathcal{P}$-values do not satisfy these axioms (for further discussion of this point see Section 5), but we condone our misuse by talking of "$\mathcal{P}$-values" rather than just "values" to emphasize that our values are defined probabilistically, not axiomatically.

**This consideration was provoked by Donald Brown.

To find these constraints we choose as our criterion that $\mu(S^\nu)$ tend in probability in some limit, where S^ν is a random subset of Π_ν picked according to $\mathcal{P}(p^\nu)$. For if this happens and $x \in$ Range μ is the limit, then the argument in the proof of Theorem 1 shows that

$$\lim_{\nu\to\infty} \mathcal{P}(p^\nu)\xi(T) = f_{\mu(T)}(x) \qquad (*)$$

for all $T \in \mathcal{C}$, and any admissible sequence of partitions.

Let μ_ℓ denote a component of μ. By the theorem in §22, p. 105 of [11],* $\mu_\ell(T^\nu)$ tends in probability to a limit if and only if

(1) $$\sum_{i=1}^{m^\nu} p_i^\nu \mu_\ell(A_i^\nu) \text{ has a limit}$$

(2) $$\sum_{i=1}^{m^\nu} p_i^\nu(1 - p_i^\nu)[\mu_\ell(A_i^\nu)]^2 \to 0$$

[(1) says that the mean weights should converge; (2) that the variance should tend to 0.] We will call (1) and (2) the G-K condition. Gnedenko and Kolmogorov in fact prove** that if the G-K condition holds, then $\mu_\ell(T^\nu)$ converges in probability to $x_\ell = \lim \sum_{i=1}^{m^\nu} p_i^\nu \mu_\ell(A_i^\nu)$. So we constrain $\{p^\nu\}_{\nu=1}^\infty$ by requiring that the G-K condition hold for every component of μ. Then $\mu(T^\nu)$ will tend in probability to x, where $x_\ell = \lim \sum_{i=1}^{m^\nu} p_i^\nu \mu_\ell(A_i^\nu)$, and the equation (*) will hold.

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*I am grateful to J. A. Hartigan for recommending [11].

**Their result immediately implies our Lemma 1. We have nevertheless given a different proof of Lemma 1 because it is self-contained and simpler.

More generally let $\lambda \in \Lambda$, and let $f^{\nu} : [0,1] \to U^{m^{\nu}}$ be a sequence of functions such that

(c) f^{ν} is measurable on $[0,1]$ w.r.t. λ.

(d) $\{f^{\nu}(t)\}_{\nu=1}^{\infty}$ satisfies the G-K condition for all $t \in [0,1]$.

Define a probability distribution ${}^{i}\mathcal{P}^{\nu}$ on the subsets of $\Pi^{\nu}\setminus\{A_i^{\nu}\}$ by

$$ {}^{i}\mathcal{P}^{\nu}(S) = \int_0^1 [{}^{i}\mathcal{P}(f^{\nu}(t))](S)d\lambda(t) $$

[(c) ensures that this integral is well-defined). Denote $\{{}^{1}\mathcal{P}^{\nu}, \ldots, {}^{m^{\nu}}\mathcal{P}^{\nu}\}$ by $\mathcal{P}^{\nu}$; and $\lim_{\nu\to\infty} \sum_{i=1}^{m^{\nu}} f_i^{\nu}(t)\mu(A_i^{\nu})$ by $x(t)$. Then, if $x : [0,1] \to \text{Range } \mu$ is measurable, the arguments in the proof of Theorem 1 show that

$$ \lim_{\nu\to\infty} {}^{\mathcal{P}^{\nu}}\xi(T_{\nu}) \qquad (**) $$

exists for all $T \in \mathcal{C}$, and for all admissible partitions, and in fact is given by

$$ \int_0^1 f_{\mu(T)}(x(t))d\lambda(t) . $$

When the set $\{t \in [0,1] : x(t) \notin \text{diagonal in Range } \mu\}$ has positive λ-measure, then we will call the limit in (**) an "<u>off-diagonal $\mathcal{P}$-value</u>" of v.

4. $\mathcal{P}$-Values and Competitive Payoffs of Non-Atomic Economies

The main result that we wish to establish is: diagonal $\mathcal{P}$-values coincide with the competitive payoffs of non-atomic economies. Further, we give an example to show that this does not hold for off-diagonal $\mathcal{P}$-values in general.

Consider a non-atomic, finite type exchange economy with transferable utilities of the type described in Ch. 6 of [8]. There it is shown that the game associated with the economy may be written as $v = f \circ \mu$, where μ is a non-atomic vector measure and f is a homogeneous function on Range μ. If we assume that the utility functions are C^1, then f can also be shown to be C^1. In this case the Shapley value ϕ of v is given by

$$\phi(T) = \int_0^1 f_{\mu(T)}(t\mu(I))dt \ .$$

Moreover (Proposition 31.5 in [8]) ϕ coincides with the core of v, hence with the competitive payoff. But since f is homogeneous of degree one, $f_{\mu(T)}$ is constant on the diagonal. Therefore

$$\int_0^1 f_{\mu(T)}(t\mu(I))d\lambda(t)$$

is invariant of $\lambda \in \Lambda$, i.e., the diagonal $\mathcal{P}$-values also coincide with the competitive payoff.

We would like to establish this without the finite-type assumption. To avoid repetition we shall quote rather freely from [8] and recall results upon which our arguments are built. Let A be a subset of BV (BV is a Banach space of non-atomic games endowed with the variation norm--§4, Ch. 1, in [8]) which has the properties:

(c) for any $v \in A$, and any $\lambda \in \Lambda$, ${}^{\lambda}\xi(v)$ exists (as an asymptotic limit).

(d) for any $v \in A$, ${}^{\lambda}\xi(v) = \varphi(v)$, where φ denotes the Shapley value, i.e., corresponds to the case where λ is the uniform distribution on $[0,1]$.

Lemma 4. Let $\bar{A}$ denote the closure of the linear span of A. Then properties (c) and (d) hold for any $v \in \bar{A}$.

Proof. It is obvious that if A is any set satisfying (c) and (d), then the linear space of A ($LS[A]$) also satisfies (c) and (d). We must show that (c) and (d) hold on the closure of $LS[A]$.

Consider the sequence $\{v^k : k = 1, 2, \ldots\}$ in $LS[A]$ and suppose $\|v^k - v\| \to 0$, where $\| \|$ denotes the BV-norm. By Proposition 18.1 in [8], $\|\varphi(v^k)\| \leq \|v^k\|$. Since (c) holds for all $v \in LS[A]$, we have $\|{}^{\lambda}\xi(v^k)\| \leq \|v^k\|$ for all $\lambda \in \Lambda$. But then $\{{}^{\lambda}\xi(v^k) : k = 1, 2, \ldots\}$ is a Cauchy sequence in BV, and hence it has a limit ζ in the BV-norm. We now prove, exactly as in Proposition 18.4 of [8], that ζ is the asymptotic limit ${}^{\lambda}\xi(v)$. For the sake of completeness we repeat the argument of Proposition 18.4.

For given $\epsilon > 0$, choose v^k so that both $\|v^k - v\| < \epsilon$, and $\|{}^{\lambda}\xi(v^k) - \zeta\| < \epsilon$.

Let $\bar{\Pi}$ be an admissible sequence $(\Pi_1, \Pi_2, \ldots)$ starting with $\{T, I/T\}$, and for each ν let $T_\nu = \{j \in \Pi_\nu : j \subset T\}$. Then*

*The BV-norm, $\|v\|$, for a finite game v is defined in the obvious way.

$$|{}^{\lambda}\xi(v_{\Pi_\nu})(T_\nu) - {}^{\lambda}\xi(v^k_{\Pi_\nu})(T_\nu)| \leq \|{}^{\lambda}\xi(v_{\Pi_\nu} - v^k_{\Pi_\nu})\|$$
$$\leq \|v_{\Pi_\nu} - v^k_{\Pi_\nu}\|$$
$$\leq \|v - v^k\| \leq \epsilon ,$$

hence

$$|{}^{\lambda}\xi(v_{\Pi_\nu})(T_\nu) - \zeta(T)| \leq |{}^{\lambda}\xi(v_{\Pi_\nu})(T_\nu) - {}^{\lambda}\xi(v^k_{\Pi_\nu})(T_\nu)|$$
$$+ |{}^{\lambda}\xi(v^k_{\Pi_\nu})(T_\nu) - {}^{\lambda}\xi(v^k)(T)|$$
$$+ |{}^{\lambda}\xi(v^k)(T) - \zeta(T)|$$
$$\leq 2\epsilon + |{}^{\lambda}\xi(v^k_{\Pi_\nu})(T_\nu) - {}^{\lambda}\xi(v^k)(T)| .$$

But $v^k \in LS[A]$, and (c) holds on $LS[A]$, therefore

$$\limsup_{\nu\to\infty} |{}^{\lambda}\xi(v_{\Pi_\nu})(T_\nu) - \zeta(T)| \leq 2\epsilon .$$

This proves that ζ is the asymptotic limit ${}^{\lambda}\xi(v)$, i.e., (c) holds for v. But note that ${}^{\lambda}\xi(v^k)$ is the same for all $\lambda \in \Lambda$, since $v^k \in LS[A]$, and (d) holds on $LS[A]$. Consequently the limit ζ is the same for all $\lambda \in \Lambda$. This proves that (d) holds for v.

Q.E.D.

We now establish a variation of Theorem 1. (Compare with Proposition 10.17 in [8].)

<u>Lemma 5</u>. Let f be a continuous, non-decreasing real function on R^m_+, vanishing at the origin, such that, for each i, $\partial f/\partial x_i$ exists and is continuous whenever $x_i > 0$. Also assume that f is homogeneous

of degree one. Let μ be an m-dimensional non-atomic vector measure. Then $v = f \circ \mu$ satisfies (c) and (d).

Proof. Let e denote the vector $\mu(I)/|\mu(I)|$. Define $f^{\delta} : R^m_+ \to R_+$ by

$$f^{\delta}(x) = f\left(\frac{x+\delta e}{1+2\delta}\right) - f\left(\frac{\delta e}{1+2\delta}\right).$$

Since f is homogeneous of degree one, $\nabla f(t_{\mu}(I))$ is constant (say c) for $0 < t \leq 1$. Moreover $\nabla f^{\delta}(t\mu(I)) = (1/1+2\delta)c$ for $0 \leq t \leq 1$, and $\delta > 0$. Let $v^{\delta} = f^{\delta} \circ \mu$. Then as shown* in Proposition 10.17 in [8] v^{δ} tends to v in the BV norm as δ tends to 0 . Now take any $\lambda \in \Lambda$. By Theorem 1, ${}^{\lambda}\xi(v^{\delta})$ exists and is given by

$$({}^{\lambda}\xi(v^{\delta}))(T) = \int_0^1 [\nabla f^{\delta}(t_{\mu}(I))]\cdot\mu(T)d\lambda(t)$$

$$= \frac{1}{1+2\delta}\int_0^1 c\cdot\mu(T)d\lambda(t) .$$

This integral is independent of $\lambda \in \Lambda$. Thus v^{δ} satisfies (c) and (d). By Lemma 4, v then also satisfies (c) and (d).

Q.E.D.

With the aid of these two lemmas we can get rid of the finite-type assumption. Consider an economy--no longer finite-type--as described in Ch. 6, pp. 182-183, of [8]. Then, as is done there, we can approximate it by a sequence of finite type economies which have associated with them

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*There e is taken to be the unit vector, but the same proof works for our e .

games $\{v^1, v^2, \ldots, \ldots\}$. Each v^k is of the form mentioned in Lemma 5 and thus satisfies (c) and (d). But v^k converges to v in the BV norm (Ch. 6 of [8]). Then by Lemma 4, v satisfies (c) and (d). Now $\varphi(v)$ coincides with core and the competitive payoff of v (Theorem J in Ch. 6 of [8]), hence so do ${}^{\lambda}\xi(v)$, $\lambda \in \Lambda$.

Given a non-atomic economy in which utilities are not transferable, we may define Diagonal $\mathcal{P}$-value allocations exactly as the (Shapley) value allocations were defined in [12]. The foregoing discussion shows that they coincide. Then the result in [12] tells us that Diagonal $\mathcal{P}$-value allocations coincide with the competitive allocations.

From the integral representation of the off-diagonal $\mathcal{P}$-values it is clear that they will not in general coincide with the competitive payoffs of non-atomic economies. We give a specific example of one such. It is based on example 33.3 in [8]. Let $\{I, \mathcal{B}, \lambda\}$ be the underlying measure space of traders, where $I \equiv [0,1]$, $\mathcal{B} \equiv$ the Borel sets of I, $\lambda \equiv$ the Lebesque measure. Suppose there is only one commodity. The utility function of all traders is the same: $u(x) = \sqrt{x+1} - 1$. Their initial endowment is given by $\bar{a} : I \to R$, $\bar{a}(s) = 8s$. Then, as shown in [8], the trading game v arising from the economy can be represented by:

$$v(S) = g(\lambda(S), \mu(S))$$

where $\mu(S) = \int_S \bar{a}\, d\lambda$, and $g(y,z) = \sqrt{y(y+z)} - y$. We shall construct an off-diagonal asymptotic value for v. Let $\hat{\Pi}_\nu$ be any admissible sequence of partitions, and pick p^ν as follows:

$$p_i^\nu = \begin{cases} 1 & \text{if} \quad \sum_{j=1}^{i} \lambda(A_j^\nu) \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Clearly p^ν satisfies the G-K conditions, and $(\lambda, \mu)(T^\nu)$ [where T^ν is a random subset of Π^ν picked according to $\mathcal{P}(p^\nu)$] tends in probability to the point $(1/2, 1)$. [This is not on the diagonal in the range of (λ, μ) , because the diagonal joins $(0,0)$ to $(1,4)$.] Now the asymptotic value ξ defined by $\{p^\nu\}_{\nu=1}^\infty$ is

$$\begin{aligned} \xi(T) &= (\nabla g)(1/2, 1)\cdot(\lambda, \mu)(T) \\ &= \left(\frac{7}{2\sqrt{3}} - 1\right)\lambda(T) + \frac{1}{2\sqrt{3}}\mu(T) \ . \end{aligned}$$

On the other hand--see example 33.3--the Shapley value ϕ is given by

$$\phi(T) = \left(\frac{3}{2\sqrt{2}} - 1\right)\lambda(T) + \frac{1}{2\sqrt{2}}\mu(T) \ .$$

The two clearly do not coincide.

5. Concluding Discussion

We will launch into a (desultory) discussion of some open questions. Having nailed down the set of diagonal $\mathcal{P}$-values as those $\mathcal{P}$-values which will coincide with the competitive payoffs of non-atomic economies one is led to wonder if they can be characterized axiomatically, i.e., shown to be precisely the set of values which satisfy certain intuitively plausible properties. Regarded as functions over the vector space of games (for a fixed player-set) they satisfy the symmetry, dummy, and linearity axioms, but the efficiency axiom breaks down. To discuss this precisely, we recall some definitions.

<u>Definitions</u>. $\mathcal{G}_N$ will denote the set of all games on N. It is a vector space of dimension $2^n - 1$. $\mathcal{C}_N$ will be the finite subset of $\mathcal{G}_N$ which consists of simple games on N, i.e., those v which take on values 0 or 1, which are monotonic [$v(S) = 1$, and $S \subset T \Longrightarrow v(T) = 1$], and for which $v(N) = 1$.

For any permutation π of N, and $v \in \mathcal{G}_N$, let πv be the game given by $(\pi v)(S) = v(\pi^{-1}(S))$ for all $S \subset N$. Given any v and w in $\mathcal{C}_N$ define $v \vee w$ and $v \wedge w$ as follows:

$$(v \vee w)(S) = \begin{cases} 1 & \text{if } v(S) = 1 \text{ or } w(S) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$(v \wedge w)(S) = \begin{cases} 1 & \text{if } v(S) = 1 \text{ and } w(S) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Note that $\mathcal{C}_N$ is closed under the operations $\vee$, $\wedge$, π; and $\mathcal{G}_N$ under π.

We will say that a real-valued function f from $\mathcal{G}_N$ (or $\mathcal{C}_N$) to R is symmetric if, for any permutation π of N and any v in $\mathcal{G}_N$ (or $\mathcal{C}_N$) $f(\pi v) = f(v)$.

Finally we define symmetric and dummy players. i is said to be a dummy in v if $v(S \cup \{i\}) = v(S)$ for all $S \subset N$. i and j are called symmetric in v if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subset N \{i,j\}$.

<u>Theorem</u>. Let $f : \mathcal{G}_N \to R$ be linear and symmetric. Then there is a unique $\phi : \mathcal{G}_N \to R^n$ which satisfies the following axioms (for any permutation π, and any v and w):

(A1) If i is a dummy in v, $\phi_i(v) = 0$

(A2) If i and j are symmetric in v, $\phi_i(v) = \phi_j(v)$

(A3) $\sum_{i \in N} \phi_i(v) = f(v)$

(A4) $\phi(v+w) = \phi(v) + \phi(w)$.

Proof. Obvious modification of the proof in [4].

Any diagonal $\mathcal{P}$-value (i.e. such that $\psi(\mathcal{P}) \in \bar{C}$) can be generated as above if we put

$$f(v) = \sum_{i \in N} \sum_{i \notin S \subset N} {}^{i}\mathcal{P}(S)[v(S \cup \{i\}) - v(S)] .$$

Question: Is there a meaningful way of characterizing those f (or some subset of those f) that arise from diagonal $\mathcal{P}$-values?

Looking at the matter in a totally different, and possibly simpler, way:

Question: Is there a meaningful characterization of the set $\{\mathcal{P} : \psi(\mathcal{P}) \in \bar{C}\}$?

Clearly the $\mathcal{P}$ here will be symmetric, i.e. ${}^{i}\mathcal{P}(S) = {}^{i}\mathcal{P}(T)$ if $|S| = |T|$. But this alone does not suffice. Consider the probability distribution ${}^{i}\mathcal{P}^{s}$ which gives (for all $i \in N$) equal probability to all subsets (of $N \setminus \{i\}$) of a fixed size $s \leq n-1$ and zero probability to all other subsets. ${}^{i}\mathcal{P}^{s}$ is symmetric but $\psi(\mathcal{P}^{s}) \notin \bar{C}$.

The values obtained by considering arbitrary symmetric f are not as arbitrary as may appear. Indeed they consist of linear combinations of $\mathcal{P}$-values, where $\mathcal{P}$ is symmetric (i.e. $\mathcal{P}(S) = \mathcal{P}(T)$ if $|S| = |T|$). To see this, let $\mathcal{F}$ be the vector space of all symmetric linear functions from $\mathcal{G}_N$ to R ; Φ the vector space of all functions from $\mathcal{G}_N$ to

R^n that satisfy (A2) and (A4). Our theorem gives us a mapping M from $\mathcal{F}$ to Φ if we put M(f) = the unique $\phi \in \Phi$ that satisfies (A1) and (A4). It can be easily checked that M is linear, and that the dimension $\mathcal{F} = n$. Now let $f_s \in \mathcal{F}$ be the f in the theorem required to generate the ρ^s-value, where $\rho^s = ({}^1\rho^s, \ldots, {}^n\rho^s)$. Then $\{f_s : 0 \leq s \leq n-1\}$ is a basis of $\mathcal{F}$.

There is an analogue of the above theorem in the context of $\mathcal{C}_N$. Define $\mathcal{B}_N = \{v_S : S \subset N\}$, where v_S is the game in $\mathcal{C}_N$ defined by

$$v_S(T) = \begin{cases} 1 & \text{if } S \subset T \\ 0 & \text{otherwise.} \end{cases}$$

First we state the following purely mathematical lemma.

Lemma . Let $f : \mathcal{B}_N \to R^k$ be any function. There is a unique extension $\hat{f}$ of f to $\mathcal{C}_N$ such that $\hat{f}(v \vee w) + \hat{f}(v \wedge w) = \hat{f}(v) + \hat{f}(w)$ for all v and w in $\mathcal{C}_N$. Moreover $\hat{f}$ is the restriction to $\mathcal{C}_N$ of the linear extension of f to $\mathcal{H}_N$.

Proof. This is essentially contained in the proof of Theorem 2 in [13]. Only changes in notation are required.

With the help of this Lemma and the previous Theorem one can easily prove:

Theorem. Let $f : \mathcal{C}_N \to R$ be a symmetric function which satisfies

$$f(v \vee w) + f(v \wedge w) = f(v) + f(w) \qquad (1)$$

for any v and w in $\mathcal{C}_N$. There is a unique $\phi : \mathcal{C}_N \to R^n$ which the following axioms (for any π, and any v and w in $\mathcal{C}_N$):

(A1) If i is a dummy in v, $\phi_i(v) = 0$

(A2) If i and j are symmetric in v, then $\phi_i(v) = \phi_j(v)$

(A3) $\sum_{i \in N} \phi_i(v) = f(v)$

(A'4) $\phi(v \vee w) + \phi(v \wedge w) = \phi(v) + \phi(w)$

(A'4) could be replaced by another requirement* to state which we shall first make a few definitions.

Let $(v_1, \ldots, v_\ell) = V$ and $(w_1, \ldots, w_k) = W$ be two sets of simple games; and think of v_i (or w_j) as being played with probability α_i (or β_j). We are interested in the case when a player will be indifferent between the choice of V or W.* Let us define the distribution $\mathcal{D}_V$ of U,

$$2^N \xrightarrow{\mathcal{D}_V} R \text{ , by:}$$

$$\mathcal{D}_V(S) = \sum_{i=1}^{\ell} \alpha_i v_i(S)$$

for all $S \subset N$. Thus $\mathcal{D}_V(S)$ is the probability that S will be a winning coalition in V. ($\mathcal{D}_W$ is defined in the same way.) If $\mathcal{D}_V = \mathcal{D}_W$ we wish to say that $i \in N$ will be indifferent between the choice of V or W since his expected value (or "power" in this case as we are in the context of "voting" games) in either set is the same. This is tantamount to the assumption that a voter's power depends solely on the probabilities of winning coalitions. The expected power of j in V is $\sum_{i=1}^{\ell} \alpha_i \phi_j(v_i)$. Thus we have the axiom

*For yet another version of (A'4) see [2].

(A''4) If $\mathcal{D}_V = \mathcal{D}_W$, then for all $j \in N$,

$$\sum_{i=1}^{\ell} \alpha_i \phi_j(v_i) = \sum_{i=1}^{k} \beta_i \phi_j(w_i) .$$

(A''4) implies (A'4) because for $V = (v \vee w, v \wedge w)$ and $W = (v,w)$, $\mathcal{D}_V = \mathcal{D}_W$ if we consider the probability of each game in V or W to be 1/2. We could have replaced (A'4) by (A''4) in the theorem.

Finally let us consider an imaginative approach to values spelled out by A. Roth in [14, 15]. He obtained the Shapley value on $\mathcal{C}_N$ $(\mathcal{U}_N)$ as the utility function that represents players' preferences on the mixture set generated by the set $\mathcal{C}_N \times N$ $(\mathcal{U}_N \times N)$ of strategic positions. ((v,i) is the prospect of being player i in the game v .) One of the conditions that Roth imposed on the preferences was:

$$\left[\frac{1}{r}(v_{\{i\}}, i); \left(1 - \frac{1}{r}\right)(v_0, i)\right] \sim [(v_R, i)]$$

for $i \in R$, where $r = |R|$ and v_0 is the game in which all conditions have zero payoff. We get $M(f)$ in Roth's framework if we replace this condition by

$$\left[\frac{f(v_R)}{rf(v_{\{i\}})}(v_{\{i\}}, i); 1 - \frac{f(v_R)}{rf(v_{\{i\}})}(v_0, i)\right] \sim [(v_R, i)] .$$

For this to make sense, however, in the context of $\mathcal{C}_N$, $f(v_R)/rf(v_{\{i\}})$ must be a probability, i.e., we must have

$$f(v_R) \leq rf(v_{\{i\}}) ,$$

which gives us* a nice restriction on f .

*This observation is due to Donald Brown.

APPENDIX

(This is an outcome of suggestions made by L. S. Shapley.)

Since C is connected we can improve upon Caratheory's theorem: any point in $\overline{C}$ is the convex combination of at most n points picked from C . But in fact* [n+2/2] points from C would suffice. To see this observe that the set of functions $(u_0, \ldots, u_{n-1})$, $[0,1] \xrightarrow{u_i} R$, given by

$$u_i(t) = {}^{n-1}C_i t^i (1-t)^{n-1-i} \quad \text{for} \quad t \in [0,1]$$

forms a Tchebycheff System (p. 1 of [16]). Then by Theorem 3.1 in [16] the assertion follows.

It would be nice if we could find a simple formula which describes these [n+2/2] points in C --i.e. [n+2/2] points in [0,1] --and their weights. We have not succeeded in this as yet.

However we note an alternative method for computing the Shapley value: Let $t_1, \ldots, t_n$ be points equally spaced** on [0,1] . We can find weights $\beta_1, \ldots, \beta_n$, <u>not necessarily non-negative</u>, such that

$$\sum_{i=1}^{n} \beta_i \begin{pmatrix} u_0(t_i) \\ \vdots \\ u_{n-1}(t_i) \end{pmatrix} = \begin{pmatrix} 1/n \\ \vdots \\ 1/n \end{pmatrix} .$$

This is because the matrix M given by

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* [n+2/2] is the largest integer less than or equal to n+2/2 .

** Any set of distinct points would do.

$$\begin{pmatrix} u_0(t_1) & \cdots & u_0(t_n) \\ \vdots & & \\ u_{n-1}(t_1) & \cdots & u_{n-1}(t_n) \end{pmatrix}$$

has a positive determinant. We can see this by observing that if we divide each column i by $(1-t_i)^{n-1}$, and then divide each row i by ${}^{n-1}C_{i-1}$, we get the familiar Vandermonde determinant. Thus a solution $(\beta_1, \ldots, \beta_n)$ can be found. But then

$$\phi_j(v) = \sum_{i=1}^{n} \beta_i f^j(t_i)$$

where f is the multilinear extension of v. This involves approximating f^j only at the n points $t_1, \ldots, t_n$. The numbers β_i can be computed without much difficulty because inverting the matrix M involves computing Vandermonde determinants.

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