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A THEORY OF MONEY AND FINANCIAL INSTITUTIONS

PART 28

THE NONCOOPERATIVE EQUILIBRIA OF A CLOSED TRADING ECONOMY WITH MARKET SUPPLY AND BIDDING STRATEGIES

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THE NONCOOPERATIVE EQUILIBRIA OF A CLOSED TRADING ECONOMY WITH MARKET SUPPLY AND BIDDING STRATEGIES*

by

Pradeep Dubey and Martin Shubik

1. INTRODUCTION

In several previous papers Shubik,[1] Shapley[2] and Shapley and Shubik[3] have proposed and investigated a model of exchange in many markets with the manner of price formation completely formulated and with trade utilizing a money.

The basic model analyzed previously was extremely simple; in some ways not completely satisfactory and certainly not unique in its portrayal of price formation. Shapley and Shubik[4] and Shubik[5] have suggested several alternative models. This paper is devoted to examining a noncooperative equilibrium solution to one of the alternative models and to contrasting this with the noncooperative equilibrium solution to the original model.

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2. THE "ALL FOR SALE" MODEL

A simple model of a trading economy with organized markets and money has been constructed and analyzed for the noncooperative equilibrium solution. This model is as follows:

Let there be n individuals, each with an endowment of $m+1$ commodities. Thus individual i has a vector of resources $(A_1^i, A_2^i, \ldots, A_{m+1}^i)$ and a set of preferences represented by a concave utility function $\varphi_i(x_1^i, x_2^i, \ldots, x_{m+1}^i)$.

Each individual i is required by the rules of the game to offer all of his endowments of his first m commodities for sale at m warehouses or points of sale. The $m+1^{st}$ commodity (which may be only a fiction) is used as a money. As all goods except the money must completely pass through the markets, the meaning of ownership is modified to imply that ownership of a commodity is really a title to the money receipts from the sale of the commodity rather than the commodity itself.

A strategy for player i is a vector of bids $b^i = (b_1^i, \ldots, b_m^i)$ such that $b_j^i \geq 0$ for $j = 1, \ldots, m$ and

$$(1) \qquad \sum_{j=1}^{m} b_j^i \leq A_{m+1}^i \ .$$

At each trading post j the amount for sale is $A_j = \sum_{i=1}^{n} A_j^i$ and the amount of money bid for good j is $b_j = \sum_{i=1}^{n} b_j^i$ hence price is:

$$(2) \qquad p_j = b_j/A_j \ .$$

The amount of commodity j obtained by i is:

$$(3) \qquad x_j^i = \begin{cases} b_j^i A_j / b_j & \text{if} \quad b_j > 0 \\ 0 & \text{if} \quad b_j = 0 \quad \text{for} \quad j = 1, \ldots, m \end{cases}$$

and

$$(4) \qquad x_{m+1}^i = A_{m+1}^i - \sum_{j=1}^{m} b_j^i + \sum_{j=1}^{m} A_j^i b_j / A_j \ .$$

For this model it has been shown elsewhere[6] that a noncooperative equilibrium in pure strategies always exists and that, if in an appropriately defined sense there is "enough money"[4] there will exist noncooperative equilibria which for a sufficiently high replication of the market lie within an arbitrarily close region of the competitive equilibrium points of the economy.

3. THE "OFFER FOR SALE" MODEL

It may be argued that for many markets and certainly for markets involving production and the carrying of inventory that the restriction forcing an individual to offer all of his goods for sale is unreasonable. This restriction is relaxed here. Instead we assume that a trader will enter each market as a seller or buyer, as both or neither.

A strategy for a trader i is a pair of vectors (s^i, b^i) such that:

$$(5) \qquad b_j^i \geq 0 \ , \quad 0 \leq s_j^i \leq A_j^i \qquad \text{for} \quad j = 1, \ldots, m$$

and

$$(6) \qquad \sum_{j=1}^{m} b_j^i \leq A_{m+1}^i$$

where the amount of commodity j in the final possession of trader i is:

$$(7) \qquad x_j^i = \begin{cases} A_j^i - s_j^i + b_j^i s_j / b_j & \text{if } b_j > 0 \\ A_j^i - s_j^i & \text{if } b_j = 0 \end{cases}$$

for $j = 1, \ldots, m$ and

$$(8) \qquad x_{m+1}^i = A_{m+1}^i - \sum_{j=1}^{m} b_j^i + \sum_{j=1}^{m} s_j^i p_j$$

where $p_j = b_j / s_j$ and $s_j^i p_j = 0$ if $s_j = 0$

3.1. The Strategy Set

In order to give some insight into the nature of the strategy sets of an individual and their relationship to final outcomes we consider a market with one commodity being bought or sold with trade in a commodity money.

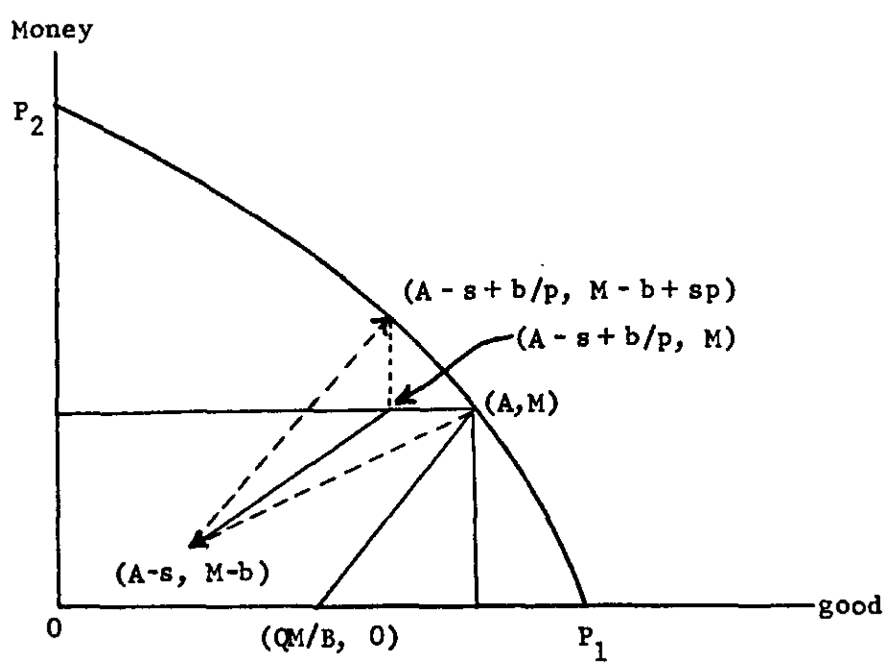

FIGURE 1

In Figure 1 the point with coordinates (A,M) indicates the initial endowment of an individual. His strategy set is a rectangle given by $0 \leq s \leq A$ and $0 \leq b \leq M$. Any point in this rectangle is a strategy.* Consider the strategy (s,b); we wish to study how it transforms the initial point (A,M). Suppose that all other traders have offered Q units of the good and a total of B units of money. Then a bid of (s,b) takes the initial holding of (A,M) to a final holding of:

$$(A - s + b/p,\ M - b + sp)$$

where

$$p = \frac{B+b}{Q+s} \quad (Q > 0) \ . \tag{9}$$

We may observe that had the individual bid $(s - b/p,\ 0)$ this too maps** (A,M) into

$$(A - s + b/p,\ M - b + sp) \ .$$

The bids which map (A,M) into itself are given by

$$A = A - s + b/p \quad \text{and} \quad M = M - b + sp$$

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*For simplicity in describing the strategies of a single individual the superscript i has been dropped as the meaning should be clear.

**We denote by $\psi_j^i(w_j^i,\ Q_j,\ B_j)$ the set of all strategies which give w_j^i of commodity j to i assuming that the total supply (bid) of the others in the j^{th} market is Q_j (B_j). This is a convex set, and is clearly continuous in its variables.

or

$$(10) \qquad b = s\left(\frac{B+b}{Q+s}\right) \quad \text{or} \quad Qb = Bs \ .$$

All bids on the line connecting $(QM/B, 0)$ to (A,M) map (A,M) onto itself. Any strategy to the right of this line maps AM onto a point on the curve P_1P_2 to the right.

All points on the curve P_1P_2 can be obtained by bids on the "L-shaped" boundary of the strategy set, i.e. bids of the form $(s,0)$ or $(0,b)$ where $0 \leq s \leq A$ and $0 \leq b \leq M$.

Let (q,m) be a point on the curve P_1P_2 then it can be checked that the equation for this surface is given by:

$$(11) \qquad m = M + \frac{(A-q)B}{Q+A-q} \ .$$

This is a concave curve.

In an analogous manner we can generalize these observations for m commodities and a money. Suppose the initial holdings of individual i are given by $(A_1^i, \ldots, A_m^i, M^i)$. All others together have offered $(Q_1, \ldots, Q_m)$ for sale and bid $(B_1, \ldots, B_m)$. A strategy by individual i is a point in a $2m$ dimensional rectangular set of the form $(s_1^i, \ldots, s_m^i, b_1^i, \ldots, b_m^i)$.

The initial holdings are transformed to final holdings of:

$$(12) \qquad (A_1^i - s_1^i + b_1^i/p_1, \ \ldots, \ A_m^i - s_m^i + b_m^i/p_m; \ M^i - \sum_{j=1}^{m} b_j^i + \sum_{j=1}^{m} s_j^i p_j) \ .$$

We may derive the equation for the set of final holdings which can be reached by all bids. Let $(q_1^i, q_2^i, \ldots, q_m^i, m^i)$ be a point on the set of final holdings. Then the equation of this surface is given by:

$$(13)\qquad m^i = M^i + \sum_{j=1}^{m}\left\{\frac{B_j(A_j^i - q_j^i)}{Q_j + A_j^i - q_j^i}\right\} .$$

This is concave. Furthermore whereas in the example with one good any final outcome could be achieved by a line of strategies (as shown in Figure 1), here any outcome can be achieved by the set of strategies formed by the product of m lines.*

In order to fully check that (13) is the surface of final allocations produced by the use of any strategy we can consider an arbitrary strategy (b_j^i, s_j^i) and show that it produces a final allocation satisfying (13). A strategy (b_j^i, s_j^i) produces a final allocation as shown in (12), substituting these values for q_j^i and m^i into (13) we obtain

$$(14)\qquad \sum_{j=1}^{m}(s_j^i p_j - b_j^i) = \sum_{j=1}^{m}\left\{\frac{\frac{B_j}{p_j}\{s_j^i p_j - b_j^i\}}{Q_j + s_j^i - \frac{b_j^i}{p_j}}\right\} = \sum_{j=1}^{m}\left\{\frac{\frac{B_j}{p_j}\{s_j^i p_j - b_j^i\}}{\frac{(Q_j + s_j^i)(b_j^i + B_j) - b_j^i(Q_j + s_j^i)}{b_j^i + B_j}}\right\}$$

$$= \sum_{j=1}^{m}\left\{\frac{\frac{B_j}{p_j}\{s_j^i p_j - b_j^i\}}{\frac{B_j}{p_j}}\right\} = \sum_{j=1}^{m}(s_j^i p_j - b_j^i) .$$

Although the careful analysis of credit is not done here (we intend

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*Implicit in our discussion is that either money is sufficiently plentiful that credit to finance the bids is not needed, or that credit is costlessly granted and repaid at the end of the period.

to investigate this separately). At least for one individual we can show how the availability of banking could enlarge his strategy set, but in this model without uncertainty would not enlarge his set of feasible final allocations that are nonnegative in all components.

Suppose that the individual has resources of (A,M) and that the bank (say through its economic activity forecast) were able to accurately estimate the aggregate bids and offers of all other traders. The meaning of the granting of credit is that the individual i can bid a sum $\sum_{j=1}^{m} b_j^i > M^i$. Given that the bank has a forecast of what aggregate bids and offers are going to be it can calculate the effect of trader i on the market even if he is a large oligopolist. Figure 2 shows this in curve P_1P_2.

The point E which represents the strategy $(0\ M)$ maps the endowment (A,M) into the point P_1 which is a final endowment of $(A + M\{Q/B+M\}, 0)$. If the trader could borrow, then all strategies of the form $(s, M + (B+M/Q)s)$ for $0 \leq s \leq A$ also map (A,M) to P_1. Thus a trader could borrow as much as $A(B+M)/Q$ and still be able to pay back his loan after trade. This amount is represented by the length HI. The strategy set of the trader without borrowing was given by the rectangle $EFGH$. With borrowing that he will be able to pay back the strategy set now becomes the trapezoid $EFGHI$.

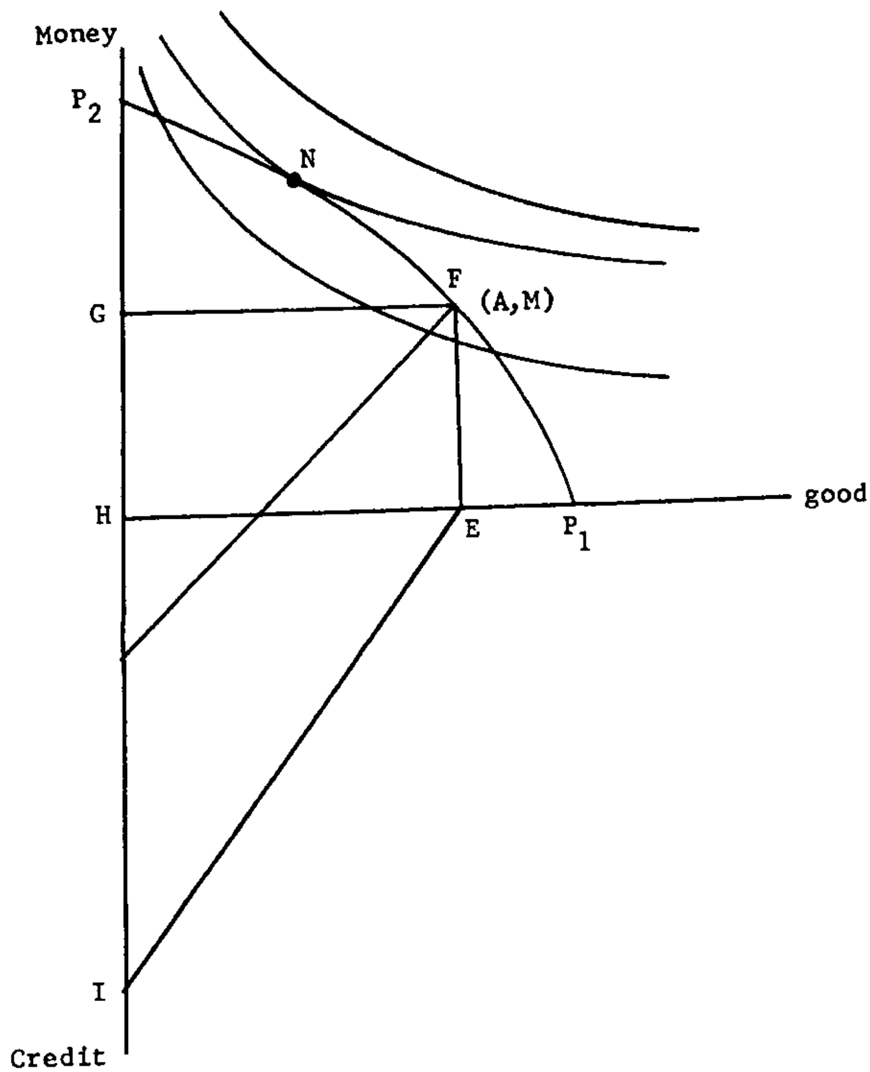

FIGURE 2

3.2. Preferences

There are two points concerning preferences and this type of bidding model which can be seen from Figure 2. First as the curve P_1P_2 is strictly concave this tells us that given the aggregate strategies of the others, the individual i will have a unique maximum he can achieve.* This is denoted by the point N .

*This is true only if the indifference sets are not "thick."

Suppose that there were only one commodity for trade and that money were fiat money. The indifference contours for the individual would be parallel to the money axis hence the optimum would occur at the point P_1 . But this is on the assumption that others would offer positive amounts of goods while the individual i would offer no goods and all his fiat money. It is easy to check that when there is only one good for sale and trade is in fiat no goods will be offered in the one period model and no trade will take place. This is not true for more than one commodity <u>if</u> the appropriate bankruptcy laws accompany the use of fiat for trade. This point is discussed in detail elsewhere.[7] 7

3.3. <u>A Simple Example</u>

A simple two trader type, example of the "offer for sale" model is provided to show the nature of our mechanism prior to discussing the general results.

It is important to maintain a clear distinction among the several features which fully define the game. They are

(a) the bids--which describe the moves

(b) the market--which transform moves to outcomes

and (c) the information conditions--which influence the domain of the strategies.

Here we assume that the individual can both bid and offer in the same market. The reader might wonder if this is a reasonable condition. We would like to prove that if an individual wishes to buy a commodity he will not simultaneously sell it; rather than make this an assumption. In a world with taxes, transactions costs and other imperfections (especially in information conditions) it is not uncommon to have an individual buy and sell the same item in the same time period. For example one may sell a stock and (with luck) buy it back 32 days later at the same price in order to establish a tax loss while maintaining one's inventory.

Consider two sets of traders who differ only in their endowments. They all have utility functions of the form

$$u^i = \alpha \log x^i + m^i \; . \tag{15}$$

Let $\alpha = 10$ and the initial holdings of traders of type 1 be (1000, 10) and of type 2 (200, 10).

The competitive solution for this market is $p = 1/60$ and $x_1 = x_2 = 600$, $m_1 = 16\frac{2}{3}$ and $m_2 = 3\frac{1}{3}$, i.e. 400 units of the first good are sold by each trader of type 1 and are bought by each of type 2.

Suppose that there are n traders of each type. The payoff for a trader of type 1 in the noncooperative games with strategies of the form (s_1^i, b_1^i) is given by:

$$\Pi_1^i = 10 \log\left\{1000 - s_1^i + b_1^i\left(\frac{Q^i + s_1^i}{B^i + b_1^i}\right)\right\} + \left\{10 - b_1^i + s_1^i\left(\frac{B^i + b_1^i}{Q^i + s_1^i}\right)\right\} \tag{16}$$

where the subscripts denote the player type and Q^i and B^i stand for the sum of all offers and bids of all traders except i .

Similarly the payoff for a trader j of type 2 is:

$$\Pi_2^j = 10 \log\left\{200 - s_2^j + b_2^j\left(\frac{Q^j + s_2^j}{B^j + b_2^j}\right)\right\} + \left\{10 - b_2^j + s_2^j\left(\frac{B^j + b_2^j}{Q^j + s_2^j}\right)\right\} . \tag{17}$$

The first order conditions for an interior maximum are:

(18) $$\frac{\partial \Pi_1^i}{\partial s_1^i} = \frac{10\left(-1 + \frac{b_1^i}{B^i + b_1^i}\right)}{1000 - s_1^i + \frac{b_1^i}{p}} + (B^i + b_1^i)\frac{Q^i}{(Q^i + s_1^i)^2} = 0$$

(19) $$\frac{\partial \Pi_1^i}{\partial b_1^i} = \frac{10\left((Q^i + s_1^i)\frac{B^i}{(B^i + b_1^i)^2}\right)}{1000 - s_1^i + \frac{b_1^i}{p}} + \left\{-1 + \frac{s_1^i}{Q^i + s_1^i}\right\} = 0 .$$

These both give the same equation of the form:

(20) $$(1000 - s_1^i)Q^i p^2 + b_1^i Q^i p - 10B^i = 0$$

and similarly for a trader of type j we obtain

(21) $$(200 - s_2^j)Q^j p^2 + b_2^j Q^j p - 10B^j = 0 .$$

The coincidence of (18) and (19) signals to us that there may be a degeneracy or indeterminacy in the equilibrium conditions hence let us guess that there may exist a type-symmetric noncooperative equilibrium (T.S.N.E.) at which traders of the first type offer the good for sale and traders of the second type bid to buy it. Replace Q^i by $(n-1)s$, s_1^i by s , b_j^i by 0 , B^j by nb and so forth, we obtain from (20) and (21)

(22) $$(1000 - s)\left(\frac{b}{s}\right) = 10\left(\frac{n}{n-1}\right)$$

and

(23) $$200\left(\frac{b}{s}\right) + b = 10\left(\frac{n-1}{n}\right) .$$

Solving we obtain:

$$(24) \qquad s = 200\,\frac{\left[5 - \left(\frac{n}{n-1}\right)^2\right]}{\left[1 + \left(\frac{n}{n-1}\right)^2\right]} \quad \text{and} \quad b = 10\left(\frac{n-1}{n}\right)\left(\frac{s}{200+s}\right) .$$

For large n, s approaches $200(5-1)/2 = 400$ and $b = 6\frac{2}{3}$ hence $p = 1/60$ which is the competitive equilibrium solution.

For $n = 2$ we obtain $s = 40$, $b = 5/6$ and $p = 5/240$. We see that with duopoly-duopsony the volume of trade is lower, prices are higher and the outcome is not Pareto optimal.

If, instead of using the strategies $(40,0)$ and $(0, 5/6)$ the traders decided to use the strategies $(40 + s_1^i, 5s_1^i/240)$ and $(s_2^j, 5/6 + 5s_2^j/240)$ where $-40 \leq s_1^i \leq 960$ and $0 \leq s_2^j \leq 200$ and:

$$5s_1^i/240 < 10 , \quad 5/6 + 5s_2^j/240 < 10 \quad \text{or} \quad s_1^i < 480 , \quad s_2^j < 440 ,$$

then any selection of s_1^i and s_2^j $(i = 1, \ldots, n$ and $j = 1, \ldots, n)$ will give an equilibrium set of strategies. This assertion can be proved generally.

<u>Lemma 1</u>. Suppose that in an n-person hold back market with trade in m goods and a money the set of strategies $(\hat{s}^1, \ldots, \hat{s}^n; \hat{b}^1, \ldots, \hat{b}^n)$ form an N.E. All strategy sets of the form

$$(25) \qquad (\hat{s}^1 + s^1, \ldots, \hat{s}^n + s^n; \hat{b}^1 + \frac{B}{Q}s^1, \ldots, \hat{b}^n + \frac{B}{Q}s^n) \text{ with } -\hat{s}^i \leq s^i \leq A^i - \hat{s}^i$$

also form an N.E. with all N.E.'s of this form having the same payoffs. Above in (25) we use the symbols $s^i = (s_1^i, s_2^i, \ldots, s_m^i)$ and $\sum_{i=1}^n \hat{s}_j^i = Q_j$; $\sum_{i=1}^n \hat{b}_j^i = B_j$; $p_j = B_j/Q_j$, $B = (B_1, B_2, \ldots, B_m)$. The initial endowments of individual i are $(A_1^i, \ldots, A_m^i, M^i)$.

Let the final endowments of traders using their original N.E. strategies be $\hat{q}_j^i$ for $i = 1, \ldots, n$, $j = 1, \ldots, m$ and $\hat{w}^i$ for money. Then given any new strategy set that is a candidate as an N.E. obtained by modifying strategies as in (25), the new prices will be:

$$(26) \qquad p_j = \frac{B_j + \sum_{i=1}^{n} \frac{B_j}{Q_j} s_j^i}{Q_j + \sum_{i=1}^{n} s_j^i} = \frac{B_j(Q_j + \sum_{i=1}^{n} s_j^i)}{Q_j(Q_j + \sum_{i=1}^{n} s_j^i)} = \hat{p}_j \ .$$

Price does not change. We may now check to see if payoffs have changed.

$$(27) \qquad q_j^i = A_j^i - (\hat{s}_j^i + s_j^i) + (\hat{b}_j^i + \hat{p}_j s_j^i)/\hat{p}_j = A_j^i - \hat{s}_j^i + \hat{b}_j^i/\hat{p}_j = \hat{q}_j^i$$

and

$$(28) \qquad w^i = M^i - \sum_{k=1}^{m} \{\hat{b}_k^i + \hat{p}_k s_k^i\} + \sum_{k=1}^{m} (\hat{s}_k^i + s_k^i)\hat{p}_k = \hat{w}^i \ .$$

From (26), (27) and (28) it can be seen that all strategy sets obtained from modifying the original N.E. shown in (25) give the same market price and distributions as it does. The only strategic interaction of the players is transmitted via price. Hence all of these strategy sets form N.E.'s with the same payoffs.

3.4. Cash Flow Minimization

A simple way to get rid of this large indeterminacy among the equilibrium points is to impose an extra condition on the model. A natural condition would be cash flow minimization. This immediately limits the selection of strategies to buying or selling (or doing neither) in each market. In terms of Figure 2 the strategies set used becomes the "L" shape of GFE (except for an ϵ-region in the neighborhood of the point F).

It is straightforward to check that if an equilibrium exists then either the strategies employed use at most only bids or offers for each commodity or there exists an equivalent equilibrium with this property. But it follows immediately from (25) that all N.E.'s strategically equivalent to the one employing only bids or offers use more cash.

In a multiperiod model with a nonzero money rate of interest the cash flow minimization condition may appear endogenously as part of utility maximization. In order however to investigate fully this phenomenon a detailed consideration of credit and bankruptcy conditions is called for.[7] Furthermore in an in depth discussion of the use of money and credit in a multiperiod economy, a distinction must be made between financing for the coverage of the float and financing to effect intertemporal trade.[7] Another equivalent way to remove indeterminacy is to minimize "goods flow."

In Section 4 which contains the formal proofs of existence for an N.E. and convergence for a T.S.N.E.* there are some notational changes from that used in Section 3.

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*There are a large number of T.S.N.E.s not all of which converge under replication to C.E.s. We will use the notation T.S.E.P. to refer to those T.S.N.E.s which are obtained from N.E.s of ϵ-games (this is spelled out in the next section in proof of Theorem 1).

4. THE EXISTENCE OF A NONCOOPERATIVE EQUILIBRIUM

4.1. The Existence Proof

For a positive integer r we shall denote by I_r the set $\{1, \ldots, r\}$, by E^r the Euclidean space of dimension r, and by Ω^r the non-negative orthant of E^r.

Let

$$I_n = \text{the set of traders}$$

$$I_{m+1} = \text{the set of commodities}$$

The initial allocation of trader i is a vector $a^i \in \Omega^{m+1}$, where a^i_j is the amount of commodity j available to i (for $j \in I_m$), and a^i_{m+1} represents the money held by i. The traders' utility functions are real-valued:

$$u^i : \Omega^{m+1} \longrightarrow \Omega^1 , \quad i \in I_n$$

and are assumed to be concave, continuous and non-decreasing. We shall say that trader i "desires" good j if $u^i(x^i)$ is an increasing function of the variable x^i_j, for any fixed choice of the other variables x^i_ℓ. A trader i for whom $a^i_{m+1} > 0$ will be called "moneyed"; and when $a^i_j > 0$ we will say that he is "j-furnished."

When we drop an index and use a bar it will indicate summation over the indexing set. Thus for $x^i \in \Omega^m$, $\bar{x}^i$ means $\sum_{j \in I_m} x^i_j$, etc. We assume that $\bar{a}_j > 0$ for all $j \in I_m$.

To cast the rules of the market in the form of a game we define the strategy set of trader i to be

$$S^i = \{(q^i, b^i) : q^i \in \Omega^m, \; b^i \in \Omega^m, \; q^i_j \le a^i_j, \; \bar{b}^i \le a^i_{m+1}\} .$$

Here q_j^i is the quantity of commodity j that trader i offers for sale. b_j^i is i's bid on good j.

The product $S^1 \times \ldots \times S^n$ will be denoted S; it is a compact convex set of dimension at most $2mn$. $\overline{S}^i$ denotes $S^1 \times \ldots \times S^{i-1} \times S^{i+1} \times \ldots \times S^n$. s (and, s^i, $\overline{s}^i$) will stand for elements of S (and, S^i, $\overline{S}^i$).

The outcome engendered by a particular $s \in S$ is determined in three simple steps. First we calculate a "price vector" $p \in \Omega^m$ by dividing the amount bid for each good by the total supply:

$$p_j = \frac{\overline{b}_j}{\overline{q}_j}, \quad \text{all } j \in I_m, \quad \text{where } s^i = (q^i, b^i) .$$

(If $\overline{q}_j = 0$, we set $p_j = 0$.) Finally we calculate the final allocation that results when the bids are executed

$$\xi_j^i(s) = \begin{cases} b_j^i/p_j & , \text{ if } j \in I_m \text{ and } p_j > 0 \\ 0 & , \text{ if } j \in I_m \text{ and } p_j = 0 \\ a_j^i - \sum_{j \in I_m} b_j^i + \sum_{j \in I_m} q_j^i p_j & , \text{ if } j = m+1 \end{cases}$$

Finally we calculate the payoff to the traders:

$$P^i(s) = u^i(\xi^i(s)) .$$

In this way, we obtain an n-person game in the standard "strategic" (or "normal" form).

A "Nash equilibrium" (or "N.E.") of this game is defined to be a $\hat{s} \in S$ with the property that, for each $i \in I_n$,

(29) $$P^i(\hat{s}) = \max\{P^i(\hat{s}|s^i) : s^i \in S^i\}$$

where $(\hat{s}|s^i)$ is $\hat{s}$ but with $\hat{s}^i$ replaced by s^i .

Theorem 1.* Assume that (1) all traders desire money; (2) for any good $j \in I_m$ there are at least two moneyed traders who desire j , and at least two j-furnished traders. Then an E.P.** exists.

To prove this we shall first establish some lemmas. By an "ϵ-game" we mean one in which some outside agency places a fixed bid of $\epsilon > 0$, and a fixed supply of $\epsilon > 0$, in each of the m markets. This does not change the players' strategy spaces, but it does of course change the outcomes and payoffs, which we will denote ξ^i_ϵ and P^i_ϵ .

Lemma 1. Under the hypotheses of Theorem 1, a N.E. exists for every ϵ-game, $\epsilon > 0$.

For the proof, it will help to build up some notation. Let

$$R^i_j = [0, a^i_j] \times [0, a^i_{m+1}]$$

$$R^i = R^i_1 \times \dots \times R^i_m$$

$$L^i = \{b \in \Omega : \sum_{j \in I_m} b_j \leq a^i_{m+1}\}$$

For $Q \in \Omega^m$, $B \in \Omega^m$, let

*We hope to weaken the hypotheses to: for each $j \in I_m$ there are at least (a) two moneyed traders who desire j , (b) two j-furnished traders who desire money.

**Recall that an E.P. is a N.E. which is the limit (as $\epsilon \to 0$) of N.E.'s of the ϵ-games.

$$D^i(Q,B,\epsilon) = \left\{ w \in \Omega^m : w^i = a^i_j - q^i_j + \frac{b^i_j(Q_j+\epsilon)}{(B_j+\epsilon)} \text{ , where } s^i \in S^i \right\} .$$

[Here $s^i = (q^i_1, \ldots, q^i_m, b^i_1, \ldots, b^i_m)$.] It is straightforwardly verified that $D^i(B,Q,\epsilon)$ is convex.*

$$H^i(Q,B,\epsilon) = \left\{ w \in \Omega^{m+1} : w_{m+1} = a^i_{m+1} + \sum_{j \in I_m} \frac{(a^i_j - w_j)(B_j+\epsilon)}{Q_j+\epsilon} , \right.$$
$$\left. (w_1, \ldots, w_m) \in D^i(Q,B,\epsilon) \right\} .$$

The analysis of the previous section shows that $H^i(Q,B,\epsilon)$ is the set of holdings that trader i can obtain with his strategies in the ϵ-game if the total supply (bid) due to others is Q_j (B_j) in the j^{th} market. Thus, in fact, $H^i(Q,B,\epsilon)$ is

$$\left\{ w \in \Omega^{m+1} : w_i = a^i_j - q^i_j + \frac{b^i_j(Q_j+\epsilon)}{B_j+\epsilon} \text{ for } j \in I_m \text{ and} \right.$$
$$\left. w_{m+1} = a^i_{m+1} - \bar{b}^i + \sum_{j \in I_m} q^i_j \left(\frac{B_j+\epsilon}{Q_j+\epsilon} \right) , \text{ where } s^i \in S^i \right\} .$$

Finally, let

$$\bar{H}^i(Q,B,\epsilon) = \{ \hat{w} \in H^i(Q,B,\epsilon) : u^i(\hat{w}) = \max_{w \in H^i(Q,B,\epsilon)} u^i(w) \} .$$

We will denote by "Proj" the projection map. Thus, for $(w_1, \ldots, w_{m+1}) = w \in \Omega^{m+1}$, $\text{Proj}_j(w)$ is w_j , etc.

---

*See Appendix.

Proof of Lemma 1

For $\bar{s}^{i} \in \bar{S}^{i}$, let $Q_j(\bar{s}^{i}) = \sum_{k\in I_n\setminus\{i\}} q_j^k$, and $B_j(\bar{s}^{i}) = \sum_{i\in I_n\setminus\{i\}} b_j^k$, where $s^k = (q_1^k, \ldots, q_m^k, b_1^k, \ldots, b_m^k)$ for $k \in I_n\setminus\{i\}$. Then it is straightforward, though laborious, to verify that the correspondence $\phi_1^i : \bar{S}^i \Longrightarrow \Omega^{m+1}$, given by:

$$\phi_1^i(\bar{s}^i) = H^i(Q(\bar{s}^i), B(\bar{s}^i), \epsilon)$$

is continuous. Since u^i is continuous on Ω^{m+1}, we see by the "Maximum Theorem" ([8], p. 116) that the correspondence ϕ_2^i given by $\phi_2^i(\bar{s}^i) = \bar{H}^i(Q(\bar{s}^i), B(\bar{s}^i), \epsilon)$ is u.s.c. We show that, in fact, $\bar{H}^i(\ )$ consists of a single point. For suppose that w and w' both belong to $\bar{H}^i(\ )$. Consider $w'' = \frac{1}{2}w + \frac{1}{2}w'$. Since u^i is concave

$$u^i(w'') \geq \frac{1}{2}u^i(w) + \frac{1}{2}u^i(w')$$

$$= u^i(w) .$$

But w_{m+1} is a <u>strictly</u> concave function of $w_1, \ldots, w_m$. Hence there is a $\gamma > 0$ such that $w'' + \gamma e^{m+1} \in \bar{H}^i(\ )$. However u^i is strictly increasing in the $(m+1)^{st}$ variable, therefore $u^i(w''+\gamma e^{m+1}) > u^i(w)$, a contradiction.

Put $\tilde{S}_j^i(\bar{s}^i) = \psi_j^i(\mathrm{Proj}_j\tilde{H}(Q(\bar{s}^i), B(\bar{s}^i), \epsilon), Q(\bar{s}^i), B(\bar{s}^i))$ and $\tilde{S}^i(\bar{s}^i) = \tilde{S}_1^i(\bar{s}^i) \times \ldots \times \tilde{S}_m^i(\bar{s}^i)$. This is a convex subset of $R_1^i \times \ldots \times R_m^i$ as shown in the preceding section. From the fact that ϕ_2^i is u.s.c. and ψ is continuous, we easily deduce that $\phi_3^i : \bar{S}^i \Longrightarrow R^i$ given by

$$\phi_3^i(\bar{s}^i) = \tilde{S}^i(\bar{s}^i)$$

is also u.s.c.

We will now define a correspondence ϕ_4^i from $\bar{S}^i$ to S^i as follows:

$$\phi_4^i(\bar{s}^i) = \phi_3^i(\bar{s}^i) \cap L^i .$$

(We interpret $r_j^i \in R_j^i$ as (q_j^i, b_j^i) and see that $\phi_4^i(\bar{s}^i) \subset S^i$.) It is obvious from the definition of H^i that this intersection is non-empty.

$$\phi : S \Longrightarrow S$$

where

$$\phi(s) = \phi_4^1(\bar{s}^1) \times \dots \times \phi_4^n(\bar{s}^n)$$

ϕ satisfies the conditions of Kakutani's fixed point theorem. Hence there is a $\hat{s} \in S$ such that

(1) $$\hat{s} \in \phi(\hat{s}) .$$

This is clearly a N.E. of the ϵ-game.

Q.E.D.

Lemma 2. Assume that for each $j \in I_m$, there exist at least two moneyed traders who desire j , and at least two j-furnished traders who desire money. Let p_j^ϵ denote the price of the j^{th} commodity in the ϵ-game at a N.E. There exist positive constants C_j and D_j , for all $j \in I_m$, such that

$$C_j < p_j^\epsilon < D_j$$

for any $\epsilon > 0$.

In the proof, we will need the following lemma from [4].

> Lemma C. (Uniform monotonicity). *Let* $j \in I_m$, *let* $f(x)$ *be a continuous, nondecreasing function from* Ω^m *to the reals that is actually increasing in the variable* x_j, *and let* H *be a positive constant. Then a positive number* $h = h(f, j, H)$ *exists such that for all* x *and* y *in* Ω^m, *if*
>
> $$\|x\| \leqq H \quad \text{and} \quad \|y-x\| \leqq h ,$$
>
> *then*
>
> $$f(y + e^j) > f(x) .$$

Proof of Lemma 2.* W.l.o.g. let $j = m$; and assume that 1 and 2 are moneyed and desire m, and 3 and 4 are m-furnished and desire money.

Put

$$H = \max\{\bar{a}_j : j \in I_{m+1}\}$$

$$h = \min[h(u^1, m, H), h(u^2, m, H)]$$

$$A = \frac{1}{m+1}\min[a^1_{m+1}, a^2_{m+1}]$$

$$\hat{h} = \min[h(u^3, m+1, H), h(u^4, m+1, H)]$$

$$\hat{A} = \frac{1}{2}\min[a^3_m, a^4_m] .$$

Note that these constants are positive under our assumptions.

First we establish the existence of C_m. Put

$$\delta = p^{\epsilon}_m = \frac{\bar{b}_m + \epsilon}{\bar{q}_m + \epsilon} .$$

*This closely follows the proof of Lemma 4 in [4].

Suppose first that the condition

$$\text{(30)} \qquad b_m^i \le \frac{\bar{b}_m}{2} \quad \text{and} \quad a_{m+1}^i - \sum_{j \in I_m} b_j^i \ge A$$

holds for at least one of $i = 1, 2$; say for $i = 1$. Then an increase Δ in 1's bid for m would be feasible if Δ is sufficiently small, say $0 < \Delta \le \min(\epsilon, A)$, and would have the following incremental effect on his final holding:

$$x_j^1(\Delta) - x_j^1 = 0 \quad \text{for} \quad j \in I_{m-1} ;$$

$$\text{(31)} \qquad x_m^1(\Delta) - x_m^1 = \frac{(\bar{q}_m + \epsilon)(\bar{b}_m^1 + \Delta)}{\bar{b}_m + \Delta + \epsilon} - \frac{(\bar{q}_m + \epsilon) b_m^1}{\bar{b}_m + \epsilon}$$

$$= (\bar{q}_m + \epsilon)\Delta \left[\frac{\bar{b}_m + \epsilon - b_m^1}{(\bar{b}_m + \epsilon)(\bar{b}_m + \epsilon + \Delta)} \right]$$

$$\ge (\bar{q}_m + \epsilon)\Delta \left[\frac{\bar{b}_m/2 + \epsilon/2 + \Delta/2}{(\bar{b}_m + \epsilon)(\bar{b}_m + \epsilon + \Delta)} \right]$$

$$= \frac{(\bar{q}_m + \epsilon)\Delta}{2(\bar{b}_m + \epsilon)} = \frac{\Delta}{2\delta} .$$

(The "$\ge$" above follows from:

$$\bar{b}_m + \epsilon - b_m^1 \ge \frac{\bar{b}_m}{2} + \epsilon \ge \frac{\bar{b}_m}{2} + \frac{\epsilon}{2} + \frac{\Delta}{2} \quad);$$

$$x_{m+1}^1(\Delta) - x_{m+1}^1 = \left(\frac{q_m^1}{\bar{q}_m + \epsilon} - 1 \right) \Delta \ge -\Delta .$$

Define*

---

* e^j is the vector in Ω^{m+1} whose j^{th} component is 1, and all others are 0.

$$z = -2\delta e^{m+1}$$

and note that we have the vector inequality

$$(32) \qquad x^1(\Delta) \geq x^1 + \frac{\Delta}{2\delta}(z + e^m) \ .$$

We are now in a position to apply Lemma C, taking $f = u^1$, $j = m$, and $y = x^1 + z$. We have $x^1 \in \Omega^m$ and $\|x^1\| \leq H$. So, by the lemma, if both $x^1 + z \geq 0$ and $z \leq h$, then

$$u^1(x^1 + z + e^m) > u^1(x^1) \ .$$

Since u^1 is concave, this implies that

$$u^1(x^1 + \frac{\Delta}{2\delta}(z + e^m)) > u^1(x^1)$$

holds for sufficiently small Δ , and hence, by (32) and the monotonicity of u^i , that

$$u^1(x^1(\Delta)) > u^1(x^1)$$

for such Δ . But this means that trader 1 could have improved, contradicting (29). Hence either $x^1 + z < 0$, or $\|z\| > h$. If the former, we have

$$x^1_{m+1} - 2\delta < 0 \ .$$

But $x^1_{m+1} \geq A$; hence

(33) $$\boxed{2\delta > A}\ .$$

If the latter, we have

(34) $$\boxed{2\delta > h}\ .$$

We now consider the case where (30) fails for both $i = 1, 2$. W.l.o.g. we may assume that

$$b_m^1 \leq \bar{b}_m/2 \ .$$

From the failure of (30) for $i = 1$, we therefore have

$$\sum_{j \in I_m} b_j^1 > a_{m+1}^1 - A \geq mA \ .$$

Hence $b_j^1 > A$ for at least one $j \in I_m$. If $j = m$, we have $b_m^1 > A$ and so <u>a fortiori</u>

(35) $$\boxed{\delta \geq A/\bar{a}_m}\ .$$

If $j \neq m$, then $m \geq 2$ and so w.l.o.g. assume $j = 1$. Trader 1 could then decrease b_1^1 by a small $\Delta > 0$ and increase b_m^1 by the same amount, with the incremental effect:

$$x_1^1(\Delta) - x_1^1 = \frac{(\bar{q}_1 + \varepsilon)(b_1^1 - \Delta)}{\bar{b}_1 - \Delta + \varepsilon} - \frac{(\varepsilon + \bar{q}_1)b_1^1}{\bar{b}_1 + \varepsilon}$$

$$\geq \frac{-(\bar{q}_1 + \varepsilon)\Delta}{\bar{b}_1 + \varepsilon} \ ;$$

$$x_j^1(\Delta) - x_j^1 = 0 \quad \text{for} \quad j = 2, \ldots, m-1 \ ;$$

$$x_m^1(\Delta) - x_m^1 \geq \frac{\Delta}{2\delta}$$

(by the same calculation as at (31), and

$$x_{m+1}^1(\Delta) - x_{m+1}^1 = \left(\frac{q_m^1}{\bar{q}_m + \varepsilon} - \frac{q_1^1}{\bar{q}_1 + \varepsilon}\right)\Delta - \left(\frac{q_1^1}{\bar{q}_1 + \varepsilon}\right)\Delta \ .$$

If we define

$$z = \left(-2\delta\frac{(\bar{q}_1 + \varepsilon)}{(\bar{b}_1 + \varepsilon)}\right)e^1 - \left(\frac{2\delta q_1^1}{\bar{q}_1 + \varepsilon}\right)e^{m+1} \tag{36}$$

then (32) is satisfied as before; so, arguing as before, we find that either $x^1 + z < 0$ or $\|z\| > h$. If the former, then either $|z_1| > x_1^1$ or $|z_{m+1}| > x_{m+1}^1$. But we have

$$x_1^1 \geq \frac{(\bar{q}_1 + \varepsilon)b_1^1}{\bar{b}_1 + \varepsilon} \geq \frac{(\bar{q}_1 + \varepsilon)A}{\bar{b}_1 + \varepsilon}$$

and

$$x_{m+1}^1 \geq \frac{b_1^1 q_1^1}{\bar{q}_1 + \varepsilon} \geq \frac{Aq_1^1}{\bar{q}_1 + \varepsilon} \ .$$

Thus, referring to (36), we see that in both cases

$$(37) \qquad \boxed{2\delta > A} \ .$$

If the latter, then either $|z_1| > h$ or $|z_{m+1}| > h$. In the first case the inequalities $\bar{b}_1 + \epsilon > b_1^1 > A$ and $\bar{a}_1 \leq H$ yield

$$(38) \qquad \boxed{2\delta > hA/H} \ ;$$

in the second, $q_1^1 \leq \bar{q}_1$ yields

$$(39) \qquad \boxed{2\delta > h} \ .$$

This exhausts all cases. Combining (33)-(35) and (37)-(39), we see that it will suffice to take C_m to be the minimum of $A/2$, $h/2$, $2A$ and $Ah/2H$.

The existence of D_m is established in a similar manner.

Suppose first that the condition

$$(40) \qquad q_m^i \leq \bar{q}_m/2 \quad \text{and} \quad a_m^i - q_m^i \geq \hat{A}$$

holds for at least one of $i = 3, 4$; say, for $i = 3$. Then an increase of Δ in 3's supply of m would be feasible if $0 < \Delta \leq \min(\epsilon, \hat{A})$, and would have the following incremental effect on his final holding:

$$x_j^3(\Delta) - x_j^3 = 0 \quad \text{for} \quad j \in I_{m-1} \ ;$$

$$x_m^3(\Delta) - x_m^3 = \left(\frac{b_m^3}{\bar{b}_m + \varepsilon} - 1\right)\Delta \geq -\Delta \ ;$$

$$x_{m+1}^3(\Delta) - x_{m+1}^3 = \frac{(\bar{b}_m + \varepsilon)}{\bar{q}_m + \Delta + \varepsilon}(q_m^3 + \Delta) - \frac{(\bar{b}_m + \varepsilon)(q_m^3)}{\bar{q}_m + \varepsilon}$$

$$\geq \frac{(\bar{b}_m + \varepsilon)\Delta}{2(\bar{q}_m + \varepsilon)} = \frac{\Delta\delta}{2} \ .$$

Define $z = -\frac{2}{\delta}e^m$ and note that then

$$x^3(\Delta) \geq x^3 + \frac{\Delta\delta}{2}[z + e^{m+1}] \ .$$

Arguing as before, we get that either $x^3 + z < 0$ or $\|z\| > \hat{h}$. If the former,

$$\frac{2}{\delta} > x_m^3 \geq \hat{A}$$

hence

(41) $$\boxed{\delta < 2/\hat{A}}$$

if the latter,

$$\frac{2}{\delta} > \hat{h}$$

hence

(42) $$\boxed{\delta < 2/\hat{h}} \ .$$

Now consider the case where (40) fails for both $i = 3$ and $i = 4$. W.l.o.g. let

$$q_m^3 \leq \bar{q}_m/2 .$$

From the failure of (40) for $i = 3$ we have

$$q_m^3 > a_m^3 - \hat{A} \geq a_m^3/2 \geq \hat{A}/2 .$$

Hence

$$\delta \leq \frac{\bar{b}_m + \epsilon}{q_m^3 + \epsilon}$$

i.e.

$$(43) \qquad \boxed{\delta \leq 2\bar{a}_{m+1}/\hat{A}} \quad .$$

Pick D_m to be the maximum of $2\bar{a}_{m+1}/\hat{A}$, $2/\hat{h}$, $2/\hat{A}$.

Q.E.D.

<u>Proof of Theorem 1</u>

Let s_ϵ be a N.E. of the ϵ-game. Consider the sequence $\{s_{\epsilon_i}\}_{i=1}^{\infty}$ where $\epsilon_i \to 0$. Lemma 2 enables us to pick a subsequence $\{s'_{\epsilon_i}\}_{i=1}^{\infty}$ such that

$$p_j^\epsilon(s'_{\epsilon_i}) \to p_j^*$$

where $0 < p_j^* < \infty$. Let s^* be a cluster point of this subsequence. Then s^* is an E.P. since it is a point of continuity of the payoff functions P^i. (If the total bid and supply at s^* are both 0 in market j, there is obviously no problem because a trader will lose j if he only supplies j, will lose money if he only bids on j, and will merely retain what he has if he both bids and supplies.)

Q.E.D.

Remarks

1. As seen earlier (see Figure 1), we may pick $s_{\epsilon_k} = \{({}^{\epsilon_k}b^1, {}^{\epsilon_k}q^1), \ldots, ({}^{\epsilon_k}b^n, {}^{\epsilon_k}q^n)\}$ such that ${}^{\epsilon_k}b^i_j \cdot {}^{\epsilon_k}q^i_j = 0$ for every $i \in I_n$ and $j \in I_m$, i.e., a trader enters any market either as a buyer or a seller, but not both. (It may, of course, happen that he neither buys nor sells.) But since s^* is a limit of $\{s_{\epsilon_k}\}$, this will hold for s^* as well.

2. If there is a trader i who is moneyed and j-furnished for some $j \in I_m$, then there will exist an E.P. at which the market for j is 'active,' i.e., the total bid and the total supply of j are both positive. From Figure 1 we observe that for a small $\eta > 0$, if we restrict S^i by requiring $q^i_j + b^i_j \geq \eta$, trader i can still obtain all his final holdings. Thus we may ensure that ${}^{\epsilon_k}b^i_j + {}^{\epsilon_k}q^i_j \geq \eta$ for each s_{ϵ_k}; and, taking limits, this holds at s^* as well. By Lemma 2, $0 < p_j(s^*) < \infty$. But $p_j(s^*) = \bar{b}^*_j/\bar{q}^*_j$. So $\bar{b}^*_j + \bar{q}^*_j > 0$ iff $\bar{b}^*_j > 0$ <u>and</u> $\bar{q}^*_j > 0$.

4.2. Limit E.P.'s and the C.E.'s

Let us consider a "replication sequence" of economies $\Gamma_1, \ldots, \Gamma_k, \ldots$. There are a fixed number t of types of traders, characterized by their utilities u^i and endowments a^i . The economy Γ_k has $n = kt$ traders, k of each type. An E.P. in which traders of the same type choose the same strategies is called a type-symmetric E.P. (denoted T.S.E.P.)* and such a T.S.E.P. ν can be represented as a vector $\bar{\nu}$ in $S = \overset{t}{\underset{i=1}{X}} S_i$ where S_i is the strategy set of trader i in Γ_1 . Thus, for each k , a T.S.E.P. in Γ_k gives us a price and an allocation in Γ_1 via $\bar{\nu}^k \in S$. We will say that ν^k is an "interior" E.P. if, for $\bar{\nu}^k = (s^1, \ldots, s^n)$ where $s^i = (q^i, b^i)$, $\bar{b}^i < a^i_{m+1}$, all $i \in I_n$.

Given a price vector $p \in \Omega^m$, we define the budget set of trader i to be

$$B^i(p) = \{x \in \Omega^{m+1} : \tilde{p}\cdot x \leq \tilde{p}\cdot a^i\}$$

with $\tilde{p}_j = p_j$ for $j \in I_m$ and $\tilde{p}_{m+1} = 1$ (i.e. the "price" of money is set equal to 1).

A competitive equilibrium (C.E.) then consists of a price $p \in \Omega^m$ and an allocation $(x^1, \ldots, x^n)$, $x^i \in \Omega^{m+1}$, such that

$$\sum_{i=1}^{n} x^i = \sum_{i=1}^{n} a^i ,$$

and

$$u^i(x^i) = \max_{y \in B^i(p)} u^i(y) \quad \text{for all } i .$$

*That such a T.S.E.P. always exists is shown in [4].

Lemma 3. Let p_j^k be the price of $j \in I_m$ at a T.S.E.P. in Γ_k. There exist, for each $j \in I_m$, positive constants C_j and D_j such that

$$C_j < p_j^k < D_j$$

for all $k = 1, 2, \ldots$.

Proof. First observe that for Γ_k, the inequalities of Lemma 2 hold with the same H (hence h) as for Γ_1. For suppose $x_j^i > \bar{a}_j$ for any trader i at a T.S.E.P. in Γ_k. (Here $\bar{a}_j = \sum_{i=1}^{t} a_j^i$, i.e., the summation is over the trader-types, not over all traders.) Then, since $x_j^i = x_j^\ell$ if i and ℓ are of the same type, we have

$$\sum_{\substack{\ell \text{ of the} \\ \text{same type} \\ \text{as } i}} x_j^\ell > k\bar{a}_j = \text{total of } j \text{ in } \Gamma_k ,$$

a contradiction. Thus (33), (34), (37), (39), (41), (42) hold. (Note that A and $\hat{A}$ do not vary with k.) To complete the proof, we need to check that (35) and (43) also hold.

In the case of (35),

$$b_m^1 > A$$

$$\Longrightarrow kb_m^1 > kA$$

$$\Longrightarrow \delta \geq \frac{kA}{k\bar{a}_m} = \frac{A}{\bar{a}_m} .$$

In the case of (43),

$$q_m^3 > \hat{A}/2$$

$$\Longrightarrow kq_m^3 > k\hat{A}/2$$

$$\Longrightarrow \delta \leq \frac{2k\bar{a}_{m+1}}{k\hat{A}} = \frac{2\bar{a}_{m+1}}{\hat{A}} .$$

Q.E.D.

Theorem 2. Suppose we have a symmetric, interior* sequence ν^k of T.S.E.P. such that $\bar{\nu}^k \to \bar{\nu} \in S$. Let p be the price and $(x^1, \ldots, x^t)$ the allocation resulting from $\bar{\nu}$. Then $(p, {}^1x^1, \ldots, {}^kx^1, \ldots, {}^1x^n, \ldots, {}^kx^t)$ is a C.E. for each Γ_k , where ${}^jx^i = {}^\ell x^i$ for all ℓ and j .

In the light of Lemma 2, the proof of this is similar to the proof of Theorem 2 in [6], and involves only the obvious changes.

5. CONCLUDING REMARKS

5.1. A Comparison with the All for Sale Model

In the "all for sale" model[3] all goods in the economy must go through the markets even if many (or all) of them are bought back by their original owners. In the model here, only the amounts intended for sale go through the markets.

We are led immediately to some macro-economic distinctions. The "all for sale" model measures total national wealth "at market" whereas the model here measures national income, or national product.

$$\text{National wealth} \quad W = \sum_{i=1}^{n} \sum_{j=1}^{m} p_j A_j^i$$

$$\text{National product = National income} \quad F = \sum_{i=1}^{n} \sum_{j=1}^{m} p_j (A_j^i - q_j^i)$$

(where q_j^i is the amount of good j offered for sale by i).

*Conditions for interiority are given in [4].

In particular, we obtain the strange, but accurate observation that if we started this system at a point of equilibrium the national product and income would be zero! This merely says that as the system begins in equilibrium nothing can be added by trade or production.

Another interpretation which can be given to the amount:

$$F = \sum_{i=1}^{n} \sum_{j=1}^{m} p_j (A_j^i - q_j^i)$$

is that it is the amount of float required to run trade in a monetary economy where velocity of money per period is at most one. If all trade were absolutely simultaneous and all individuals had only enough money to cover trade then we observe that the velocity of money would be exactly one and the size of the float would equal the national income.

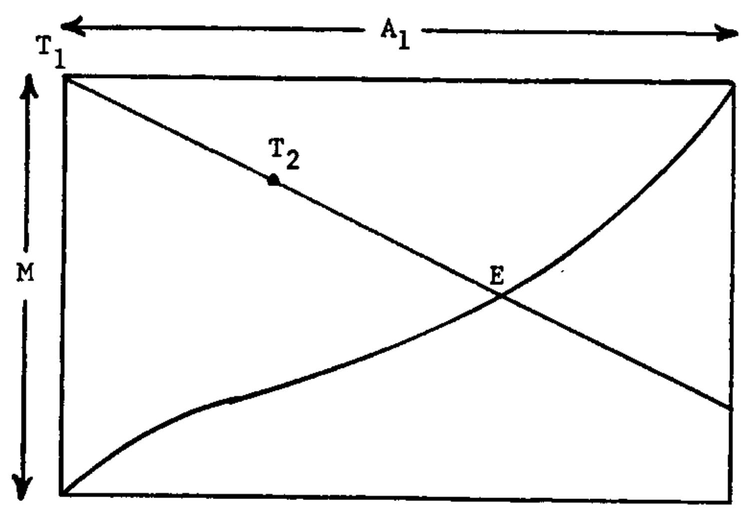

FIGURE 3

In Figure 3 we show a classical Edgeworth box for trade in two commodities where we may regard one of them as a money. Suppose E is the competitive equilibrium. Then any initial distribution of resources along the line T_1E will yield the same equilibrium point at E. Yet associated with each one will be different levels of trade. In particular if we chose the initial distribution of resources to be at point E then there will be no trade* as there are no gains to be had from trade. If the initial distribution is at T_1 trade will be larger than at T_2.

*Until we explicitly introduce cash flow minimization there will also be an associated equilibrium where all individuals offer for sale all or most of their initial resources and buy them back.

APPENDIX

We verify that $D^i(Q,B,\varepsilon)$ is convex. Take w' and w'' in $D^i(\ )$ and consider $\hat{w} = \lambda w' + (1-\lambda)w''$ where $0 \leq \lambda \leq 1$. We want to show that $\hat{w} \in D^i(\ )$.

Suppose w' $[w'']$ is obtained through the strategy (q',b') $[(q'',b'')]$. Define $(\hat{q},\hat{b})$ by:

$$\hat{q}_j = \min(q_j', q_j'') \quad \text{for} \quad j \in I_m$$

$$\hat{b}_j = \lambda b_j' + (1-\lambda)b_j'' \quad \text{for} \quad j \in I_m .$$

Since w_j is a decreasing function of q_j (if we keep b_j fixed),

$$\hat{w}_j' = \xi_j^i(\hat{q}, b') \geq \xi_j^i(q', b') = w'$$

$$\hat{w}_j'' = \xi_j^i(\hat{q}, b'') \geq \xi_j^i(q'', b'') = w'' .$$

But each component of w is concave in b (keeping q fixed), hence

$$\hat{w} = \lambda w' + (1-\lambda)w'' \leq \lambda\hat{w}' + (1-\lambda)\hat{w}''$$

$$\leq \xi^i(\hat{q}, \lambda b' + (1-\lambda)b'') = w^* .$$

By the intermediate value theorem in calculus, observe that since setting $\hat{q}_i = 0$, $b_i' = 0$, and $b_i'' = 0$ would make $w_i^* = 0$, we may reduce $\hat{q}_i$, b_i' and b_i'' appropriately to get $\hat{w}_i$ for all $i \in I_m$.

Q.E.D.

REFERENCES

[1] Shubik, M., "Commodity Money, Oligopoly, Credit and Bankruptcy in a General Equilibrium Model," Western Economic Journal 10 (1972): 24-38.

[2] Shapley, L. S., "Noncooperative General Exchange," in S. A. Y. Lin, ed., Theory and Measurement of Economic Externalities, Academic Press, to appear. (Also P-5286, Santa Monica, California: The Rand Corporation, 1974.)

[3] Shapley, L. S. and M. Shubik, "Trade Using One Commodity as a Means of Payment," Rand Corporation, 1975.

[4] Shapley, L. S. and M. Shubik, "Models of Noncooperative Exchange" (in process, partially available from authors).

[5] Shubik, Martin, "On the Number of Types of Markets with Trade in Money," CFDP 416, January 14, 1976.

[6] Dubey, P. and M. Shubik, "Trade and Prices in a Closed Economy with Exogenous Uncertainty, Different Levels of Information, Money and Special Futures Markets," CFDP 410(Revised), January 1976.

[7] Shubik, M., "The Optimal Bankruptcy Rule in a Trading Economy Using Fiat Money," CFDP 424, February 9, 1976.

[8] Berge, Claude, "Topological Spaces," The MacMillan Co., New York, 1963.