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ACYCLIC CHOICE

Donald J. Brown

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by

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I. Introduction

Traditionally, with few exceptions, economists have assumed that individual's preference orderings are complete and transitive. Transitivity has the untenable consequence that indifference is a transitive relation. The notable exceptions are Georgescu-Roegen [16], and Armstrong [3]. They argue that intransitivities can arise out of a threshold in the perception of preference. In another vein May [21] has suggested that if the alternatives are multidimensional then preferences may be intransitive. An excellent survey of intransitive indifference is given in Fishburn [12]. Following Fishburn, we shall call complete and transitive orderings weak orders. In the most general case where we do not assume any additional structure on the set of alternatives, such as convexity, three other families of relations have been suggested as models of individual's preferences. They are partial orders, semiorders, and suborders or acyclic orders. Partial orders have been systematically explored by Aumann [5] and his colleagues Peleg [24] and Schmeidler [29]. Semiorders have been studied primarily by mathematical psychologists in particular

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Luce [19] and Tversky [23]. Acyclic orders were introduced into the economic literature by von Neumann and Morgenstern [36] and have received some attention from Sen [32], Adams [2] and Fishburn [13]--acyclic orders are called suborders by Fishburn. We should mention that Sonnenshein [34] has proved the existence of demand functions where the transitivity of preferences has been replaced by the convexity of preferences.

The need to study orderings other than weak orders is also occasioned by several theorems pertaining to the aggregation of individual's preference orderings, the most important theorem being Arrow's Possibility Theorem. Arrow has shown that even weak orders can not be aggregated into a weak order, if the method of aggregation must satisfy certain intuitively appealing conditions. A fortiori more general individual's orderings, say acyclic orderings, can not be aggregated subject to the remainder of Arrow's conditions. Even if we were willing to accept the assumption of weak orders for individuals, the Arrow theorem still suggests that we ought to consider a broader class of orderings for social preferences. This has been the approach of Sen and Mas-Colell and Sonnenschein. Sen [30] has replaced Arrow's condition that the social ordering be a weak order with the condition that it be a partial order. Mas-Colell and Sonnenschein [20] have only required the social ordering to be acyclic. Their work and its relationship with Arrow's Possibility Theorem is discussed in detail in Fishburn [15].

The need for studying the lattice-theoretic or combinatorial properties of acyclic relations is evidenced by the observation that many social decision functions can be expressed as 'polynomials' over the family of acyclic relations, on the set of social alternatives. Where the operations of combination are union and intersection of relations. For example majority rule is such a function. We, of course, wish to find sufficient conditions for given combinations of acyclic relations to be an acyclic relation.

This is a special instance of the more general desire to characterize what properties of individual's preferences are preserved by various group decision functions. If we are to make precise the notion that the social ordering produced by majority rule or any other social choice function is typical or representative of the preferences in society. Then we shall have to introduce a formal language, such as the first order predicate calculus, where the properties of individual's preferences are expressed as formalized sentences and determine the class of sentences which are preserved by the aggregation procedure. This is done in a paper by Bloom and Brown [7].

II. The Lattice of Acyclic Relations

Let A be a nonempty set and $P \subseteq A \times A$. P is said to be a partial order if it is asymmetric and transitive. A relation P is anti-symmetric if $\langle a,b\rangle \in P$ and $\langle b,a\rangle \in P$ then $a = b$. A relation P is transitive if $\langle a,b\rangle \in P$ and $\langle b,c\rangle \in P$ implies $\langle a,c\rangle \in P$. A partially ordered, p.o., set is a pair $\langle A,P\rangle$ where P is a partial order on A. We will often denote the partial order as $\succ$, and $\langle a,b\rangle \in P$ as aPb or $a \succ b$. The least upper bound of two elements a and b in a p.o. set A is denoted $a \vee b$ and is defined as the element in A such that for all $c \in A$ if $c \succ a$ and $c \succ b$ then $c \succ a \vee b$, and where $a \vee b \succ a$, $a \vee b \succ b$. The greatest lower bound of two elements a and b in a p.o.

set is denoted <u>$a \wedge b$</u> and is defined in an analogous fashion. A <u>lattice</u> $\mathcal{L}$ is a p.o. set where every pair of elements has both a least upper bound (<u>join</u>) and a grestest lower bound (<u>meet</u>). The classical example of a lattice is the family of subsets of a nonempty set ordered under set inclusion. In this case the least upper bound of two elements is their union and the greatest lower bound is their intersection. A lattice is said to be <u>complete</u> if the join and meet of every set of elements in the lattice exists.

An element a in a lattice is called <u>meet irreducible</u> if it cannot be expressed as the meet of all the elements which contain it. A subset B of a p.o. set A is said to be <u>directed</u> if every finite subset of B has an upper bound in B. Where a is an <u>upper bound</u> for C in the p.o. set $\langle A, \succ \rangle$ if for all $b \in C$, $a \succ b$. The meet of any subset, B, of A will be denoted $\bigwedge B$ and the join as $\bigvee B$. Let $\mathcal{L} = \langle A, \succ \rangle$ be a complete lattice. An element $c \in \mathcal{L}$ is called <u>compact</u> if whenever $D \subseteq A$ is a directed set such that $\bigvee D \succ c$, then $a \succ c$ for some $a \in D$. A lattice $\mathcal{L}$ is <u>compactly generated</u> if it is complete and every element of $\mathcal{L}$ is the join of a set of compact elements. Compactly generated lattices have an abundance of meet irreducible elements as is shown by the next theorem.

<u>Theorem 1</u>, Pierce [25]. Let $\mathcal{L}$ be a compactly generated lattice. Then every element of $\mathcal{L}$ is the meet of the set of all meet irreducible elements that contain it. That is, if M is the set of all meet irreducible elements of $\mathcal{L}$, then for every $a \in \mathcal{L}$, $a = \bigwedge\{m \in M | m \succ a\}$.

If X is a nonempty set and P a binary relation over X, i.e. $P \subseteq X \times X$. Then P is <u>acyclic</u> if there does not exist a finite subset

$x_1, x_2, \ldots, x_n$ such that $x_1Px_2, x_2Px_3, \ldots, x_{n-1}Px_n$, and x_nPx_1. An equivalent definition is that every finite subset of X has a maximal element with respect to P. If $x \in Y$, then x is <u>maximal</u> if Y with respect to P is there does not exist $y \in Y$ such that yPx.

Let $\mathcal{Y}_X$ be the family of acyclic relations on X and $\bar{\mathcal{Y}}_X = \mathcal{Y}_X \cup \{X \times X\}$. Then $\bar{\mathcal{Y}}_X$ ordered under set includion is a complete lattice. Also the union of any directed family of relations belonging to $\bar{\mathcal{Y}}_X$ belongs to $\bar{\mathcal{Y}}_X$. Hence by the following theorem $\bar{\mathcal{Y}}_X$ is a compactly generated lattice.

<u>Theorem 2</u>, Pierce [25]. Let Z be any set. Suppose that $\mathcal{L} \subseteq \mathcal{P}(Z)$ (where $\mathcal{P}(Z)$ is the power set of Z) satisfies the conditions

(i) $\mathcal{L}$ is a complete lattice under the partial ordering of set inclusion

(ii) $\mathcal{D} \subseteq \mathcal{L}$, $\mathcal{D}$ directed implies $\bigcup\mathcal{D} \in \mathcal{L}$.

Then $\langle \mathcal{L}; \subseteq \rangle$ is a compactly generated lattice.

<u>Theorem 3</u>. Every acyclic relation has a maximal extension.

<u>Proof</u>. The union of a directed family of acyclic relations is acyclic. Hence the theorem follows from Zorn's lemma.

<u>Theorem 4</u>. P is a maximal acyclic relation on X iff P is transitive, asymmetric and complete, i.e. a total order. Where a relation is complete if for all distinct $x, y \in X$ either xPy or yPx.

<u>Proof</u>. Suff. is obv. so suppose $\langle\alpha,\beta\rangle \notin P$, then $\exists \langle\beta, \rho_0\rangle, \langle\rho_0, \rho_1\rangle, \ldots, \langle\rho_m, \alpha\rangle \in P$, which we shall denote by $\langle\beta \to \alpha\rangle$, since P is maximal. Again by the maximality of P, $\langle\beta \to \alpha\rangle \in P$ implies that $\langle\beta, \alpha\rangle \in P$. If $\langle\alpha, \beta\rangle, \langle\beta, \gamma\rangle \in P$, then since P is total either $\langle\alpha, \gamma\rangle \in P$ or $\langle\gamma, \alpha\rangle \in P$. But $\langle\gamma, \alpha\rangle \in P$ contradicts the acyclicity of P.

This is a minor modification of the proof of the same proposition for finite A given in Harary [18].

Clearly every maximal acyclic relation is meet irreducible, but there exists meet irreducible elements in $\bar{\mathcal{U}}_X$ which are not maximal. This follows from Theorem 2, since the meet of transitive relations is transitive and there exists intransitive acyclic relations. If R and S are relations on X, then R o S is defined as $\{<x,y> \in X \times X \mid (\exists w \in X)<x,w> \in R$ and $<w,y> \in S\}$.

Theorem 5. $P \in \bar{\mathcal{U}}_X$ is irreducible and not maximal iff $P = Q - \{<\alpha,\beta>\}$, where Q is maximal and $<\alpha,\beta> \in P \circ P$.

Proof. Since P is acyclic it has a maximal extension P', by hypothesis $<\alpha \rightarrow \beta> \in P'$. Hence $<\alpha,\beta> \in P'$, but $P \cup \{<\alpha,\beta>\} = Q$, i.e. $Q \subseteq P'$. Therefore by the maximality of Q, $Q = P'$, which implies that Q is the only acyclic relation which properly contains P. Suppose P is irreducible and not maximal. The P has a maximal extension P'. Suppose P is intransitive, then $<\alpha,\beta>, <\beta,\gamma> \in P$ and $<\alpha,\gamma> \notin P$. But $<\alpha,\gamma> \in P'$, hence let $S = P' - \{<\alpha,\gamma>\}$. Note that $P \subseteq S$ and S acyclic. If $P \subset S$, then define $\mathcal{J} = \{V \mid V = P \cup \{<n,\theta>\}\}$, $<n,\theta> \in P' - P$. Consequently $P = \bigcap \mathcal{J}$, which contradicts the complete meet irreducibility of P. The intransitivity of P follows as a corollary to the next theorem.

A sublattice of $\mathcal{U}_X$ of particular importance is the lattice of transitive acyclic relations or partial orders, which we shall denote $\mathcal{H}_X$. Letting $\bar{\mathcal{H}}_X = \mathcal{H}_X \cup \{X \times X\}$, it is easy to show that $\bar{\mathcal{H}}_X$ is a compactly generated lattice. But in $\bar{\mathcal{H}}_X$ a relation P is meet irreducible iff it is maximal. Where P is maximal iff P is a total order. These observations are immediate consequences of the following theorem of Dushnik and Miller [11].

Theorem 6. Any intersection of total orders on a set X is a partial order and any partial order is the intersection of the total orders containing it.

Corollary 6.1. If $P \in \mathcal{G}_X$ and P is meet irreducible but not maximal, then P is intransitive.

III. Choice Structures and Acyclic Preferences

If P is an acyclic relation over X , xPy is to be interpreted as x is (strictly) preferred to y . Economists usually assume that the element chosen by an individual is maximal with respect to his preference relation P . With each finite nonempty subset B of X we can associate a finite nonempty subset $A \subseteq B$ which is the set of maximal elements with respect to the acyclic relation P . This is a special instance of a choice structure. Choice structures were first defined by Arrow [4].

Let X be a nonempty set, $\mathcal{F}$ a family of nonempty subsets of X ; d a function from $\mathcal{F}$ into the family of nonempty subsets of X , where for every $B \in \mathcal{F}$, $d(B) \subseteq B$. The pair $<\mathcal{F},d>$ is called a choice structure. In particular we will be interested in acyclic choice structures. $<\mathcal{F},d>$ is said to be acyclic if there exists an acyclic relation P on X such that for all $B \in \mathcal{F}$, $d(B) = \{a \in B | (\nexists b \in B) bPa\}$, i.e. the set of maximal elements in B with respect to P . In the voting literature, X is usually finite and $\mathcal{F}$ is the family of nonempty subsets of

X , but in economics where X is infinite and the subsets belonging to $\mathcal{J}$ are infinite, one has to assume some additional structure. If X is a topological space and a P a relation over X , then P is said to be upper semicontinuous if for all $y \in X$, $\{z \in X | yPz\}$ is open.

Theorem 7. If P is an upper semicontinuous acyclic relation over the topological space X and B a compact subset of X . Then P has a maximal element in B .

Proof. Let $P^{-1}(b) = \{a | bPa\}$. If B has no maximal elements with respect to P , then $B \subseteq \bigcup_{b \in B} P^{-1}(b)$. By assumption $P^{-1}(b)$ is open. Hence by the compactness of B , there exists a finite subset $B_f \subseteq B$ such that $B \subseteq \bigcup_{b \in B_f} P^{-1}(b)$. Since P is acyclic, B_f contains a maximal element b_0 . But $b_0 \notin \bigcup_{b \in B_f} P^{-1}(b)$, a contradiction.

Therefore every upper semicontinuous acyclic relation on a topological space X defines a choice structure where the domain $\mathcal{J}$ is the family of compact subsets of X and for all $B \in \mathcal{J}$, $d(B) = \{a \in B | \nexists b \in B) bPa\}$.

It is important to note that there exist choice structures on finite sets X , with domain $\mathcal{J}$ the family of nonempty subsets of X , which are not acyclic choice structures. The following example--due to Plott [26]--is such a choice structure. Let $X = \{a,b,c\}$ and for all proper subsets B of X , let $C(B) = B$. Let $C(X) = \{a\}$. That C is not generated by an acyclic relation is a consequence of the next theorem.

Theorem 8. If $<\mathfrak{F},d>$ is a choice structure over X, where $\mathfrak{F}$ is the family of all nonempty finite subsets of X. Then $<\mathfrak{F},d>$ is an acyclic choice structure iff $<\mathfrak{F},d>$ satisfies the following conditions:

(i) If $E, F \in \mathfrak{F}$ and $E \subseteq F$ then $E \cap d(F) \subseteq d(E)$

(ii) If $B_1, B_2, \ldots, B_n$ a finite family of sets belonging to $\mathfrak{F}$ then $\bigcap_{i=1}^{n} d(B_i) \subseteq d(\bigcup_{i=1}^{n} B_i)$.

Proof. Suppose $<\mathfrak{F},d>$ is acyclic. Then the necessity of (i) and (ii) is obvious. Suppose $<\mathfrak{F},d>$ satisfies (i) and (ii). Define aRb if $a \in d(\{a,b\})$. If $E \in \mathfrak{F}$, let $\tilde{d}(E) = \{a \in E | aRb, \text{ for all } b \in E\}$. Suppose $\alpha \in d(F)$ and $\beta \in F$, then let $E = \{\alpha,\beta\}$. By (ii) $\{\alpha,\beta\} \cap d(F) \subseteq d(\{\alpha,\beta\})$, but $\alpha \in \{\alpha,\beta\} \cap d(F)$. Hence $\alpha \in \tilde{d}(F)$. Suppose $\alpha \in \tilde{d}(F)$ and let $B_\beta = \{\alpha,\beta\}$ for each $\beta \in F$, where $F \in \mathfrak{F}$. Consequently $\alpha \in \bigcap_{\beta \in F} d(B_\beta)$ and $F = \bigcup_{\beta \in F} B_\beta$. This implies by (ii) that $\alpha \in d(F)$.

Therefore $\tilde{d}(F) \subseteq d(F)$, and we have shown that $d = \tilde{d}$. We have defined aRb as $a \in f(\{a,b\})$. Let aPb if $<a,b> \in R$ and $<b,a> \notin R$. Suppose P is cyclic, then there exists $\{a_1, a_2, \ldots, a_n\}$ such that $a_1Pa_2, a_2Pa_3, \ldots, a_{n-1}Pa_n$, and a_nPa_1 . But then $\tilde{d}(\{a_1, a_2, \ldots, a_n\}) = \emptyset$ which contradicts $\tilde{d} = d$. If $E \in \mathfrak{F}$, then $\tilde{d}(E) = \{b \in E | \nexists a \in E)aPb\}$. Hence $<\mathfrak{F},d>$ is an acyclic choice structure.

Another representation problem, of some historical interest, is the existence of utilities for acyclic relations. φ is said to be a <u>utility</u> for P if xPy implies $\varphi(x) > \varphi(y)$, where φ is a real valued function. Every acyclic relation P can be uniquely extended to the smallest partial order, denoted $\tilde{P}$, which contains it. $\tilde{P}$ is just the transitive closure of P . Since any utility for $\tilde{P}$ is a utility for P , conditions on P which are sufficient to guarantee a utility for $\tilde{P}$ suffice for P . For example the proof that every countable acyclic relation has a utility is the same as the proof for countable partial orders. P is said to be <u>0-separable</u> if there exists a countable subset $\{\gamma_i\}_1^\infty$ in X such that for all $\langle\alpha,\beta\rangle \in P$, there exists some γ_j where $\langle\alpha, \gamma_j\rangle \in P$ and $\langle\gamma_j, \beta\rangle \in P$. It is easy to show that if P is 0-separable, then $\tilde{P}$ is 0-separable, hence $\tilde{P}$ has a representation--see Milgram [22]. For more results of this type, see Fishburn [12].

IV. Aggregation of Acyclic Preference Relations

Let A be a nonempty set, to avoid special cases we shall assume that A is denumerably infinite. We will consider three classes of relations or orderings over A . Weak orders, partial orders, and acyclic orders. They will be denoted as $\mathcal{W}$, $\mathcal{P}$, and $\mathcal{A}$ respectively. Clearly $\mathcal{W} \subsetneq \mathcal{P} \subsetneq \mathcal{A}$. Let I be a nonempty set having at least three elements. We define three kinds of societies: $\mathcal{S}_{\mathcal{W}} = \mathcal{W}^I$, $\mathcal{S}_{\mathcal{P}} = \mathcal{P}^I$, and $\mathcal{S}_{\mathcal{A}} = \mathcal{A}^I$. Where for nonempty sets C and B , $C^B = \{g|g : B \to C\}$. The intended interpretations of A and I are the set of social alternatives and the set of individuals in society, respectively.

Throughout we will ignore the strategic aspects of voting and assume that individuals vote their true preferences. The problem of social choice or aggregation is to show the existence of functions which we shall call social choice functions, say from $\mathcal{S}_{\mathcal{W}}$ into $\mathcal{W}$, which have certain desirable ethical and institutional properties. In his classic, <u>Social Choice and Individual Values</u>, Arrow proposed the following necessary conditions for a social choice function φ .

(i) *Individual rationality*: Domain of φ is $\mathcal{S}_{\mathcal{W}}$.

(ii) *Collective rationality*: Range of φ is $\mathcal{W}$.

(iii) *Pareto Optimality*: If for all $i \in I$, xP_iy then xP_sy . Where P_i is the i^{th} individual's (strict) preference relation and P_s is society's (strict) preference relation.

(iv) *Independence of Irrelevant Alternatives*: The social relation for any pair of alternatives x and y depends only on the individuals' preferences between x and y .

Arrow's Possibility Theorem is that if φ is a social choice function satisfying these four conditions, then there exists an individual i_0 such that $xP_{i_0}y$ implies xP_sy . That is, the social choice function is dictatorial.

Sen [30] allowed the range of φ to be $\mathcal{P}$ instead of $\mathcal{W}$. He then demonstrated that the relation of Pareto dominance or the unanimity voting rule was a nondictatorial social choice function satisfying the remainder of Arrow's conditions. In a later paper [31], Sen announced a result of A. Gibbard, which stated that in any society whose social choice function satisfied all of Sen's conditions there exists an oligarchy. Where an oligarchy is a set of individuals who if they unanimously (strictly) prefer α to β then society (strictly) prefers α to β , and if one individual in the oligarchy (strictly) prefers α to β then society does not (strictly) prefer β to α . Gibbard's Theorem is proven in Fishburn [15].

Implicit in the results quoted thus far is the assumption that I is finite. A fact first explicitly pointed out by Fishburn [14] who constructed a non-dictatorial social choice function satisfying all of Arrow's conditions for any society having an infinite number of individuals. In

Arrow's proof of the Possibility Theorem he defined the notion of a decisive set of individuals. A set of individuals, J, is decisive for a given social choice function if for all $i \in J$, xP_iy implies xP_sy, regardless of the preferences of the rest of society. In the case of a society of three people where the social choice function is majority rule, then any group having more than one individual is decisive. The interesting aspect of Fishburn's example was not that he had managed to avoid dictatorship by positing an infinite society, but that the family of decisive sets for his social choice function had a very well defined internal structure. They were, in fact, a free ultrafilter over the set of individuals.

Let I be a nonempty set and $\mathfrak{F}$ a family of subsets of I. $\mathfrak{F}$ is a filter if (i) $I \in \mathfrak{F}$, (ii) $A \in \mathfrak{F}$, $A \subseteq B$ implies $B \in \mathfrak{F}$, (iii) $A, B \in \mathfrak{F}$ implies $A \cap B \in \mathfrak{F}$, (iv) $\emptyset \notin \mathfrak{F}$. An ultrafilter, $\mathfrak{F}$, is a filter such that each subset of I or its complement belong to $\mathfrak{F}$. A free ultrafilter is an ultrafilter where the intersection of all the sets belonging to the ultrafilter is the empty set. That free ultrafilters exist only on infinite sets is well known. Fishburn simply took a free ultrafilter $\mathfrak{F}$ over the infinite set of individuals and defined the social relation: xP_sy iff $\{i \in I | xP_iy\} \in \mathfrak{F}$. In a sense, Fishburn picked his decisive sets first and used them to define his social choice function. Although free ultrafilters do not exist on finite sets, ultrafilters do exist, which are called fixed. A filter on a finite set is an ultrafilter iff the intersection of all the sets in the filter consists of a single element.

These observations suggest that a coherent treatment of the work of Arrow, Sen, and Fishburn might be given within the theory of filters. Hansson [17] provides an extremely elegant treatment in his recent paper.

Hansson's major theorems are: (1) the decisive sets of any social choice function satisfying all of Arrow's conditions form an ultrafilter; (2) the decisive sets of any social choice function satisfying all of Sen's conditions form a filter.

Sen [32] recognized that requiring the social preference relation be a partial order was a sufficient but not a necessary condition for the existence of social equilibria, i.e. maximal elements of the social preference relation, in any finite set of social alternatives. As noted earlier, the necessary and sufficient condition is that P_s be acyclic. In order to extend Hansson's analysis to acyclic social choice, we shall have to consider families of decisive sets which are not, in general, filters.

Let I be a nonempty set and $\mathfrak{F}$ a family of subsets of I. $\mathfrak{F}$ is a <u>prefilter</u> if (i) $I \in \mathfrak{F}$, (ii) $A \in \mathfrak{F}$, $A \subseteq B$ implies $B \in \mathfrak{F}$, (iii) Every finite family of sets in $\mathfrak{F}$ has nonempty intersection. For a finite set I, (iii) is equivalent to assuming that there is some individual who belongs to every set in $\mathfrak{F}$.

In a society of three people $\{a,b,c\}$, the collection of sets $\{a,b\}$, $\{a,c\}$, $\{a,b,c\}$ is a prefilter which is not a filter. But, of course, every filter is a prefilter. Given a prefilter $\mathfrak{F}$, following Fishburn, we define the social relation: xP_sy iff $\{i \in I | xP_iy\} \in \mathfrak{F}$. In the example above, society (strictly) prefers x to y if individual a and at least one other individual (strictly) prefer x to y. In the first part of the paper, we suggested that individual preference relations need only be acyclic. Hence we will weaken both Arrows individual and collective rationality conditions to acyclity. These together with the remainder of Arrow's conditions will be called the "$\mathfrak{a}$" conditions.

<u>Theorem 9</u>. If $\mathfrak{F}$ is a prefilter over I, then the social choice function defined as xP_sy iff $\{i \in I|xP_iy\} \in \mathfrak{F}$ satisfies the "$\mathcal{a}$" conditions.

<u>Proof</u>. We need only check that P_s is acyclic, the other conditions are obvious. Suppose P_s is cyclic, i.e. there exists $\{a_1, a_2, \ldots, a_n\}$ such that $a_1P_sa_2$, $a_2P_sa_3$, ..., $a_{n-1}P_sa_n$, and $a_nP_sa_1$. Then there exists sets $E_1, E_2, \ldots, E_n$ in $\mathfrak{F}$ such that for all $i \in E_j$, $a_jP_ia_{j+1}$ for $j = 1, 2, \ldots, n-1$ and for all $i \in E_n$, $a_nP_ia_1$. Since $\mathfrak{F}$ is a prefilter there exists $i_0 \in \bigcap_{i=1}^{n} E_i$, hence P_{i_0} is cyclic which contradicts the hypothesis that every individual's preference relation is acyclic.

<u>Theorem 10</u>. If φ is a social choice function satisfying the "$\mathcal{a}$" conditions then the family of decisive sets form a prefilter.

<u>Proof</u>. We need only check the intersection property. Let $\mathfrak{F}$ be the family of decisive sets for φ. Suppose there exists $\{E_i\}_{i=1}^{n}$, where each $E_i \in \mathfrak{F}$, such that $\bigcap_{j=1}^{n} E_j = \emptyset$. Define a society where for all $i \in E_j$, $a_jP_ia_{j+1}$ for $j = 1, 2, \ldots, n-1$ and for all $i \in E_n$, $a_nP_ia_1$. All the a_j are assumed to be distinct. Extend the preferences of the individuals belonging to $\bigcup_{j=1}^{n} E_j$ to acyclic preferences. Give the remainder of society arbitrary acyclic preferences. Then this is a society whose individual preference relations are acyclic, but the social preference relation has the cycle $a_1P_sa_2$, $a_2P_sa_3$, ..., $a_{n-1}P_sa_n$, and $a_nP_sa_1$.

In Hansson's demonstration that the decisive sets for a social choice function satisfying Sen's conditions form a filter, the only property of individual's preference relations he used was that (strict) individual preference was a partial order. This allows us to summarize the relationships between aggregation of weak, partial, and acyclic orders in the following manner: If $\mathfrak{F}$ is a prefilter we define the social choice function $\varphi_{\mathfrak{F}}$, where given any society $\{P_i\}_{i \in \mathcal{I}}$, $xP_s y$ iff $\{i \in \mathcal{I} \mid xP_i y\} \in \mathfrak{F}$. We will denote the Pareto optimality condition as " P " and the Independence of irrelevant alternatives as " I ".

<u>Theorem 11</u>.

(i) If $\varphi : \mathcal{S}_{\mathcal{W}} \to \mathcal{W}$ and satisfies conditions " P " and " I ", then the decisive sets of φ are an ultrafilter.

(ii) If $\mathfrak{F}$ is an ultrafilter over I , then $\varphi_{\mathfrak{F}} : \mathcal{S}_{\mathcal{W}} \to \mathcal{W}$ and satisfies conditions " P " and " I ".

(iii) If $\varphi : \mathcal{S}_{\mathcal{P}} \to \mathcal{P}$ and satisfies conditions " P " and " I ", then the decisive sets of φ are a filter.

(iv) If $\mathfrak{F}$ is a filter over I , then $\varphi_{\mathfrak{F}} : \mathcal{S}_{\mathcal{P}} \to \mathcal{P}$ and satisfies conditions " P " and " I ".

(v) If $\varphi : \mathcal{S}_{\mathcal{A}} \to \mathcal{A}$ and satisfies conditions " P " and " I ", the decisive sets of φ are a prefilter.

(vi) If $\mathfrak{F}$ is a prefilter over I , then $\varphi_{\mathfrak{F}} : \mathcal{S}_{\mathcal{A}} \to \mathcal{A}$ and satisfies conditions " P " and " I ".

We should note in passing that prefilters on finite sets, I , aggregate upper semicontinuous relations into upper semicontinuous relations. The generic example of a social choice function which is generated by a prefilter is given by considering a society of k people. Choose m people as the collegium and then pick an n such that $m+n \leq k$. Let the social preference be that α is (strictly) socially preferred to β iff every one in the collegium (strictly) prefers α to β and at least n other individual in society (strictly) prefer α to β. The interesting cases are when both m and n are positive and $m+n < k$.

The social choice functions generated by the prefilters on a set of individuals range from dictatorship to oligarchy. If we consider a nontrivial prefilter, i.e. one which is not a filter, we have those polities which are neither dictatorships or oligarchies. Let us call these polities collegial polities. Certainly the distinguishing feature of collegial polities is the marked asymmetry between a coalition's power to carry a motion and a coalition's power to block a motion. Given any proper prefilter, we define the collegium as the intersection of all the sets in the prefilter. Each member of the collegium has an absolute veto. Individuals outside the collegium have no veto power. A necessary but insufficient condition for a motion to carry is that the members of the collegium must vote in the affirmative, not just concur by abstaining. If this condition is met, then their vote must be ratified by a coalition of individuals, none of whom belongs to the collegium, such that the coalition comprised of the collegium and the given coalition belong to the prefilter.

The requirement that each member of the collegium must vote in the affirmative before a motion can carry seems a little strong. Consider the voting rule used in the Security Council of the United Nations prior to August 31, 1965. The Security Council consisted of eleven members; each of the five permanent members had an absolute veto. On substantive matters, a motion required seven affirmative votes and the concurring votes of the five permanent members. Of course, we wish to know if this voting rule is acyclic. It is not if we only require individuals' preferences to be acyclic. But, if we assume that the preference relations of individuals are weak orders. Then it is acyclic.

Theorem 12. Let φ be the Security Council voting rule. Then $\varphi : \mathcal{S}_W \to \mathcal{A}$ and φ satisfies conditions " P " and " I ".

Proof. That φ satisfies conditions " P " and " I " is immediate. Suppose for some society $\{P_i\}_{i=1}^{n}$, the P_s generated by φ has a cycle $\{a_1, a_2, \ldots, a_n\}$. A permanent member voted affirmatively on a_1 versus a_2, say individual 1. For each pair of alternatives (a_j, a_{j+1}) for $j = 2, \ldots, n-1$, either individual 1 (strictly) prefers a_j to a_{j+1} or he is indifferent. In any case, because his preference order is assumed to be a weak order, $a_1 P_1 a_n$. Hence society can not (strictly) prefer a_n to a_1, since individual 1 has an absolute veto and he (strictly) prefers a_1 to a_n.

The fact that φ can cycle if we have acyclic individual preference relations implies that the Security Council is not a collegial polity, oligarchy, or dictatorship. That is, there does not exist a prefilter which will generate φ. Of course the family of decisive sets is a prefilter, this follows from Theorem 10. But the preference relation generated by the prefilter of decisive sets is properly contained in the social preference relation. The difference between the Security Council and the polities generated by prefilters is analogous to the difference between a strict majority voting rule and a majority or plurality voting rule. In a strict majority x defeats y if more than one half of the people vote for x over y. In a majority or plurality voting rule x defeats y if it gets more affirmative votes than y. Again it's a matter of how abstentions are counted. The next theorem gives necessary and sufficient conditions for the (strict) social preference relation to be completely determined by the relation generated from the decisive sets.

Theorem 13. Let $\varphi : \mathcal{A}_{\mathcal{Q}} \to \mathcal{Q}$ and $\mathcal{F}$ the family of decisive sets of φ. Then $\varphi = \varphi_{\mathcal{F}}$ iff φ satisfies the following conditions:

(i) Pareto Optimality

(ii) Independence of Irrelevant Alternatives

(iii) Independence of Irrelevant Individuals: For all alternatives x and y. If $xP_s y$, let $A(x,y) = \{i \in I | xP_i y\}$. Then $xP_s y$ depends only on $A(x,y)$.

(iv) Neutrality: If a set of individuals is decisive for some pair of alternatives then they are decisive for every pair of alternatives.

The Security Council voting rule has the following extension. Suppose the society or group has k people. Choose m people as the collegium and then pick an ℓ such that $0 < k-m \leq \ell < k$. Let the social preference be that α is (strictly) socially preferred to β if ℓ individuals (strictly) prefer α to β and all the members of the collegium concur. That this family of voting rules are acyclic, if individuals' preference orderings are weak orders, follows immediately from Theorem 12. These rules and the generic example of a collegial polity are uniquely characterized by the properties of Pareto optimality, Independence of irrelevant alternatives, Anonymity among the collegium, Anonymity among the individuals outside the collegium, and Neutrality. Anonymity with respect to set of individuals means that the voting rule is invariant under permutations of members of this set.

One would like to characterize, in some sense, all the collegial polities which come closest to satisfying all of Arrow's conditions. The next three theorems are one way of formulating this problem. Let I be a finite nonempty set, Λ the family of prefilters over I, and $\bar{\Lambda} = \Lambda \cup \{\mathcal{P}(I)\}$. Where $\mathcal{P}(I)$ is the power set of I.

<u>Theorem 14</u>. $\bar{\Lambda}$ ordered under set inclusion is a compactly generated lattice.

<u>Proof</u>. It is clear that $\bar{\Lambda}$ satisfies the conditions of Theorem 2.

<u>Theorem 15</u>. A necessary condition that a prefilter $\mathfrak{F}$ be meet irreducible in the lattice $\langle\bar{\Lambda}, \subseteq\rangle$ is that $\cap\mathfrak{F} = \{i_0\}$ for some $i_0 \in I$.

<u>Proof</u>. Suppose $\cap\mathfrak{F}$ contains at least two elements $\{i_1, i_2\}$. Let $A = \{i_1\}$. Let $\mathfrak{F}'$ be the smallest prefilter containing $\mathfrak{F} \cup \{A\}$ and $\mathfrak{F}''$ the smallest prefilter containing $\mathfrak{F} \cup \{A^c\}$. $\mathfrak{F}'$ and $\mathfrak{F}''$ exist since both $\mathfrak{F} \cup \{A\}$ and $\mathfrak{F} \cup \{A^c\}$ have the finite intersection property. Clearly $\mathfrak{F} \subseteq \mathfrak{F}' \cap \mathfrak{F}''$. Suppose $\mathfrak{F}' \cap \mathfrak{F}''/\mathfrak{F} \neq \emptyset$. If $B \in \mathfrak{F}' \cap \mathfrak{F}''/\mathfrak{F}$, then $B \supseteq A \cup A^c = I$. Hence $B \in \mathfrak{F}$ a contradiction. Therefore $\mathfrak{F} = \mathfrak{F}' \cap \mathfrak{F}''$ and $\mathfrak{F}$ is not meet irreducible.

<u>Theorem 16</u>. If $\mathfrak{F}$ is an ultrafilter or $\mathfrak{F} = \mathfrak{U}/\cap\mathfrak{U}$ where $\mathfrak{U}$ is an ultrafilter then $\mathfrak{F}$ is meet irreducible.

These theorems suggest that the collegial polities defined by the meet irreducible prefilters are the 'best' approximations to nondictatorial social choice functions satisfying Arrow's conditions. For example the one given in Theorem 16 is of the form where one individual is chosen as the collegium and society (strictly) prefers α to β if he (strictly) prefers α to β and at least one other individual in society (strictly) prefers α to β.

In the normative literature of mathematical political theory, Buchanan and Tullock [8] have argued that among the linear voting rules unanimity "minimizes" the asymmetry between the power to enact and the power to veto. Their argument has been challenged by Baumol [6] who asserts that majority rule "minimizes" the difference between these two powers. Both of these analyses are heuristic, but they do suggest that some quantitative measure of an individual's power in a polity might provide a means of ethically discriminating between collegial polities.

At least two methods of measuring individual political power strike me as appealing. One is the method of Rae[27] and Taylor [35], they suggest that for a given class of voting rules--say collegial polities--we compute the _a priori_ probability that an individual supports a proposal which is rejected and opposes a proposal which is adopted. Under this measure, the best voting rule is that which minimizes this probability. In a much quoted article [33], Shapley and Shubik have proposed that the _a priori_ probability that the individual is critical to the success of a winning coalition be used as an index of power. They in fact made such a calculation for the Security Council. It would be interesting to compute Rae-Taylor and Shapley-Shubik indices for the collegial polities on societies of different sizes.

In the descriptive literature of political theory, Dahl and Lindbloom [10] have been concerned with what they call the conflict between the tendency in small groups away from equality of control to inequality of control and the tendency away from purely unilateral control. The first tendency they call the "iron law" of oligarchy and the second tendency they call the "law" of reciprocity. The "iron law" of oligarchy is due to Roberto Michels, he along with Mosca and Pareto developed the theory of political elites. Their central tenent is that all governments are oligarchies, i.e. government is always by a few for the interests of a few. In particular, they argue that democracy is a fraud. An excellent summary of the theory of political elites appears in Runciman [28].

We do not suggest that the Sen-Gibbard theorem is a "proof" of the theory of elites, but point out that in political theory the suggested checks against the tyranny of a few are not only within the polity. For example presidential veto. But also in the manner in which the polities are chosen. A serious defect of our approach is that it does not include how people choose the members of the polity, what are the terms of office, how are they recalled, etc.

The "law" of reciprocity, due to Dahl and Lindbloom [10], is simply the observation that in small groups leaders can only lead if they are supported by the group. The leader must be responsive to his followers. Of course, Dahl and Lindbloom, are concerned with an informal and ill defined ratification of a leader's choices. When they apply their law to governments, they suggest as does Runciman that the mechanics of reciprocity are open competitive political elections. Both mention in passing that the 'checks and balances' of traditional constitutional theorists is one way of limiting oligarchical rule, hence supporting reciprocity.

Collegial polities are distinguished from the oligarchies of Sen principally because of the asymmetry between the power to enact and the power to veto. The necessary ratification of the collegium's preferences act as a check against oligarchy and the absolute vetos of the collegium act as a check against Madison's "tyrannical majority."

Collegial polities most commonly occur in defining the relationship between the executive and the legislature in the passing of laws. Clearly in any country where the executive has an absolute veto over the bills passed by the legislature, we have a collegial polity, i.e., the executive belongs to every decisive set. For example, the parliamentary governments of Spain and Ghana. Those governments, like the United States, where the executive veto can be over-ridden are not acyclic, hence not collegial polities.

The cabinet government affords us with two examples of collegial polities. By a cabinet government we mean a parliamentary form of government where the cabinet is chosen either by the parliament or head of state. In the cabinet, we have the executive or prime minister. One collegial polity is defined by the voting rule that a motion passes if and only if it receives the support of the prime minister, some fixed fraction of the cabinet--say a third--and a majority of parliament. Another collegial polity is defined by the rule that a law passes if it is supported by the prime minister and a majority of parliament. In the second case the members of the cabinet are merely administrators and informal advisors. Crick [9] has argued that the British parliamentary system has changed from the strong cabinet government of our first example to a parliament dominated by the prime minister as in our second example.

The British form of a government provides a novel solution to the problem of social incomparability arising out of using a collegial polity. On issues basic to the government's program, either the prime minister is supported by a majority of Parliament or new elections are called. That is, in the event of a social incomparability, produced by an irresolvable conflict between the executive and the legislature, a new collegial polity is chosen. This is similar to impaneling a new jury in the event of a hung jury.

Finally, the Roman Republic used the power of veto along with separation of powers as their primary check on abusive political power. We mention one such check which is discussed in Abbott [1]. The Roman senate was attended by ten tribunes and three hundred senators. Each tribune had an absolute veto and a motion required a majority vote of the senators. Note this voting rule differs from that in the Security Council of the United Nations. In both cases concurring votes of the five permanent members and the tribunes are necessary conditions for a motion to pass. But affirmative votes of the tribunes did not count for passage.

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