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A DUALITY THEORY FOR CONVEX ∞-HORIZON PROGRAMMING

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A DUALITY THEORY FOR CONVEX ∞-HORIZON PROGRAMMING*

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The programming model is based on a sequence of closed convex sets in R^{2m+1}, each representing triples of inputs, outputs and utility which are feasible in the corresponding period. The usual balance of goods and the usual objective function consisting of the ∞-horizon sum of discounted utilities, complete the model. Under free disposal and a rather weak assumption concerning the existence of particular primal and dual feasible solutions, the following results are derived: the primal and dual problems both possess an optimal solution, the well-known sufficient conditions for optimality appear to be necessary as well. Finally, some approximation methods are presented, which are based on finite horizon programs. The treatment emphasizes symmetry between the primal and dual problems.

1. The Primal Problem**

The programming model is based on a sequence of sets $\{E_t\}_1^\infty \subset R^{1+m+m}$; each of them satisfies the following assumptions:

A1: $E_t \subset R^1 \times R_+^m \times R_+^m$, $t = 1(1)$

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**A list of symbols is added at the end.

A2: $(\mu,x,y) \in E_t \implies \forall \tilde{\mu} \leqq \mu$, $\tilde{x} \geqq x$, $\tilde{y} \in [0,y]$: $(\tilde{\mu},\tilde{x},\tilde{y}) \in E_t$

A3: each E_t is convex

A4: each E_t is closed.

Economic interpretation: Each triple $(\mu_t, x_t, y_t) \in E_t$ may be understood as follows: (1) x_t are inputs at the beginning of a period t . (2) y_t are outputs produced by the system during the period t , and which are available at the end of period t . (3) μ_t is a feasible profit, given inputs/outputs for that period; i.e.: with given inputs/outputs x_t/y_t , there is at least one activity which produces a profit equal or larger than μ_t . In this context A2 may be taken as a free disposal assumption.

Starting from a given amount of initial outputs $y_0 \in R^m_+$, a sequence of triples $\{(\mu_t, x_t, y_t)\}_1^\infty \subset R^{1+2m}$ is called a feasible path if $(\mu_t, x_t, y_t) \in E_t$, $t = 1(1)$, $x_t \leqq y_{t-1}$, $t = 1(1)$. The inequalities $x_t \leqq y_{t-1}$, $t = 1(1)$ simply represent the balance of goods. The objective function associated with such feasible paths consists of the discounted sum $<\pi^t \mu_t>_1^\infty := \Sigma_{t=1}^\infty \pi^t \mu_t$, where the discount factor $\pi \in]0,1[$. The formal structure of the programming problem is given by the following definitions.

Definitions (1-D1 to 4):

D1: (sub-bar convention): every sequence $\{a_t\}_1^\infty$ of finite dimensional vectors shall be denoted by $\underline{a}$. Note: the initial index of a sequence denoted by $\underline{b}$ is always 1 . In case of a sequence $\{b_t\}_0^1$, $\underline{b}$ stands for the subsequence $\{b_t\}_1^\infty$.

D2: $\underline{E} := \{(\underline{\mu},\underline{x},\underline{y}) \in \ell^1 \times \ell^m \times \ell^m \mid (\mu_t, x_t, y_t) \in E_t$, $t = 1(1)\}$.

D3: The set of feasible paths (or solutions) with a given initial vector y_0 :

$$PF(y_0) := \{(\underline{\mu},\underline{x},\underline{y}) \in \underline{E} \mid x_1 \leqq y_0,\ x_{t+1} \leqq y_t,\ t = 1(1)\} .$$

D4: The programming problem:

$$\text{(1.1)} \qquad \sup_{h\to\infty} \langle \pi^t \mu_t \rangle_1^h \quad \text{w.r.t.} \quad (\underline{\mu},\underline{x},\underline{y}) \in PF(y_0) ,$$

where optimality concept is defined as follows (viz. Halkin [6]): $(\hat{\underline{\mu}},\hat{\underline{x}},\hat{\underline{y}}) \in PF(y_0)$ is called an optimal solution if there is no $(\underline{\mu},\underline{x},\underline{y}) \in PF(y_0)$ such that, for some $\epsilon > 0$ and some period r :

$$\text{(1.2)} \qquad \langle \pi^t \mu_t \rangle_1^h \geqq \epsilon + \langle \pi^t \hat{\mu}_t \rangle_1^h , \quad h = r(1) .$$

In other words, if a triple $(\underline{\mu},\underline{x},\underline{y}) \in PF(y_0)$ exist satisfying (1.2) for some $\epsilon > 0$ and some period r, we say that $(\hat{\underline{\mu}},\hat{\underline{x}},\hat{\underline{y}})$ is dominated by $(\underline{\mu},\underline{x},\underline{y})$. Note: if, for all $(\underline{\mu},\underline{x},\underline{y}) \in PF(y_0)$, the series $\{\langle \pi^t \mu_t \rangle_1^h\}_{h=1}^{\infty}$ converges then this concept of optimality coincides with the usual notion of optimality.

2. The Dual Problem

Definition (2-D1): To each E_t, we associate a "dual" set:

$$D_t := \{(\nu,u,v) \in R^1 \times R^m \times R_+^m \mid \forall (\mu,x,y) \in E_t : \mu - u'x + \pi v'y \leqq \nu\} .$$

Proposition (2-P1 to 4): The assumptions 1-A1 and 2 concerning the sets E_t imply, for each corresponding D_t :

P1: $D_t \subset R^1 \times R_+^{2m}$.

P2: $(\nu,u,v) \in D_t \Longrightarrow \forall\tilde{\nu} \geqq \nu$, $\tilde{u} \geqq u$, $\tilde{v} \in [0,v]$: $(\tilde{\nu},\tilde{u},\tilde{v}) \in D_t$.

P3: D_t is convex.

P4: D_t is closed.

Note: comparing (1-A1 to 4) with (1-P1 to 4), is is seen that the sets E_t and D_t possess a similar structure.

Proof:

P1: Since, by 1-A2 every x in $(\mu,x,y) \in E_t$ may be chosen arbitrarily large, $(\nu,u,v) \in D_t$ implies $u \geqq 0$.

P2: Since, by 1-A1: $E_t \subset R^1 \times R_+^{2m}$, this property immediately follows from the definition 2-D1.

P3: Let $(\bar{\nu},\bar{u},\bar{v})$, $(\tilde{\nu},\tilde{u},\tilde{v}) \in D_t$, and let $\alpha \in [0,1]$. Then 2-D1 implies: $\forall(\mu,x,y) \in E_t$: $\mu - (\alpha\bar{u} + (1-\alpha)\tilde{u})'x + \pi(\alpha\bar{v} - (1-\alpha)\tilde{v})'y \leqq \alpha\bar{\nu} + (1-\alpha)\tilde{\nu}$. Thus, the convex combination is a triple of D_t .

P4: Let $\{(\nu^i, u^i, v^i)\}_1^\infty \subset D_t$ be a sequence which converges to (ν^0, u^0, v^0) . Suppose $(\nu^0, u^0, v^0) \notin D_t$. Then a triple $(\mu,x,y) \in E_t$ exists such that $\mu - u^{0'}x + \pi v^{0'}y > \nu^0$, $\mu - u^{i'}x + \pi v^{i'}y \leqq \nu^i$, $i = 1(1)$. However, this contradicts the convergence.

Proposition (2-P5): If $(\hat{\mu},\hat{x},\hat{y}) \in E_t$, $(\hat{\nu},\hat{u},\hat{v}) \in D_t$ satisfy: $\hat{\mu} - \hat{u}'\hat{x} + \pi\hat{v}'\hat{y} = \hat{\nu}$, then $(\hat{\mu},\hat{x},\hat{y})$ is optimal for:

$$(2.1) \qquad \sup(\mu - \hat{u}'x + \pi\hat{v}'y) \ , \quad \text{w.r.t.} \ (\mu,x,y) \in E_t \ ,$$

and $(\hat{\nu},\hat{u},\hat{v})$ is optimal for:

$$(2.2) \qquad \inf(\nu + u'\hat{x} - \pi v'\hat{y}) \ , \quad \text{w.r.t.} \ (\nu,u,v) \in D_t \ .$$

<u>Proof</u>: The definition of D_t (2-D1) implies: $\forall(\mu,x,y) \in E_t$, $(\nu,u,v) \in D_t : \mu - u'c + \pi v' \leqq \nu$. Clearly, $(\hat{\mu},\hat{x},\hat{v}) \in E_t$, $(\hat{\nu},\hat{u},\hat{v}) \in D_t$ implies: supremum in (2.1) is not larger than $\hat{\nu}$, and the infimum in (2.2) is not smaller than $\hat{\mu}$. Hence, $\hat{\mu} - \hat{u}'\hat{x} + \pi\hat{v}'\hat{y} = \hat{\nu}$ implies optimality.

<u>Economic interpretation</u>: Each triple $(\nu_t, u_t, v_t) \in D_t$ may be understood as follows: (1) u_t input prices acting at the beginning of period t. (2) v_t output prices at the end of period t. (3) ν_t an upper bound for the discounted feasible profits $(\mu - u_t'x + \pi v_t'x)$, given the intput/output prices. In that context, problem (2.1) may be considered as profit maximization, with given input/output prices. The meaning of min. problem (2.2), to be elaborated later, can be deduced from proposition 2-P5, which presents a sufficient condition for optimality with respect to (2.1). In the following definitions, the dual programming problem is constructed in such a way that a sequence of feasible triples $(\nu_t, u_t, v_t) \in D_t$, $t = 1(1)$ with respect to the separate periods, are connected by the additional requirement $u_t \leqq v_{t-1}$. In connection with property 2-P2, these inequalities may be replaced by equalities expressing the reasonable condition that input and output prices acting at the same moment have to be equal.

<u>Definition (2-D2 to 4)</u>:

D2: $\underline{D} := \{\underline{\nu},\underline{u},\underline{v}) \in \ell^1 \times \ell^m \times \ell^m | (\nu_t, u_t, v_t) \in D_t , t = 1(1)\}$.

D3: The set of dual feasible solutions, with a given initial vector $v_0 \in R_+^m$:

$$DF(v_0) := \{(\underline{\nu},\underline{u},\underline{v}) \in \underline{D} | u_1 \leqq v_0, u_{t+1} \leqq v_t , t = 1(1)\} .$$

D4: The dual problem with a given initial vector $v_0 \in R^m_+$:

$$\inf <\pi^t \nu_t>_1^h \quad \text{w.r.t.} \quad (\underline{\nu},\underline{u},\underline{v}) \in DF(v_0) ,$$

where, changing the sign, the similar optimality concept is used as in 1-D4. Problem 1-D4 and 2-D4 shall be treated as two aspects of one single programming problem. In that context 1-D4 is called the primal problem, and, feasible/optimal solutions of 1-D4 are called primal feasible/optimal (briefly P-feasible/optimal solutions). Consequently, 2-D4 gives the dual problem. Its feasible/optimal solutions are called dual (or briefly D-) feasible/optimal solutions. Note that, with respect to 1-A1,2,3,4 and 2-P1,2,3,4, both problems possess a similar structure. Thus symmetry implies that both possess the same properties. In what follows, we assume that both problems possess a feasible solution.

3. Sufficient Conditions for Optimality

Proposition (3-P1 to 3): $\forall(\underline{\mu},\underline{x},\underline{y}) \in PF(y_0)$, $(\underline{\nu},\underline{u},\underline{v}) \in DP(v_0)$:

P1: $<\pi^t \mu_t>_1^h \leqq \pi v_0' y_0 + <\pi^t \nu_t>_1^h - <\pi^t (v_{t-1}' y_{t-1} - u_t' x_t)>_1^h - \pi^{h+1} v_h' y_h$, $h = 1(1)$.

P2: $v_{t-1}' y_{t-1} - u_t' x_t \geqq 0$, $t = 1(1)$.

P3: $<\pi^t \mu_t>_1^h \leqq \pi v_0' y_0 + <\pi^t \nu_t>_1^h$, $h = 1(1)$.

Proof: 2-D1, 1-D3, and 2-D3 imply: $\forall(\underline{\mu},\underline{x},\underline{y}) \in PF(y_0)$, $(\underline{\nu},\underline{u},\underline{v}) \in DF(v_0)$: $<\pi^t \mu_t>_1^h \leqq <\pi^t \nu_t>_1^h - <\pi^t (\pi v_t' y_t - u_t' x_t)>$, $h = 1(1)$. Shifting the terms $v_t' y_t$ yields 3-P1.

3-P2 is the consequence of the conditions: $x_t, y_t, u_t, v_t \geqq 0$, $x_t \leqq y_{t-1}$, $u_t \leqq v_{t-1}$, $t = 1(1)$.

3-P3 immediately follows from 3-P1,2 and $y_h, v_h \geqq 0$, $h = 1(1)$.

<u>Proposition (3-P4)</u>: (sufficient condition for optimality) If $(\hat{\underline{\mu}}, \hat{\underline{x}}, \hat{\underline{y}}) \in PF(v_0)$, $(\hat{\underline{v}}, \hat{\underline{u}}, \hat{\underline{v}}) \in DF(v_0)$ satisfy: $<\pi^t(\hat{\mu}_t - \hat{v}_t)>_1^h \rightarrow \pi v_0' y_0$ for $h \rightarrow \infty$, then $(\tilde{\underline{\mu}}, \tilde{\underline{x}}, \tilde{\underline{y}})$ and $(\tilde{\underline{v}}, \tilde{\underline{u}}, \tilde{\underline{v}})$ are both optimal.

<u>Proof</u>: Suppose there is a $(\tilde{\underline{\mu}}, \tilde{\underline{x}}, \tilde{\underline{y}}) \in PF(y_0)$ for which a number $\epsilon > 0$ and a period r exists such that $<\pi^t \tilde{\mu}_t>_1^h \geqq \epsilon + <\pi^t \hat{\mu}_t>_1^h$, $h = 1(1)$ (i.e.: $(\hat{\underline{\mu}}, \hat{\underline{x}}, \hat{\underline{y}})$ is dominated by $(\tilde{\underline{\mu}}, \tilde{\underline{x}}, \tilde{\underline{y}})$). Then, $<\pi^t(\hat{\mu}_t - \hat{v}_t)>_1^h \rightarrow \pi v_0' y_0$ for $h \rightarrow \infty$, implies the existence of a period $s \geqq r$ such that: $<\pi^t \tilde{\mu}_t>_1^h \geqq \frac{1}{2}\epsilon + <\pi^t \hat{v}_t>_1^h$, $h = s(1)$. This contradicts 3-P3, implying that there is no dominant solution with respect to $(\hat{\underline{\mu}}, \hat{\underline{x}}, \hat{\underline{y}})$. The dual part of the proposition may be proved in a similar way.

<u>Proposition (3-P5)</u>: $(\hat{\underline{\mu}}, \hat{\underline{x}}, \hat{\underline{y}}) \in PF(y_0)$, $(\hat{\underline{v}}, \hat{\underline{u}}, \hat{\underline{v}}) \in DF(v_0)$ satisfy: $<\pi^t(\hat{\mu}_t - \hat{v}_t)>_1^h \rightarrow \pi v_0' y_0$ for $h \rightarrow \infty$, if, and only if simultaneously: $\hat{\mu}_1' \hat{x}_1 = v_0' y_0$; $\hat{u}_{t+1}' \hat{x}_{t+1} = \hat{v}_t' \hat{y}_t$, $t = 1(1)$; $\hat{\mu}_t - \hat{\mu}_t' \hat{x}_t + \pi \hat{v}_t' \hat{y}_t = \hat{v}_t$, $t = 1(1)$; $\pi^t \hat{v}_t \hat{y}_t \rightarrow 0$ for $t \rightarrow \infty$.

<u>Proof</u>: Let $(\hat{\underline{\mu}}, \hat{\underline{x}}, \hat{\underline{y}}) \in PF(y_0)$, $(\hat{\underline{v}}, \hat{\underline{u}}, \hat{\underline{v}}) \in DF(v_0)$, and let $(\hat{v}_0, \hat{y}_0) := (v_0, y_0)$. Then:

(3.1) $\delta_t := \hat{v}_t + \hat{u}_t' x_t - \pi \hat{v}_t' y_t - \hat{\mu}_t \geqq 0$, $t = 1(1)$ (by 2-D1)

(3.2) $\gamma_t := \hat{v}_{t-1}' \hat{y}_{t-1} - \hat{u}_t' \hat{x}_t \geqq 0$, $t = 1(1)$ (by 3-P2)

(3.3) $\rho_t := \pi^{t+1} \hat{v}_h' \hat{y}_h' \geqq 0$, $h = 1(1)$ (by $\hat{v}_h, \hat{y}_h \geqq 0$).

and, finally by straightforward calculation:

(3.4) $<\pi^t(\hat{\mu}_t - \hat{v}_t)>_1^h = \pi v_0' y_0 - <\pi^t \delta_t>_1^h - <\pi^t \gamma_t>_1^h - \rho_h$, $h = 1(1)$.

The relations (3.1) to (3.4) imply equivalence between the two premises.

Proposition (3-P6): (sufficient condition for optimality) If $(\hat{\underline{\mu}},\hat{\underline{x}},\hat{\underline{y}}) \in PF(y_0)$, $(\hat{\underline{\nu}},\hat{\underline{u}},\hat{\underline{v}}) \in DF(v_0)$ satisfy: $\hat{u}_1'\hat{x}_1 = v_0'y_0$; $\hat{u}_{t+1}'\hat{x}_{t+1} = \hat{v}_t'\hat{y}_t$, $t = 1(1)$; $\hat{\mu}_t - \hat{u}_t'\hat{x}_t + \pi\hat{v}_t'\hat{y}_t = \hat{\nu}_t$, $t = 1(1)$; $\pi^t\hat{v}_t'\hat{y}_t \to 0$ for $t \to \infty$; then they are both optimal. (Corollary of 3-P4 and 3-P5.)

Definition (3-D1 and 2): (dual free starting point problem). The condition $\hat{u}_1'\hat{x}_1 = v_0'y_0$ in 3-P6 shows that, in any case, very special combinations of initial vectors y_0 , v_0 are required to met the sufficient condition. For that reason we put one of the initial vectors as an optimization variable. Since we started from the primal problem with a given initial state y_0 , it is natural to consider a free starting point version of the dual problem defined as follows:

D1: The set of feasible dual free starting point solutions, briefly D^0-feasible solutions:

$$DF^0 := \{(v_0,\underline{\nu},\underline{u},\underline{v}) \mid v_0 \in R_+^m,\ (\underline{\nu},\underline{u},\underline{v}) \in DF(v_0)\}$$

D2: The dual free starting point problem:

$$\inf_{h\to\infty} \{\pi y_0'v_0 + \langle\pi^t \nu_t\rangle_1^h\} , \quad \text{w.r.t.} \quad (v_0,\underline{\nu},\underline{u},\underline{v}) \in DF^0 .$$

The aim of this study is to prove the existence of P-and D^0-optimal solutions, under general conditions concerning the existence of P-and D^0-feasible solutions, and to show that the sufficient conditions for optimality are necessary, as well.

Economic interpretation: Suppose $(\underline{\hat{\mu}}, \underline{\hat{x}}, \underline{\hat{y}}) \in PF(y_0)$, $(\hat{v}_0, \underline{\hat{\nu}}, \underline{\hat{u}}, \underline{\hat{v}}) \in DF^0$ satisfy the sufficient condition of 3-P6, which includes the equalities: $\hat{\mu}_t - \hat{u}_t'x_t + \pi\hat{v}_t'\hat{y}_t = \hat{\nu}_t$, $t = 1(1)$. Then, by virtue of 2-P5, it appears that each $(\hat{\mu}_t, \hat{x}_t, \hat{y}_t)$ is an optimal solution of the corresponding problem:

$$(3.5) \qquad \sup(\mu - \hat{u}_t'x + \pi\hat{v}_t'y) , \quad \text{w.r.t.} \quad (\mu, x, y) \in E_t .$$

Thus, in case the optimal solutions of these separate profit maximization problems are unique, one can say that such a dual optimal solution generates a price system that allows a decomposition (or decentralization) of the infinite horizon program in a sequence of separate programs for each period. Further the sufficient condition includes: $\hat{u}_t'\hat{x}_t = \hat{v}_{t-1}'\hat{y}_{t-1}$, $t = 1(1)$, with the interpretation that, at each moment of period changing, the value of the inputs is equal to the value of the outputs.

4. Necessary Conditions for Optimality

The necessary conditions, to be deduced in this paragraph, are based on a system which, for every feasible solution, constructs a sequence of substitute feasible solutions. In a next phase, the values of the objective function of the substitute solutions are compared with that of the original feasible solution, resulting in a necessary condition for optimality. This procedure is based on an assumption concerning the existence of particular feasible solutions, expressed in the definitions 4-D1 and 2.

Definition (4-D1 to 4):

D1: P-regular solutions, defined by the set:

$$PR(y_0) := \left\{ (\underline{\mu},\underline{x},\underline{y}) \in \underline{E} \;\middle|\; \begin{array}{l} \underline{\mu} \in \ell^1_\infty,\ \underline{x},\ \underline{y} \in \ell^m_\infty; \exists\, \delta_1 > 0: \\ x_1 \leqq y_0 - \delta_1 e;\ x_{t+1} \leqq y_t - \delta_1 e,\ t = 1(1) \end{array} \right\}$$

D2: D-regular solutions, defined by the set:

$$DR(v_0) := \left\{ (\underline{\nu},\underline{u},\underline{v}) \in \underline{D} \;\middle|\; \begin{array}{l} \underline{\nu} \in \ell^1_\infty,\ \underline{u},\ \underline{v} \in \ell^m_\infty; \exists\, \delta_2 > 0: \\ u_1 \leqq v_0 - \delta_2 e;\ u_{t+1} \leqq v_t - \delta_2 e,\ t = 1(1) \end{array} \right\}$$

In the next paragraph, the existence of optimal solutions will be proved with the help of pertubations imposed on the primal problem. Under pertubations expressed by vectors $\underline{z} \in \ell^m_+$ the sets of P-feasible solutions are defined by:

D3: $PF(y_0;\underline{z}) := \{(\mu,x,y) \in \underline{E} \mid x_1 \leq y_0 + z_1;\ x_{t+1} \leqq y_t + z_{t+1},\ t = 1(1)\}$.

The corresponding primal problems are formulated:

D4: $\sup\limits_{h\to\infty} \langle \pi^t \mu_t \rangle_1^h$, w.r.t. $(\underline{\mu},\underline{x},\underline{y}) \in PF(y_0;\underline{z})$.

Consequently, we have to deduce necessary conditions for optimality which can be applied for all $\underline{z} \in \ell^m_+$. We start with the construction of substitute feasible solutions.

<u>Proposition (4-P1)</u>: Suppose $(\underline{\tilde{\mu}},\underline{\tilde{x}},\underline{\tilde{y}}) \in \underline{E}$, $\alpha \in\,]0,1[$ satisfy $\tilde{x}_{t+1} \leqq \alpha \tilde{y}_t$, $t = 1(1)$. Then, associating to every $(\underline{z},\underline{\mu},\underline{x},\underline{y})$, $\underline{z} \in \ell^m_+$ $PF(y_0;\underline{z}) \neq \emptyset$, $(\underline{\mu},\underline{x},\underline{y}) \in PF(y_0;\underline{z})$ a sequence of triples $(\underline{\mu}^h, \underline{x}^h, \underline{y}^h) \in \ell^1 \times \ell^m \times \ell^m$, $h = 1(1)$ by:

$$(4.1)\quad \begin{cases} (\mu_t^h,\ x_t^h,\ y_t^h) := \alpha^{h-t}(\tilde{\mu}_t,\ \tilde{x}_t,\ \tilde{y}_t) + (1-\alpha^{h-t})(\mu_t,\ x_t,\ y_t)\ ,\ t = 1(1)h \\ (\mu\ ,\ x\ ,\ y\ ) := (\tilde{\mu}_t,\ \tilde{x}_t,\ \tilde{y}_t)\ ,\ t = h+1(1)\ , \end{cases}$$

the assumption concerning $(\tilde{\underline{\mu}}, \tilde{\underline{x}}, \tilde{\underline{y}})$, α , and $\underline{z}$ imply:

$$(4.2) \qquad (\underline{\mu}^h, \underline{x}^h, \underline{y}^h) \in \mathrm{PF}(\alpha^{h-1}\tilde{x}_1 + (1-\alpha^{h-1})y_0; \underline{z}) , \quad h = 1(1) .$$

<u>Proof</u>: $(\tilde{\underline{\mu}}, \tilde{\underline{x}}, \tilde{\underline{y}}) \in \underline{E}$, $\alpha \in]0,1[$, $(\underline{\mu}, \underline{x}, \underline{y}) \in \mathrm{PF}(y_0;\underline{z}) \subset \underline{E}$ and the convexity of each E_t (<u>viz</u>. 1-A3) imply:

$\alpha^{h-t}(\tilde{\mu}_t, \tilde{x}_t, \tilde{y}_t) + (1-\alpha^{h-t})(\mu_t, x_t, y_t) \in E_t$, $h = 1(1)$, $t = 1(1)h$.

Together with the definition (4.1) of $(\underline{\mu}^h, \underline{x}^h, \underline{y}^h)$, this implies:

$(\underline{\mu}^h, \underline{x}^h, \underline{y}^h) \in \underline{E}$, $h = 1(1)$. Further, $\tilde{x}_{t+1} \leqq \alpha\tilde{y}_t$, $x_{t+1} \leqq y_t + z_{t+1}$,

$\tilde{x}_t, \tilde{y}_t, x_t, y_t, z_t \geqq 0$, $t = 1(1)$ implies:

$$\begin{cases} \alpha^{h-1}\tilde{x}_1 + (1-\alpha^{h-1})x_1 \leqq \alpha^{h-1}\tilde{x}_1 + (1-\alpha^{h-1})y_0 + z_1 \\ \alpha^{h-t-1}\tilde{x}_{t+1} + (1-\alpha^{h-t-1})x_{t+1} \leqq \alpha^{h-t}\tilde{y}_t + (1-\alpha^{h-t})y_t + z_{t+1}, \ t = 1(1)h-1 \\ \tilde{x}_{t+1} \leqq \tilde{y}_t + z_{t+1} , \ t = h(1) \end{cases}$$

Thus, it appears that $(\underline{\mu}^h, \underline{x}^h, \underline{y}^h) \in \mathrm{PF}(\alpha^{h-1}\tilde{x}_1 + (1-\alpha^{h-1})y_0; \underline{z})$, $h = 1(1)$.

<u>Proposition (4-P2 and 3)</u>: (preparation to 4-P4): Suppose $(\tilde{\underline{\nu}}, \tilde{\underline{u}}, \tilde{\underline{v}}) \in \underline{D}$, $\tilde{v}_0 \in R^m_+$, $\alpha \in]0,1[$, and $\delta \geqq 0$ satisfy: $\tilde{u}_t \leqq \tilde{v}_{t-1} - \delta e$, $t = 1(1)$.

Then, for every $(\underline{z}, \underline{\mu}, \underline{x}, \underline{y})$ such that: $\underline{z} \in \ell^m_+$, $\mathrm{PF}(\underline{z}) \neq \emptyset$, $(\underline{\mu}, \underline{x}, \underline{y}) \in \mathrm{PF}(\underline{z})$:

$$\text{P2:} \quad \langle(\pi/\alpha)^t\mu_t\rangle_1^h \leqq \langle(\pi/\alpha)^t\tilde{\nu}_t\rangle_1^h - \delta\langle(\pi/\alpha)^t e' y_{t-1}\rangle_1^h + $$
$$+ \langle(\pi/\alpha)^t\tilde{u}_t' z_t\rangle_1^h + \pi\tilde{v}_0' y_0 - \pi(\pi/\alpha)^h\tilde{v}_h' y_h , \quad h = 1(1) .$$

$$\text{P3:} \quad \langle\pi^t\mu_t\rangle_r^s \leqq \langle\pi^t\tilde{\nu}_t\rangle_r^s - \delta\langle\pi^t e' y_{t-1}\rangle_r^s + \langle\pi^t\tilde{u}_t' z_t\rangle_r^s + \pi^r\tilde{v}_{r-1}' y_{r-1} , \ s = 1(1) , r = 1(1)s .$$

<u>Proof</u>: $(\tilde{\underline{\nu}}, \tilde{\underline{u}}, \tilde{\underline{v}}) \in \underline{D}$ implies, for all $(\underline{\mu}, \underline{x}, \underline{y}) \in \underline{E}$:

$\mu_t - \tilde{u}_t' x_t + \pi\tilde{v}_t' y_t \leqq \tilde{\nu}_t$, $t = 1(1)$, and next with $\alpha \in]0,1[$:

$$(4.3)\quad \begin{aligned} &<(\pi/\alpha)^t \mu_t>_r^s + <(\pi/\alpha)^t (\alpha\tilde{v}'_{t-1} y_{t-1} - \tilde{u}'_t x_t)>_r^s \leqq \\ &\leqq <(\pi/\alpha)^t \tilde{\nu}_t>_r^s + \pi^r \tilde{v}'_{r-1} y_{r-1} - \pi(\pi/\alpha)^s \tilde{v}'_s y_s \ , \ s = 1(1) \ , \ r = 1(1)s \ . \end{aligned}$$

The assumption $\tilde{u}_t \leqq \alpha\tilde{v}_{t-1} - \delta e$, $x_t \leqq y_{t-1} + z_t$, $t = 1(1)$, implies:

$$(4.4)\quad <(\pi/\alpha)^t (\alpha\tilde{v}_{t-1} y_{t-1} - \tilde{u}'_t x_t)>_r^s \geqq \delta<(\pi/\alpha)^t e' y_{t-1}>_r^s - <(\pi/\alpha)^t \tilde{u}'_t z_t>_r^s \ .$$

Putting $r := 1$, $s := h$, and combining (4.3) and (4.4), 4-P2 follows. Putting $\alpha := 1$, combining (4.3), (4.4), and removing the term $\pi(\pi/\alpha)^s \tilde{v}'_s y_s \geqq 0$ in (4.3), 4-P3 follows.

Proposition (4-P4): (preparation to 4-P5): Suppose $(\underline{\tilde{\mu}}, \underline{\tilde{x}}, \underline{\tilde{y}}) \in \underline{E}$, $(\underline{\tilde{\nu}}, \underline{\tilde{u}}, \underline{\tilde{v}}) \in \underline{D}$, $\tilde{y}_0, \tilde{v}_0 \in R_+^m$, $\alpha \in]0,1[$, and $\delta \geqq 0$ satisfy: $\tilde{x}_t \leqq \alpha\tilde{y}_{t-1}$, $\tilde{u}_t \leqq \alpha\tilde{v}_{t-1} - \delta e$, $t = 1(1)$. Then, for every $(\underline{z}, \underline{\mu}, \underline{x}, \underline{y})$ such that $\underline{z} \in \ell_+^m$, $PF(y_0;\underline{z}) \neq \emptyset$, $(\underline{\mu}, \underline{x}, \underline{y}) \in PF(y_0;z)$, and every corresponding sequence $(\underline{\mu}^h, \underline{x}^h, \underline{y}^h)$, $h = 1(1)$ defined by (4.1), the following inequalities hold:

$$(4.5)\quad \begin{aligned} <\pi^t (\mu_t - \mu_t^h)>_1^s \leqq\ & \pi\alpha^h \tilde{v}'_0 y_0 - \delta\alpha^h <(\pi/\alpha)^t e' y_{t-1}>_1^h + \\ &+ \alpha^h <(\pi/\alpha)^t \tilde{u}'_t z_t>_1^h + <\pi^t \tilde{u}'_t z_t>_{h+1}^s + \\ &+ \alpha^h <(\pi/\alpha)^t \tilde{\nu}_t>_1^h + <\pi^t \tilde{\nu}_t>_{h+1}^s - \\ &- \alpha^h <(\pi/\alpha)^t \tilde{\mu}_t>_1^h - <\pi^t \tilde{\mu}_t>_{h+1}^s \ , \ h = 1(1) \ , \ s = h+1(1) \ . \end{aligned}$$

Proof: The definition of each $(\underline{\mu}^h, \underline{x}^h, \underline{y}^h)$ (viz. 4.1) implies:

$$(4.6)\quad \begin{aligned} <\pi^t (\mu_t - \mu_t^h)>_1^s =\ & \alpha^h <(\pi/\alpha)^t \mu_t>_1^h + <\pi^t \mu_t>_{h+1}^s - \\ &- \alpha^h <(\pi/\alpha)^t \tilde{\mu}_t)>_1^h - <\pi^t \tilde{\mu}_t>_{h+1}^s \ , \ h=1(1) \ , \ s=h+1(1) \end{aligned}$$

combining (4.6) with 4-P2 and 4-P3, one can find (4.5).

Theorem (4-P5) (necessary condition for optimality): Suppose the primal and the dual free starting point problem both possess a regular solution (viz. 4-D1 and 2). Then numbers $\alpha \in]\pi,1[$, $\beta_1, \beta_2 \geqq 0$ exist such that, for every $\underline{z} \in \ell^m_{1;\pi/\alpha+}$, the inequalities:

$$(4.7) \qquad \| \{(\mu_t, x_t, y_t)\}^h_1 \|_{1;\pi/\alpha} \leqq \beta_1 + \beta_2 \|\underline{z}\|_{1;\pi/\alpha} , \quad h = 1(1) ,$$

are necessary conditions for optimality with respect to the corresponding perturbed primal problem defined by 4-D4. Moreover, for every feasible solution not satisfying (4.7), there exists a dominating feasible solution which satisfies (4.7).

Proof: Let $(\underline{\tilde{\mu}}, \underline{\tilde{x}}, \underline{\tilde{y}}) \in PR(y_0)$ (viz. 4-D1) , and let $(\underline{\tilde{y}}, \underline{\tilde{u}}, \underline{\tilde{v}}) \in DR(\tilde{v}_0)$ (viz. 4-D2). Then numbers $\alpha \in]\pi,1[$ (close enough to 1) and $\delta > 0$ (close enough to 0) exist such that $\tilde{x}_1 \leqq \alpha y_0$; $\tilde{x}_{t+1} \leqq \alpha \tilde{y}_t$, $t = 1(1)$; $\tilde{u}_t \leqq \tilde{v}_{t-1} - \delta e$, $t = 1(1)$. By virtue of 4-P1, this implies: for every $\underline{z} \in \ell^m_+$, $(\underline{\mu}, \underline{x}, \underline{y}) \in PF(y_0;\underline{z})$, the triples $(\underline{\mu}^h, \underline{x}^h, \underline{y}^h)$ $h = 1(1)$ defined by (4.1) are feasible solutions of (4-D4); i.e.: $(\underline{\mu}^h, \underline{x}^h, \underline{y}^h) \in PF(y_0;\underline{z})$ $h = 1(1)$.

With respect to the values of the objective function: by virtue of the assumptions concerning regular solutions (viz. 4-D1,2) and the assumption $\underline{z} \in \ell^m_{1;\pi/\alpha+}$, the inequalities (4.5) of proposition 4-P4, may be reduced to:

$$(4.8) \quad \langle \pi^t (\mu_t - \mu^h_t) \rangle^s_1 \leqq \pi\alpha^h \tilde{v}'_0 y_0 - \delta\alpha^h \langle (\pi/\alpha)^t e' y_{t-1} \rangle^h_1 + \alpha^h \|\underline{\tilde{u}}\|_\infty \|\underline{z}\|_{1;\pi/\alpha} + \\ + \alpha^h \|\underline{\tilde{y}}\|_{1;\pi/\alpha} + \alpha^h \|\underline{\tilde{\mu}}\|_{1;\pi/\alpha} , \quad h = 1(1) , \ s = h(1) .$$

So, a necessary condition for $(\underline{\mu}, \underline{x}, \underline{y}) \in PF(y_0;\underline{x})$ not to be dominated by one of the substituting solutions $(\underline{\mu}^h, \underline{x}^h, \underline{y}^h)$ is that the right hand side of (4.8) is non-negative. Clearly, since δ is supposed to be positive, this condition can be reduced to:

$$(4.9) \qquad \| \{y_{t-1}\}_1^h \|_{1;\pi/\alpha} \leqq \gamma_1 + \gamma_2 \|\underline{z}\|_{1;\pi/\alpha}, \quad h = 1(1),$$

γ_1, γ_2 being non-negative numbers. Further, since $0 \leqq x_t \leqq y_{t-1} + z_t$, $t = 1(1)$, the necessary condition (4.9) also implies the existence of numbers γ_3, γ_4 such that:

$$(4.10) \qquad \| \{x_t\}_1^h \|_{1;\pi/\alpha} \leqq \gamma_3 + \gamma_4 \|\underline{x}\|_{1;\pi/\alpha}, \quad h = 1(1),$$

is a necessary condition for optimality. In order to deduce a necessary condition for $\underline{\mu}$, we start from the inequalities:

$$(4.11) \qquad \mu_t \leqq \rho_t := |\tilde{\nu}_t + \tilde{u}_t' x_t - \tilde{v}_t' y_t|, \quad t = 1(1),$$

implied by the definition of the sets D_t. Since by definition $\{(\nu_t, u_t, v_t)\}_1^\infty \in \ell_\infty^{1+2m}$, the necessary conditions (4.9), (4.10) imply the existence of numbers γ_5, γ_6 such that:

$$(4.12) \qquad \| \{\rho_t\}_1^h \|_{1;\pi/\alpha} \leqq \gamma_5 + \gamma_6 \|\underline{z}\|_{1;\pi/\alpha}, \quad h = 1(1),$$

is a necessary condition for optimality. A lower bound for the series $\{<(\pi/\alpha)^t \mu_t>_1^h\}_{h=1}^\infty$ can be constructed as follows: A necessary for $(\underline{\mu}, \underline{x}, \underline{y})$ not to be dominated by one of the substituting solutions is (viz. 4.6):

$$(4.13) \qquad \alpha^h <(\pi/\alpha)^t \mu_t>_1^h + <\pi^t \mu_t>_{h+1}^s \geqq \alpha^h <(\pi/\alpha)^t \tilde{\mu}_t>_1^h + <\pi^t \tilde{\mu}_t>_{h+1}^s,$$

and with $\mu_t \leqq \rho_t$, $t = 1(1)$ (viz. 4.11):

$$(4.14) \qquad <(\pi/\alpha)^t \mu_t>_1^h \geqq <(\pi/\alpha)^t \tilde{\mu}_t>_1^h + \alpha^{-h} <\pi^t \tilde{\mu}_t>_{h+1}^s - \alpha^{-h} <\pi^t \rho_t>_{h+1}^s ,$$
$$h = 1(1) , \quad s = h+1(1) .$$

From (4.14), (4.12), $\tilde{\underline{\mu}} \in \ell_\infty^1$, and $\alpha \in]\pi, 1[$, one may conclude that numbers γ_7 , γ_8 exist such that:

$$(4.15) \qquad <(\pi/\alpha)^t \mu_t>_1^h \geqq -\gamma_7 - \gamma_8 \|\underline{z}\|_{1;\pi/\alpha} , \quad h = 1(1) ,$$

is a necessary condition for optimality. Combining (4.11), (4.12), and (4.15), one may conclude there exist numbers γ_9 , γ_{10} such that:

$$(4.16) \qquad \| \{\mu_t\}_1^h \|_{1;\pi/\alpha} \leqq \gamma_9 + \gamma_{10} \|\underline{x}\|_{1;\pi/\alpha} , \quad h = 1(1) .$$

Finally, combining (4.9), (4.10), and (4.16) it appears that

$$(4.17) \qquad \| \{(\mu_t, x_t, y_t)\}_1^h \|_{1;\pi/\alpha} \leqq (\gamma_1 + \gamma_3 + \gamma_9) + (\gamma_2 + \gamma_4 + \gamma_{10}) \|\underline{z}\|_{1;\pi/\alpha} ,$$
$$h = 1(1) ,$$

is a necessary condition for $(\underline{\mu}, \underline{x}, \underline{y})$ not to be dominated by one of the substituting feasible solutions $(\underline{\mu}^h, \underline{x}^h, \underline{y}^h)$. Putting $\beta_1 := (\gamma_1 + \gamma_2 + \gamma_9) + \| \{(\tilde{\mu}_t, \tilde{x}_t, \tilde{y}_t)\}_1^\infty \|_{1;\pi/\alpha}$, $\beta_2 := (\gamma_2 + \gamma_4 + \gamma_{10})$, the latter, together with the definition of $(\underline{\mu}^h, \underline{x}^h, \underline{y}^h)$ proves the proposition.

Definition (4-D5 and 6): Similar results can be derived with respect to the dual free starting point problem. Denoting the perturbations imposed on that problem by vectors $\underline{w} \in \ell^m$, the corresponding sets of feasible solutions are defined by:

D5: $$DF^0(\underline{w}) := \{(v_0, \underline{\nu}, \underline{u}, \underline{v}) \in R^m_+ \times \underline{D} | u_t \leq v_{t-1} + z_t,\ t = 1(1)\},$$

and the corresponding dual problems by:

D6: $$\inf_{h \to \infty} \{\pi v_0' y_0 + \langle \pi^t \nu_t \rangle_1^h\},\quad \text{w.r.t.}\quad (v_0, \underline{\nu}, \underline{u}, \underline{v}) \in DF^0(\underline{w}) .$$

Apart from the difference with respect to the initial vector, problem (4-D6) possesses a similar structure as problem (4-D4). Including some modifications concerning the initial vector, that means, all previous results may be shown to hold for the dual free starting point problem. Therefore the dual version of theorem (4-P5), shall be presented without proof:

Theorem (4-P6) (dual necessary condition for optimality): Suppose the primal and the dual free starting point problem both possess a regular solution (viz. 4-D1 and 2). Then numbers $\alpha \in]\pi, 1[$, $\gamma_1, \gamma_2 \geq 0$ exist such that, for every $\underline{w} \in \ell^m_{1;\pi/\alpha+}$, the inequalities:

(4.20) $$\|v_0\|_1 + \|\{(\nu_t, u_t, v_t)\}_1^h\|_{1;\pi/\alpha} \leq \gamma_1 + \gamma_2 \|w\|_{1;\pi/\alpha},\quad h = 1(1),$$

are necessary conditions for optimality with respect to the corresponding perturbed dual free starting point problem (4.19). Moreover, for every feasible solution not satisfying (4.20), there exists a dominating feasible solution which satisfies (4.20).

5. Optimality

Definition (5-D1): The consequence of necessary condition for optimality 4-P5 is that, in case P-and D^0-regular solutions exist, we may restrict ourselves to triples $(\underline{\mu},\underline{x},\underline{y})$ of:

$$(5.1) \qquad \underline{\tilde{E}} := \underline{E} \cap \ell^{1}_{1;\pi/\alpha} \times \ell^{m}_{1;\pi/\alpha} \times \ell^{m}_{1;\pi/\alpha} ,$$

$\alpha \in \,]\pi,1[$ being the number appearing in 4-P5. Since, for such feasible solutions the series $\{<\pi^t\mu_t>_1^h\}_{h=1}^{\infty}$ converges, the perturbed primal problem (4-D4) with $\underline{z} \in \ell^m_{1;\pi/\alpha}$ can be written:

$$(5.2) \quad \sup<\pi^t\mu_t>_1^{\infty} , \quad \text{w.r.t.} \quad (\underline{\mu},\underline{x},\underline{y}) \in \underline{\tilde{E}} | x_t \leqq y_{t-1} + z_t , \quad t = 1(1) .$$

Putting $\underline{z} := 0$, (5.2) contains all optimal solutions of the original problem. With the help of the "perturbation" set:

$$(5.3) \quad \Gamma := \{(\varphi,\underline{z}) \in R^1 \times \ell^m_{1;\pi/\alpha} | \exists\, (\mu,x,y) \in \underline{\tilde{E}} : \varphi \leqq <\pi^t\mu_t>_1^{\infty},\ x_t = y_{t-1} + x_t\} ,$$

the original primal problem (i.e.: 5.2 with $\underline{z} := 0$) can be replaced by:

$$(5.4) \qquad \hat{\varphi} := \sup \varphi , \quad \text{s.t.} \quad (\varphi,0) \in \Gamma .$$

Proposition (5-P1 to 5):

P1: Γ is convex. (By convexity of $\underline{E}$.)

P2: $(\varphi,\underline{z}) \in \Gamma \Longrightarrow \forall \tilde{\varphi} \leqq \varphi , \ \underline{\tilde{z}} \geqq \underline{z} : (\tilde{\varphi},\underline{\tilde{z}}) \in \Gamma$ (by 1-A2) .

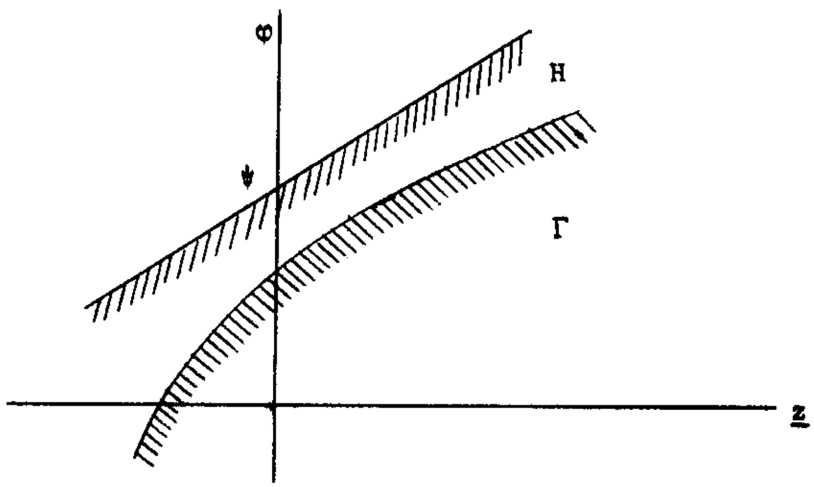
P3: The supremum in (5.4) is equal to the supremum in (5.2), with $\underline{z} := 0$.

P4: Problem (5.4) possesses an optimal solution if, and only if, $\Gamma \cap (R^1 \times \{0\})$ is non-empty and closed and if, in addition, the supremum in (5.4) is bounded.

P5: Problem (5.2) with $\underline{z} := 0$ possesses an optimal solution if, and only if, (5.4) possesses an optimal solution.

Definition (5-D2) (the dual problem): The dual space of $\ell^m_{1;\pi/\alpha}$ (i.e.: the normed space of bounded linear functionals on $\ell^m_{1;\pi/\alpha}$) is:

$$\ell^m_{\infty;\alpha/\pi} := \{\underline{x} \in \ell^m | \sup(\alpha/\pi)^t \|x_t\|_\infty < \infty\} .$$



Thus implies that, every halfspace in $R^1 \times \ell^m_{1;\pi/\alpha}$ can be expressed by:

(5.5) $$H(\underline{w}, \psi, \eta) := \{(\varphi, \underline{z}) \in R^1 \times \ell^m_{1;\pi/\alpha} | \eta\varphi - \langle w_t' z_t \rangle_1^\infty \leq \psi\} ,$$

where $(\underline{w}, \psi, \eta) \in \ell^m_{\infty;\alpha/\pi} \times R^1 \times R^1$. Clearly, in that respect:

(5.6) $$G := \{(\underline{w}, \psi, \eta) \in \ell^m_{\infty;\alpha/\pi} \times R^1 \times R^1 | \forall(\varphi, \underline{z}) \in \Gamma : \eta\varphi - \langle w_t' z_t \rangle_1^\infty \leq \psi\} ,$$

may be interpreted as the set of triples $(\underline{w}, \psi, \eta)$ which generate half-spaces that contain Γ . Consequently, the programming problem

(5.7) $$\Psi := \inf \psi , \quad \text{w.r.t.} \quad (\underline{w}, \psi, \eta) \in G | \eta = 0 ,$$

may be considered as the problem of seeking a hyperplane such that: (1) the corresponding halfspace (5.5) contains Γ, (2) the hyperplane gives a lower upper bound for points φ satisfying $(\varphi,0) \in \Gamma$. Before elaborating this interpretation, the relations between (5.7) and the dual free starting point problem (3-D2) will be stated.

<u>Proposition (5-P6)</u>: $(\underline{w},\psi) \in \ell^m_{\infty;\alpha/\pi} \times R^1$ is a feasible solution of (5.7) if, and only if, the dual free starting point problem (3-D2) possesses a feasible solution $(v_0,\underline{\nu},\underline{u},\underline{y}) \in R^m \times \ell^1 \times \ell^m_{\infty;\alpha} \times \ell^m_{\infty;\alpha}$

$$(5.8) \qquad \begin{cases} v_{t-1} = \pi^{-t} w_t \ , \ t=1(1) \\ \langle \pi^t \nu_t \rangle_1^h + \pi v_0' y_0 \leqq \psi \ , \ h=1(1) \ . \end{cases}$$

<u>Proof</u>: The following statements are equivalent:

(a) $(\underline{w},\psi)$ is a feasible solution of (5.7)

(b) $\forall (\varphi,\underline{z}) \in \Gamma : \varphi - \langle w_t' z_t \rangle_1^\infty \leqq \psi$. (by 5.6)

(c) $\forall (\underline{\mu},\underline{x},\underline{y}) \in \underline{\tilde{E}} : \langle \pi^t \mu_t \rangle_1^\infty - \langle w_t'(x_t - y_{t-1}) \rangle_1^\infty \leqq \psi$. (by 5.3)

(d) $\forall (\underline{\mu},\underline{x},\underline{y}) \in \underline{\tilde{E}}$:

$$(5.9) \qquad \langle \pi^t \mu_t \rangle_1^\infty - \langle w_t' x_t - w_{t+1}' y_t \rangle_1^\infty + w_1' y_0 \leqq \psi \ .$$

(e) There exists a sequence $\{\gamma_t\}_1^\infty \in \ell^1$ such that simultaneously:

$$(5.10) \qquad \begin{cases} \langle \gamma_t \rangle_1^h + w_1' y_0 \leqq \psi \ , \quad h=1(1) \ , \\ \forall (\underline{\mu},\underline{x},\underline{y}) \in \underline{\tilde{E}} : \pi^t \mu_t - w_t' x_t + w_{t+1}' y_t \leqq \gamma_t \ , \quad t=1(1) \ . \end{cases}$$

The equivalence of (d) and (e) can be proven by putting:

$$\gamma_t := \sup\{\mu - w_t' x + w_{t+1}' y\} \ , \quad \text{w.r.t.} \quad (\mu,x,y) \in E_t \ .$$

Clearly, (5.9) implies the boundedness of all of these suprema. Now, putting $\nu_t := \pi^{-t}\gamma_t$, $u_t := \pi^{-t}w_t$, $v_{t-1} := \pi^{-t}w_t$, $t = 1(1)$, (5.10) takes the form (5.8). Finally, since each x_t in (5.10) may be chosen arbitrarily large (viz. 1-A2), the vectors w_t must be positive. Thus, it appears that $(v_0, \underline{\nu}, \underline{u}, \underline{v})$ is a feasible solution of the dual problem (3-D2).

The relations between problem (5.4) and (5.7) shall be pointed out in the following propositions, resulting in the main theorem concerning the existence of P- and D-feasible solutions and the statement that the sufficient conditions of 3-P6 are necessary, as well.

Proposition (5-P7 and 8): If $\Gamma \cap (R^1 \times \{0\}) \neq \emptyset$, and if the supremum $\hat{\phi}$ in (5.4) is bounded, then closedness of $\Gamma_+ := \Gamma \cap (R^1 \times \mathcal{L}_+^m)$ implies:

P7: $\forall \varepsilon > 0 : (\hat{\phi}+\varepsilon) \notin cl(\Gamma)$.

P8: Problem (5.4) possesses an optimal solution.

Proof: The conditions concerning Γ and sup. $\hat{\phi}$ imply:

$[-\infty, \hat{\phi}] \times \{0\} = cl(\Gamma_+ \cap (R^1 \times \{0\})) = cl(cl(\Gamma_+) \cap (R^1 \times \{0\})) = cl(\Gamma_+) \cap (R^1 \times \{0\})$.

Proposition 5-P2 implies: $cl(\Gamma \cap (R^1 \times \mathcal{L}_-^m)) \subset [-\infty, \hat{\phi}] \times \mathcal{L}_-^m$, and so:

$cl(\Gamma \cap (R^1 \times \mathcal{L}_-^m)) \cap (R^1 \times \{0\}) \subset cl(\Gamma_+(\cap (R^1 \times \{0\}$, as well. Combining the latter with $[-\infty, \hat{\phi}] \times \{0\} = cl(\Gamma_+) \cap (R^1 \times \{0\})$, we may conclude $[-\infty, \hat{\phi}] = cl(\Gamma) \cap (R^1 \times \{0\})$, which implies 5-P7.

Since by assumption $\Gamma_+ = cl(\Gamma_+)$, 5-P8 can be proved by:

$\Gamma \cap (R^1 \times \{0\}) = \Gamma_+ \cap (R^1 \times \{0\}) = cl(\Gamma_+) \cap (R^1 \times \{0\}) = cl(\Gamma_+ \cap (R^1 \times \{0\}))$,

which implies $\Gamma \cap (R^1 \times \{0\}) = cl(\Gamma \cap (R^1 \times \{0\}))$. Since by assumption $\Gamma \cap (R^1 \times \{0\}) \neq \emptyset$ and $\hat{\phi}$ is bounded, closedness of $\Gamma \cap (R^1 \times \{0\})$ implies (viz. 5-P4) the existence of an optimal solution of (5.4).

Proposition (5-P9 and 10): If $\Gamma \cap (R^1 \times \{0\}) \neq \emptyset$ and if the supremum $\hat{\varphi}$ in (5.4) is bounded, then closedness of $\Gamma_+ := \Gamma \cap (R^1 \times \ell_+^m)$ implies:

P9: Infimum problem (5.7) possesses a feasible solution.

P10: The infimum $\hat{\psi}$ in (5.7) is equal to the supremum $\hat{\varphi}$ in (5.4).

Proof: By virtue of 5-P7, the suppositions concerning Γ and the supremum $\hat{\varphi}$ in (5.4) imply: $\forall \epsilon > 0 : (\hat{\varphi}+\epsilon, 0) \notin \Gamma$. So, by convexity of Γ (viz. 5-P1) we may conclude: for every $\epsilon > 0$ a closed halfspace $\tilde{\Gamma}_\epsilon$ exists such that $cl(\Gamma) \subset \tilde{\Gamma}_\epsilon$, $(\hat{\varphi}+\epsilon) \notin \tilde{\Gamma}_\epsilon$ (viz. Luenberger [8], page 134). Further, each of these halfspaces can be expressed by:

$$(5.11) \qquad \tilde{\Gamma}_\epsilon := \{(\varphi, \underline{z}) \in R^1 \times \ell^m_{1;\pi/\alpha} \mid \eta^\epsilon \varphi - \langle w_t^{\epsilon'} z_t \rangle_1^\infty \leqq \psi^\epsilon\} ,$$

where $(\underline{w}^\epsilon, \psi^\epsilon, \eta^\epsilon) \in G$ (viz. def. 5.6 in 5-D2), to be specified as follows:

(a) $\eta^\epsilon > 0$ (for $(\varphi, 0) \in \Gamma$ and $\eta^\epsilon \leqq 0$ imply $\forall \epsilon \geqq 0 : (\varphi+\epsilon, 0) \in \tilde{\Gamma}_\epsilon$, which is excluded by assumption $(\hat{\varphi}+\epsilon, 0) \in \tilde{\Gamma}_\epsilon$)

(b) $\eta^\epsilon := 1$ (this is allowed by $\eta^\epsilon > 0$ and by linearity of $\eta^\epsilon \varphi - \langle w_t^{\epsilon'} x_t \rangle_1^\infty \leqq \psi^\epsilon$ in η^ϵ, $\underline{w}^\epsilon$, ψ^ϵ)

(c) $\underline{w}^\epsilon \geqq 0$ (by 5-P3 and $\Gamma \subset \tilde{\Gamma}_\epsilon$).

This leads to the conclusion: $\forall \epsilon > 0 : \exists (\psi^\epsilon, \underline{w}) \in R^1 \times \ell^m_{\infty;\alpha/\pi+}$:

$$(5.12) \qquad \begin{cases} \varphi - \langle w_t^{\epsilon'} z_t \rangle_1^\infty \leqq \psi^\epsilon , & \text{for all } (\varphi, \underline{z}) \in cl(\Gamma) \\ \hat{\varphi} \leqq \psi^\epsilon \leqq \hat{\varphi}+\epsilon , & \end{cases}$$

where the first inequality is obtained by $cl(\Gamma) \subset \tilde{\Gamma}_\epsilon$, and the second by $(\hat{\varphi}+\epsilon, 0) \notin \tilde{\Gamma}_\epsilon$ and $(\hat{\varphi}, 0) \in \tilde{\Gamma}_\epsilon$.

Clearly, the first inequality implies the existence of a feasible solution of (5.7); the two inequalities together imply the infimum $\hat{\Phi}$ in (5.7) is equal to the supremum $\hat{\Phi}$ in (5.4).

<u>Proposition (5-P11)</u>: Let Γ be defined by (5.3). Suppose $\Gamma \cap (R^1 \times \{0\}) \neq \emptyset$, and suppose numbers $\eta_1, \eta_2 \geqq 0$ exist such that, for every $(\varphi, \underline{z}) \in \Gamma_+$ there is a triple $(\underline{\mu}, \underline{x}, \underline{y}) \in \tilde{\underline{E}}$ satisfying: $x_t \leqq y_{t-1} + z_t$, $t = 1(1)$ (y_0 being the given initial vector), $\varphi \leqq \langle \pi^t \mu_t \rangle_1^\infty$, $\| \{(\mu_t, x_t, y_t)\}_1^\infty \|_{1;\pi/\alpha} \leqq \eta_1 + \eta_2 \|\underline{z}\|_{1;\pi/\alpha}$. Then the set Γ_+ is closed.

<u>Proof</u>: Let $\{(\varphi^i, \underline{z}^i)\}_1^\infty \subset \Gamma_+$ be a sequence which converges to a point $(\varphi^0, \underline{z}^0)$; the existence of the sequence is ensured by assumption $\Gamma \cap (R^1 \times \{0\}) \neq \emptyset$. The other assumptions imply the existence of a sequence $\{(\underline{\mu}^i, \underline{x}^i, \underline{y}^i)\}_1^\infty \subset \tilde{\underline{E}}$ such that

$$(5.13) \qquad \varphi^i \leqq \langle \pi^t \mu_t^i \rangle_{t=1}^\infty, \quad i = 1(1),$$

$$(5.14) \qquad \left\{ \begin{array}{l} x_1^i \leqq y_0 + z_1^i \\ x_{t+1}^i \leqq y_t^i + z_{t+1}^i, \quad t = 1(1) \end{array} \right\} \quad i = 1(1),$$

$$(5.15) \qquad \| \{(\mu_t^i, x_t^i, y_t^i)\}_{t=1}^\infty \|_{1;\pi/\alpha} \leqq \eta_1 + \eta_2 \|\underline{z}\|_{1;\pi/\alpha}, \quad i = 1(1).$$

Moreover the convergence of $\{\underline{z}^i\}_1^\infty$ and (5.15) imply the existence of a number $\eta_3 \geqq 0$ such that:

$$(5.16) \qquad \| \{(\mu_t^i, x_t^i, y_t^i)\}_{t=1}^\infty \|_{1;\pi/\alpha} \leqq \eta_3, \quad i = 1(1).$$

Since the closed unit sphere in ℓ_1 is weak* compact, and so the closed unit sphere in $\ell_{1;\pi/\alpha}^{1\times 2m}$ as well, there exists (by 5.16) a subsequence

$(\underline{\mu}^{i(k)}, \underline{x}^{i(k)}, \underline{y}^{i(k)})$ which converges weak*, with respect to the $(1;\pi/\alpha)$-norm, to a point $(\underline{\mu}^0, \underline{x}^0, \underline{y}^0)$ satisfying (5.16). We denote this by:

$$(5.17)\qquad (\underline{\mu}^{i(k)}, \underline{x}^{i(k)}, \underline{y}^{i(k)}) \xrightarrow{w^*} (\underline{\mu}^0, \underline{x}^0, \underline{y}^0)\ , \quad \text{for } k \to \infty .$$

Since $\alpha \in]\pi,1[$, the expression $<\pi^t \mu_t>_1^\infty$ defines a linear functional on $\ell^1_{1;\pi/\alpha}$ which is weak* continues. That means, (5.17) implies: $\{<\pi^t \mu_t^{i(k)}>_{t=1}^\infty\}_{k=1}^\infty$ converges to $<\pi^t \mu_t^0>_{t=1}^\infty$. Thus we may conclude (by 5.13, 5.14):

$$(5.18)\qquad \varphi^0 \leqq <\pi^t \mu_t^0>_1^\infty ,$$

$$(5.19)\qquad \begin{aligned} x_1^0 &\leqq y_0 + z_1^0 \\ x_{t+1}^0 &\leqq y_t^0 + z_{t+1}^0 , \quad t = 1(1) . \end{aligned}$$

In order to prove $(\underline{\mu}^0, \underline{x}^0, \underline{y}^0) \in \tilde{\underline{E}}$, we observe that $(\underline{\mu}^i, \underline{x}^i, \underline{y}^i) \in \tilde{\underline{E}}$ $i = 1(1)$ implies:

$$(\mu_t^{i(k)}, x_t^{i(k)}, y_t^{i(k)}) \in E_t\ , \quad t = 1(1)\ , \quad k = 1(1)$$

Weak* convergence (5.17) implies that each $\{(\mu_t^{i(k)}, x_t^{i(k)}, y_t^{i(k)})\}_{k=1}^\infty$ converges to (μ_t^0, x_t^0, y_t^0) . Consequently, closedness of each E_t implies: $(\mu_t^0, x_t^0, y_t^0) \in E_t$, $t = 1(1)$. Thus, we find:

$$(5.20)\qquad (\underline{\mu}^0, \underline{x}^0, \underline{y}^0) \in \tilde{\underline{E}} .$$

Finally, the relations (5.18), (5.19), (5.20) and the definition of Γ (viz. 5.3) imply $(\varphi^0, \underline{z}^0) \in \Gamma$, and, by closedness of $\ell^m_{1;\pi/\alpha+}$, $(\varphi^0, \underline{z}^0) \in \Gamma_+$ as well. Thus, it appears that $\{(\varphi^i, \underline{z}^i)\}_1^\infty \in \Gamma_+$,

$(\varphi^i, \underline{z}^i) \to (\varphi^0, \underline{z}^0)$ for $i \to \infty$, implies $(\varphi^0, \underline{z}) \in \Gamma_+$.

Definition (5-D3): In order to formulate a dual version of some of these results, we start from theorem 4-P6 which implies that only triples $(\underline{\nu}, \underline{u}, \underline{v})$ of the set:

$$(5.21) \qquad \underline{\tilde{D}} := \underline{D} \cap \ell^1_{1;\pi/\alpha} \times \ell^m_{1;\pi/\alpha} \times \ell^m_{1;\pi/\alpha} ,$$

have to be considered, $\alpha \in]\pi, 1[$ being the number appearing in 4-P6. Taking into account the initial vector v_0 of the dual free starting point problem (3-D2), the dual version of Γ (viz. 5.3) takes the form:

$$(5.22) \qquad \Gamma_d := \left\{ (\psi, \underline{w}) \in R^1 \times \ell^m_{1;\pi/\alpha} \left| \begin{array}{l} \exists (v_0, \underline{\nu}, \underline{u}, \underline{v}) \in R^m_+ \times \underline{\tilde{D}} : \\ \pi v_0' y_0 + \langle \pi^t \nu_t \rangle_1^\infty \leq \psi \\ u_t \leq v_{t-1} + w_t \ , \ t = 1(1) \end{array} \right. \right\} ,$$

and, instead of the original dual free starting point problem (3-D2), the following min. problem will be considered:

$$(5.23) \qquad \check{\psi} := \inf \psi \ , \quad \text{s.t.} \quad (\psi, 0) \in \Gamma_d .$$

Proposition (5-P12 to 14):

P12: Problem (5.23) possesses a feasible/optimal solution if, and only if, problem (3-D2) possesses a feasible/optimal solution.

P13: Problem (5.23) possesses an optimal solution if, and only if: $\Gamma_d \cap (R^1 \times \{0\})$ is non-empty and closed, and if, in addition, the infimum in (5.23) is bounded.

Further, we observe that the propositions 5-P7,8, and 11 are valid with respect to Γ_d , as well. (Note: all other properties might be formulated in a dual version, but we don't need them.)

Theorem (5-P14 to 16): If the primal problem (1-D4) and the corresponding dual free starting point problem (3-D2) both possess a regular solution (viz. 4-D1 and 2), then:

P14: The supremum in (1-D4) is equal to the infimum in (3-D2).

P15: Both problems possess an optimal solution.

P16: P- and D^0-feasible solutions $(\hat{\underline{\mu}}, \hat{\underline{x}}, \hat{\underline{y}})$, $(\hat{v}_0, \hat{\underline{\nu}}, \hat{\underline{u}}, \hat{\underline{v}})$ both are optimal if, and only if, simultaneously: $\hat{u}_1'\hat{x}_1 = \hat{v}_0'y_0$; $\hat{u}_{t+1}'\hat{x}_{t+1} = \hat{v}_t'\hat{y}_t$, $\hat{\mu}_t - \hat{u}_t'\hat{x}_t + r\hat{v}_t'\hat{y}_t = \hat{\nu}_t$, $t = 1(1)$; $\pi^t\hat{v}_t'\hat{y}_t \to 0$ for $t \to \infty$.

Proof: By virtue of 3-P3, 4-P5, 5-P11, and the definition of Γ (viz. 5.3), the assumptions concerning the existence of regular solutions imply: (1) $\Gamma \cap (R^1 \times \{0\}) \neq \emptyset$, (2) the supremum in (5.4) is bounded, (3) the set $\Gamma_+ := \Gamma \cap (R^1 \times \ell_+^m)$ is closed. These properties have the following implications:

(a) The primal problem possesses an optimal solution (by 5-P8).

(b) The supremum in (1-D4) is equal to the infimum in (5.7) (by 5-P10).

By virtue of 3-P3, 4-P6, 5-P11-dual version, and the definition of Γ_d (viz. 5.22), the existence of regular solutions also imply: (1) $\Gamma_d \cap (R^1 \times \{0\}) \neq \emptyset$, (2) the infimum in (5.23) is bounded, (3) the set $\Gamma_{d+} := \Gamma_d \cap (R^1 \times \ell_+^m)$ is closed, which implies (by 5-P8-dual-version, and 5-P12):

(c) Dual problem (3-D2) possesses an optimal solution.

Since (by implication of 5-P6) the infimum in (3-D2) $\leq$ the infimum in (5.7), and since (by 3-P3) the supremum in (1-D4) $\leq$ infimum in (3-D2), (b) implies:

(d) The supremum in (1-D4) = infimum in (3-D2), which proves P14.

From 3-P3, 4-P5, 4-P6, and 5-P14, one may conclude that P- and D^0-feasible solutions $(\hat{\underline{\mu}}, \hat{\underline{x}}, \hat{\underline{y}})$, $(\hat{v}_0, \hat{\underline{\nu}}, \hat{\underline{u}}, \hat{\underline{v}})$ both are optimal if, and only if: $<\pi^t(\hat{\mu}_t - \hat{\nu}_t)>_1^h \to \pi\hat{v}_0' y_0$ for $h \to \infty$; which is (by 3-P5) equivalent with the conditions mentioned in 5-P16.

6. Approximations by Finite Horizon Programs

The results of Section 4 also indicate a simple way to construct approximation methods based on finite horizon programs. Let us start from an infinite horizon problem with P-and D^0-regular solutions $(\tilde{\underline{\mu}}, \tilde{\underline{x}}, \tilde{\underline{y}})$, $(\tilde{v}_0, \tilde{\underline{\nu}}, \tilde{\underline{u}}, \tilde{\underline{v}})$, and define, for every "horizon" $h = 2(1)$, the following programming problems:

D1: $$AP(y_0;h) := \left\{ \{(\mu_t, x_t, y_t)\}_1^h \in \{E_t\}_1^h \,\middle|\, \begin{array}{l} x_1 \leqq y_0,\ y_h \geqq \tilde{x}_{h+1}, \\ x_{t+1} \leqq y_t,\ t = 1(1)h-1 \end{array} \right\}$$

D2: $$\tilde{\varphi}(h) := \sup <\pi^t \mu_t>_1^h , \quad \text{w.r.t.} \quad \{(\mu_t, x_t, y_t)\}_1^h \in AP(y_0;h)\}$$

D3: $$BP(y_0;h) := \left\{ \{(\mu_t, x_t, y_t)\}_1^h \in \{E_t\}_1^h \,\middle|\, \begin{array}{l} x_1 \leqq y_0, \\ x_{t+1} \leqq y_t,\ t = 1(1)h-1 \end{array} \right\}$$

D4: $$\tilde{\psi}(h) := \sup <\pi^t \mu_t>_1^h + \pi^{h+1}\tilde{u}_{h+1}' y_t , \quad \text{w.r.t.} \quad \{(\mu_t, x_t, y_t)\}_1^h \in BP(y_0;h)$$

The dual problem with respect to D-4, can be formulated:

D5: $$BD^0(h) := \left\{ (v_0, \{(\nu_t, u_t, v_t)\}_1^h \in R_+^m \times \{D_t\}_1^\infty \,\middle|\, \begin{array}{l} v_h \geqq \tilde{u}_{h+1}, \\ u_t \leqq v_{t-1},\ t = 1(1)h \end{array} \right\}$$

D6: $$\inf \pi v_0' y_0 + <\pi^t \nu_t>_1^h , \quad \text{w.r.t.} \quad (v_0, \{(\nu_t, u_t, v_t)\}_1^h) \in BD^0(h)$$

The basis of approximations is constituted by the following properties.

Theorem (6-P1 to 6): Let $(\tilde{\underline{\mu}},\tilde{\underline{x}},\tilde{\underline{y}})$, $(\tilde{v}_0,\tilde{\underline{\nu}},\tilde{\underline{u}},\tilde{\underline{v}})$ be a P-and a D^0-regular solution. Let $\hat{\phi}$ be the supremum of the primal ∞-horizon problem. Then the finite horizon programs, defined by 6-D1 to 6, possess the following properties:

P1: If $\{(\bar{\mu}_t, \bar{x}_t, \bar{y}_t)\}_1^h \in AP(y_0;h)$ with $h \geqq 2$, then $(\underline{\mu}^*, \underline{x}^*, \underline{y}^*)$ defined by: $(\mu_t^*, x_t^*, y_t^*) := (\bar{\mu}_t, \bar{x}_t, \bar{y}_t)$, $t=1(1)h$, $(\mu_t^*, x_t^*, y_t^*) := (\tilde{\mu}_t, \tilde{x}_t, \tilde{y}_t)$, $t=h+1(1)$, satisfies: $\{(\mu_t^*, x_t^*, y_t^*)\}_1^r \in AP(y_0;r)$, $r=h(1)$, and: $(\underline{\mu}^*,\underline{x}^*,\underline{y}^*) \in PF(y_0)$

P2: If $(\bar{v}_0, \{(\bar{\nu}_t, \bar{u}_t, \bar{v}_t)\}_1^h) \in BD^0(h)$ with $h \geqq 2$, then $(\underline{\nu}^*,\underline{u}^*,\underline{v}^*)$ defined by: $(\nu_t^*, u_t^*, v_t^*) := (\bar{\nu}_t, \bar{u}_t, \bar{v}_t)$, $t=1(1)h$, $(\nu_t^*, u_t^*, v_t^*) := (\tilde{\nu}_t, \tilde{u}_t, \tilde{v}_t)$, $t=h+1(1)$, safisties: $(\bar{v}_0, \{(\nu_t^*, u_t^*, v_t^*)\}_1^r) \in BD^0(r)$, $r=h(1)$, and: $(\bar{v}_0,\underline{\nu}^*,\underline{u}^*,\underline{v}^*) \in DF^0$.

P3: $\tilde{\varphi}(h-1) + \langle\pi^t\tilde{\mu}_t\rangle_h^\infty \leqq \tilde{\varphi}(h) + \langle\pi^t\tilde{\mu}_t\rangle_{h+1}^\infty \leqq \hat{\phi}$, $h=3(1)$

P4: $\hat{\phi} \leqq \tilde{\psi}(h) + \langle\pi^t\tilde{\nu}_t\rangle_{h+1}^\infty \leqq \tilde{\psi}(h-1) + \langle\pi^t\tilde{\nu}_t\rangle_h^\infty$, $h=3(1)$

P5: $(\tilde{\varphi}(h) + \langle\pi^t\tilde{\mu}_t\rangle_{h+1}^\infty) \to \hat{\phi}$, for $h \to \infty$

P6: $(\tilde{\psi}(h) + \langle\pi^t\tilde{\nu}_t\rangle_{h+1}^\infty) \to \hat{\phi}$, for $h \to \infty$

Proof: P1 and P2 immediately follow from the definitions 6-D1 and 6-D5. P3 is implied by P1 and 6-D2, P4 is implied by P2 and 6-D6.

Proof of P5: Starting from the regular solutions $(\tilde{\underline{\mu}},\tilde{\underline{x}},\tilde{\underline{y}})$, $(\tilde{v}_0,\tilde{\underline{\nu}},\tilde{\underline{u}},\tilde{\underline{v}})$ and putting $\alpha \in]\pi,1[$ close enough to 1 such that the conditions of 4-P1 and 4-P4 are satisfied, we associate to a primal optimal solution $(\hat{\underline{\mu}},\hat{\underline{x}},\hat{\underline{y}})$ a sequence $(\hat{\underline{\mu}}^h, \hat{\underline{x}}^h, \hat{\underline{y}}^h)$, $h=1(1)$ defined by (4.1). This definition and 6-D1 implies $\{(\hat{\mu}_t^h, \hat{x}_t^h, \hat{y}_t^h)\}_1^h \in AP(y_0;h)$, $h=2(1)$, and therefore $\tilde{\varphi}(h) \geqq \langle\hat{\mu}_t^h\rangle_1^h$, $h=2(1)$, as well. Since $\hat{\varphi} - (\tilde{\varphi}(h) + \langle\pi^t\tilde{\mu}_t\rangle_{h+1}^\infty) \leqq \hat{\varphi} - \langle\pi^t\hat{\mu}_t^h\rangle_1^\infty = \langle\pi^t(\hat{\mu}_t - \hat{\mu}_h^t)\rangle_1^\infty$, we may conclude by 4-P4 (with $\underline{z} = 0$ and $s \to \infty$):

$$\hat{\varphi} - (\tilde{\varphi}(h) + \langle\pi^t\tilde{\mu}_t\rangle_{h+1}^\infty) \leqq \pi\alpha^h\tilde{v}_0' y_0 + \alpha^h\langle(\pi/\alpha)^t\tilde{v}_t\rangle_1^h + \langle\pi^t\tilde{v}_t\rangle_{h+1}^\infty - \alpha^h\langle(\pi/\alpha)^t\tilde{\mu}_t\rangle_1^h - \langle\pi^t\mu_t\rangle_{h+1}^\infty , \quad h=2(1) .$$

Using the fact that by assumption $\tilde{\underline{v}}, \tilde{\underline{\mu}} \in \ell_\infty^1$, $\alpha \in]\pi,1[$, these inequalities may be reduced to:

$$\hat{\varphi} - (\tilde{\varphi}(h) + \langle\pi^t\tilde{\mu}_t\rangle_{h+1}^\infty) \leqq \pi\alpha^h\tilde{v}_0' y_0 + \alpha^h\|\tilde{\underline{\mu}}\|_{1;\pi/\alpha} + \alpha^h\|\tilde{\underline{v}}\|_{1;\pi/\alpha} , \quad h=2(1)$$

Clearly, 6-P3, $\alpha \in]\pi,1[$ and the latter implies 6-P5.

Using similar arguments with respect to the dual problems defined by 6-D5 and 6-D6, and using 5-P14, property 6-P6 may be deduced.

Corollary: In case P-and D^θ-regular solutions exist, a procedure to find ϵ-optimal solutions (i.e. P-and D^0-feasible solutions $(\underline{\mu}^*, \underline{x}^*, \underline{y}^*)$, $(v_0, \underline{v}^*, \underline{u}^*, \underline{v}^*)$ such that some integer r : $\langle\pi^t\mu_t^*\rangle_1^s \geqq -\epsilon+\hat{\varphi}$, $s=r(1)$ and $\pi v_0^{*\prime} y_0 + \langle\pi^t v_t^*\rangle_1^s \leqq \hat{\varphi}+\epsilon$, $s=r(1)$, $\hat{\varphi}$ being the supremum of the infinite horizon problem), can be constructed by the finite horizon problems 6-D1 to 6. Since by theorem 6-P5 and 6, for every $\epsilon > 0$ an horizon h_ϵ exists such that

$$\tilde{\psi}(h_\epsilon) + \langle\pi^t\tilde{v}_t\rangle_{h_\epsilon+1}^\infty - \tilde{\varphi}(h_\epsilon) - \langle\pi^t\tilde{\mu}_t\rangle_{h_\epsilon+1}^\infty < \epsilon , \tag{6.1}$$

P-and D^0-ϵ-optimal solution can be found by solving (6-D2) and (6-D4), and shifting the horizon.

Approximations for invariant problems

In case the sets E_t are constant ever since some period c, approximations can also be obtained by the following sequences of finite horizon programs:

D7: $\bar{\varphi}(h) := \sup\left\{<\pi^t \mu_t>_1^{h-1} + \frac{\pi^h}{1-\pi} \mu_h\right\}$, w.r.t.:

$(\mu_t, x_t, y_t) \in E_t$, $t = 1(1)h$, s.t.:

$x_1 \leqq y_0$, $x_{t+1} \leqq y_t$, $t = 1(1)h-1$, $x_h \leqq y_h$.

D8: $\bar{\psi}(h) := \sup\left\{<\pi^t \mu_t>_1^{h-1} + \frac{\pi^h}{1-\pi} \mu_h\right\}$, w.r.t.:

$(\mu_t, x_t, y_t) \in E_t$, $t = 1(1)h$, s.t.:

$x_1 \leqq y_0$, $x_{t+1} \leqq y_t$, $t = 1(1)h-2$, $\frac{1}{1-\pi} x_h - \frac{\pi}{1-\pi} y_h \leqq y_{h-1}$,

and the dual programs of D8 (to be deduced by the method described in 5-D2):

D9: $\bar{\psi}(h) := \left\{\pi v_0' y_0 + <\pi^t \nu_t>_1^{h-1} + \frac{\pi^k}{1-\pi} \nu_k\right\}$, w.r.t.:

$v_0 \in R_+^m$, $(\nu_t, u_t, v_t) \in D_t$, $t = 1(1)h$, s.t.:

$u_t \leqq v_{t-1}$, $t = 1(1)h$, $u_h \leqq v_h$.

Clearly, these definitions imply similar relations as given in 6-P1 to 4. More precisely: If $\{(\bar{\mu}_t, \bar{x}_t, \bar{y}_t)\}_1^h$ is a feasible solution of 6-D7 with $h > c$, then $(\underline{\mu}^*, \underline{x}^*, \underline{y}^*)$ defined by:
$(\mu_t^*, x_t^*, y_t^*) := (\bar{\mu}_t, \bar{x}_t, \bar{y}_t)$, $t = 1(1)h$, $(\mu_t^*, x_t^*, y_t^*) := (\bar{\mu}_h, \bar{x}_h, \bar{y}_h)$, $t = h+1(1)$, has the nice properties that: each $\{(\mu_t^*, x_t^*, y_t^*)\}_1^s$ with $s \geqq h$ is a feasible solution of the corresponding problem (6-D7), and that $(\underline{\mu}^*, \underline{x}^*, \underline{y}^*)$ is a feasible solution of the primal ∞-horizon problem.

Since similar relations hold with respect to the dual programs 6-D9, we may conclude by 3-P3:

$$(6.2)\qquad \bar{\varphi}(h) \leqq \bar{\varphi}(h+1) \leqq \bar{\psi}(h+1) \leqq \bar{\psi}(h)\ ,\quad h = c(1)\ .$$

Moreover, in case P-and D^0-regular solutions exist, it can be shown that:

$$(6.3)\qquad \begin{cases} \bar{\varphi}(h) \to \hat{\varphi}\ , & \text{for } h \to \infty \\ \bar{\psi}(h) \to \hat{\varphi}\ , & \text{for } h \to \infty \end{cases}\ ,$$

$\hat{\varphi}$ being the supremum in the primal ∞-horizon program. The proof is based on the fact that it is possible to construct P-and D^0-regular solutions $(\tilde{\underline{\mu}}, \tilde{\underline{x}}, \tilde{\underline{y}})$, $(\tilde{v}_0, \tilde{\underline{\nu}}, \tilde{\underline{u}}, \tilde{\underline{v}})$ which are invariant after some period, say s . Using such regular solutions in the definition of the finite horizon programs 6-D1 to 6, it should be clear that $\tilde{\varphi}(h) \leqq \bar{\varphi}(h) \leqq \bar{\psi}(h) \leqq \tilde{\psi}(h)$, $h = s(1)$, which implies, by 6-P5 and 6-P6: $\bar{\varphi}(h) \to \hat{\varphi}$, $\bar{\psi}(h) \to \hat{\varphi}$ for $h \to \infty$. Thus, it appears that the programs 6-D7 and 6-D8 are suitable for approximation purposes too.

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LIST OF SYMBOLS

Index notations:

$i = r(1)s$, means: for each $i = r, r+1, \ldots, s$

$i = r(1)$, means: for each $i = r, r+1, \ldots$

$\{x_t\}_1^\infty$, a sequence of finite dimensional vector, the i^{th} component of a vector x_t is denoted $x_{i,t}$

Vectors and vectorspaces:

R^n , n-dimensional vectorspace

$R_+^n := \{x \in R^n | x_i \geq 0,\ i = 1(1)n\}$

e , finite dimensional vector with all component equal to 1

0 , the zero vector

$x'y$, the inner product of two finite dimensional vectors

$[x,y]$, closed interval in a finite dimensional vectorspace

$]x,y] := [x,y]/\{x\}$, $[x,y[:= [x,y]/\{y\}$, $]x,y[:=]x,y] \cap [x,y[$

$\ell^n := \{\{x_t\}_1^\infty | x_t \in R^n,\ t = 1(1)\}$, set of sequences of n-dimensional vectors

$\ell_+^n := \{\{x_t\}_1^\infty | x_t \in R_+^n,\ t = 1(1)\}$

Normed vectorspaces:

$\|x\|_1 := |x_1| + |x_2| + \ldots + |x_n|$: the ℓ_1-norm for finite dimensional vectors

$\|x\|_\infty := \max\{|x_1|, |x_2|, \ldots, |x_n|\}$: the ℓ_∞-norm for finite dim. vectors

$\|\{x_t\}_r^s\|_1 := \|x_r\|_1 + \ldots + \|x_s\|_1$

$\|\{x_t\}_r^s\|_\infty := \max\{\|x_r\|_\infty, \ldots, \|x_s\|_\infty\}$

$\|\{x_t\}_r^\infty\|_1 := \lim_{s\to\infty} \|\{x_t\}_r^s\|_1$ (possible infinite)

$\|\{x_t\}_r^\infty\|_\infty := \lim_{s\to\infty} \|\{x_t\}_1^s\|_\infty$ (possible infinite)

$cl(x)$ is the closure of a set x .

$\ell_1^n := \{\{x_t\}_1^\infty \in \ell^n \mid \|\{x_t\}_1^\infty\|_1 < \infty\}$, $\ell_{1+}^n = \ell_1^n \cap \ell_+^n$

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<u>Particular normed vectorspaces</u>: For every $\rho > 0$:

$\|\{x_t\}_r^s\|_{1;\rho} := \|\{\rho^t x_t\}_r^s\|_1$

$\|\{x_t\}_r^s\|_{\infty;\rho} := \|\{\rho^t x_t\}_1^s\|_\infty$

$\|\{x_t\}_1^\infty\|_{1;\rho} := \lim_{h\to\infty} \|\{\rho^t x_t\}_1^h\|_1$ (possible infinite)

$\|\{x_t\}_1^\infty\|_{\infty;\rho} := \lim_{h\to\infty} \|\{\rho^t x_t\}_1^h\|_\infty$ (possible infinite)

$\ell_{1;\rho}^n := \{\{x_t\}_1^\infty \in \ell^n \mid \|\{x_t\}_1^\infty\|_{1;\rho} < \infty\}$

$\ell_{\infty;\rho}^n := \{\{x_t\}_1^\infty \in \ell^n \mid \|\{x_t\}_1^\infty\|_{\infty;\rho} < \infty\}$

Note: for every $\rho > 0$, the $\ell_{1;\rho}^n$-space is congruent to ℓ_1^n and the $\ell_{\infty;\rho}^n$-space is congruent to ℓ_∞^n .

<u>Summation</u>

For every sequence of numbers $\alpha_1, \alpha_2, \ldots$:

$\langle\alpha_t\rangle_r^s := \sum_{t=r}^{s} \alpha_t$

$\langle\alpha_t\rangle_r^\infty := \lim_{s\to\infty} \langle\alpha_t\rangle_r^s$