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FIAT MONEY IN AN ECONOMY WITH ONE NONDURABLE GOOD AND NO CREDIT

(A NONCOOPERATIVE SEQUENTIAL GAME)

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1. Introduction

Our purpose is to provide satisfactory connections between current theories in macroeconomics and microeconomics. This study differs from the majority of past efforts in this direction because in a microeconomic context it focuses on money, credit, and financial institutions. From the point of view of monetary theory, the novelty lies in our efforts to connect individual economic behavior with the behavior of the economy as a whole and to apply mathematical models for this purpose. From the point of view of the more closely related mathematical literature on n-person games and competitive equilibrium analysis, the novelty lies in our efforts to explicitly consider money, credit, and financial institutions. We believe that an explicit treatment of money, credit, and financial institutions is necessary for forging links between macroeconomics and microeconomics.

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The study of money and credit poses many different problems. These concern information, uncertainty, risk, trust, convenience of exchange, and so forth, cf. [12]. Our object is to construct and analyze mathematical models which will yield adequate theories to cover these different features. We believe that in appropriate mathematical models the prototypes of institutions familiar to modern economics will naturally emerge. Thus we claim that although particular institutions such as commercial, central, and investment banks, insurance companies, loan societies, and stock markets may reflect particular institutional details pertaining to specific societies, the essential functions that these institutions perform call for the existence of entities to perform these functions in any complex economy. This premise has a significant bearing on our modelling approach. Instead of incorporating as many as possible financial institutions into our models at the outset, we want to see how each of these institutions arises out of necessity. We thus begin with an overly-simplified model, not only because it is easier to analyze, but because we want to explicitly represent economic behavior when important features are missing. We want to establish the fundamental need for the various financial institutions. Furthermore, we are thus able to evaluate the possible forms these institutions can take. For example, in [11] the notion of an optimal bankruptcy law is introduced. (See also [10].)

Having set forth our general objectives, we must say that the results to be reported here constitute only a first step in the overall program. We consider a highly-simplified model which only addresses a few of the problems posed by money and credit. While it is our intent to extend the analysis to more sophisticated models in the future, we believe it is important to stress the virtues of simple models. Among these virtues are:

(i) The simplicity enables us to obtain a rather complete mathematical solution.

(ii) The effect of missing features can be observed in the solution.

(iii) By considering only one or two of the features, the effects of different features are isolated and, hopefully, better understood.

We now give a brief overview. We begin by regarding our simplified economy as a deterministic noncooperative sequential game. The modelling is discussed in Section 2 so we shall not dwell on it here. As a consequence of viewing the economy as a noncooperative sequential game, we confront a system of interdependent dynamic programs, cf. (2.2). We first investigate the case of 2 players and n periods. While we could start by normalizing our game, that is, we could convert it immediately into an equivalent static game, it does not appear to be fruitful to do so. For example, our game is never constant-sum, and even if we converted it to constant-sum, which is possible in some cases, the number of strategies is infinite. Furthermore, the normalization seems to destroy the nice properties in the payoff function. Hence, instead of normalizing, we apply the standard backward recursion of dynamic programming. In this way, we verify under fairly general conditions the existence of a unique noncooperative equilibrium solution. We also describe this solution in detail. These answers were discovered by rather painful analysis of the four-period problem, but of course the proofs here are by induction.

We next consider infinite-horizon extensions of our two-player sequential game. It is not difficult to see that with discounting the infinite-horizon version possesses all the properties of the finite-horizon versions. In fact, the infinite-horizon solution coincides with the n-period solutions for all n sufficiently large because then the (n+1)-period and n-period solutions agree. These answers are obtained by direct argument employing our finite-horizon solutions. Again, the existing (sequential) game theory literature does not appear to be very helpful.

We next extend our model by adding more players. Of course, this is the way in which we intend to relate micro and macro theory. It turns out that the analysis of even a 2-period game for 3-5 players is quite complicated, but great simplicity is achieved when many players are present. If the number of players is sufficiently large with each player sufficiently small in relation to the economy as a whole, then we verify the existence of a unique noncooperative equilibrium solution which has a very simple form. In this solution each player uses his myopic or one-period optimal strategy which dictates spending all his available money each period. It is significant that the nonuniqueness possibilities for two players disappear in the large economies. Furthermore, the complex dynamics in which players exploit their money advantage over several time periods also disappears.

For comparison, we conclude by investigating the set of constrained competitive equilibria. We show for economies of any size that there is always a unique competitive equilibrium solution. This solution also involves each player spending all his money in each period. Thus, for sufficiently large economies, the set of noncooperative equilibrium solutions coincides with the set of competitive equilibrium solutions. This is to be distinguished from recent related work by Aumann [2], Brown and Robinson [3], Hildenbrand [6], [7], and others involving the core and the set of competitive equilibria because here "large" means finite instead of infinite. Our two solution concepts coincide in large finite economies. Hence, there is no need to introduce infinite spaces of players and there is no need to prove a limit theorem (which is not to say that these devices are not very useful in other contexts).

We now turn to Section 2 for a definition of our model. The results appear in Section 3 and the proofs appear in Sections 4-6.

2. A Simple Money Game

2.1. Goods, Money, and the System of Exchange

We shall study the distribution of real goods and money over time in a fiat money economy. We assume there is a single nondurable good which goes on the market in constant quantity each of a finite or denumerably infinite number of time periods. (In [12] a model was studied in which players could choose to keep goods off the market, thus causing only part of the real goods to be monetized. This will not be the case here.) Each player owns a fraction of this market or, equivalently, each player brings this fraction of the real good to the market. We keep these fractions of ownership fixed over time. Each period all of the commodity on the market is distributed to the players for consumption. By "nondurable," we mean that all of the real good available each period is consumed during that period. Nondurable real goods can not be inventoried.

We also assume the existence of an "institutional stuff" called money whose quantity is fixed. It is neither created nor destroyed over time. Furthermore, each period all the money is initially in the hands of the players. The important point is that we prohibit barter. All real goods must be purchased at the market with money. This money may be thought of as having a physical existence such as poker chips or green pieces of paper; or it may be thought of as a set of accounting numbers whose ownership may be transferred. Its value is established by fiat; i.e., by our assumption here that goods can only be obtained in exchange for money.

We can now describe the exchange over time. Each player begins with certain fractions of the total money supply and of the total ownership of the market. Each period each player must make just one decision: how much money to spend to purchase the real good that period. We assume that the real good is distributed to the players in proportion to the money spent. Furthermore, the money taken in by the market is given back to the players in proportion to their ownership. A player's money at the end of the period thus equals the money he had left over after spending plus his share of the market take. This model has interest because of each player's conflicting desires: to spend more now to get more real goods now or to spend less now to get more real goods later.

2.2. Credit

We distinguish sharply between money and credit. In this initial model no credit is granted. In particular, this means that there is no market for current money in exchange for claims on future money or goods. In an economy which has neither credit nor barter all exchanges are in the form of a payment of current money for real goods. Obviously, this should make money play a more prominent role than it otherwise would, and this is confirmed by our results. First, players starting with a fraction of money greater than their fraction of ownership are often able to reap a significant advantage in real goods over time which would not be possible with credit. Second, the noncooperative equilibrium solution and the prospective competitive equilibrium allocations turn out not to be Pareto optimal if players have different time discounts. Without credit, the model has difficulty responding to time preferences for goods. Thus, when money is introduced without credit, there exist motivations for introducing credit.

2.3. Individual Preferences

We assume that individual i has preferences which can be represented by a utility function of the form

$$U_i(\underset{\sim}{x}) = \sum_{k=0}^{k=n-1} \beta_i^k \varphi_i(x_k) , \tag{2.1}$$

where n $(1 \leq n \leq \infty)$ is the number of periods, $\underset{\sim}{x} = (x_1, \ldots, x_n)$ is the vector of real goods to be received (and consumed) in successive periods, β_i is the discount factor for individual i, $0 < \beta_i \leq 1$, k in β_i^k is an exponent instead of a superscript, and φ_i is the one-period utility function. For the most part, we assume the one-period utility function φ_i is of the special form $\varphi_i(x) = x$, but the principal results for the case of many players hold for quite general φ_i. The important point is that money does not appear in (2.1). Money is valued only as a means to obtain more real goods.

2.4. Solution Concepts

2.4.1. Noncooperative Game

For the most part, we shall view our market as a noncooperative game. Therefore, our object is to identify the set of noncooperative equilibrium state strategy solutions. Such a solution consists of an n-period strategy for each player with the property that no individual acting alone can improve his position by altering his strategy. For specified initial conditions, an equilibrium point is determined by an appropriate schedule of spending for all the players. It is important to distinguish between equilibrium in this game theoretic sense and equilibrium as it may exist in the dynamics of a particular schedule of spending. For example, within our equilibrium solution the situation in which each individual's fraction of money equals his fraction of ownership is a stationary state or an equilibrium point. In other words, if at any period this situation prevails, then it will prevail in every period thereafter if we follow the schedule of spending associated with the noncooperative equilibrium solution.

Since we are viewing the market as an m-person n-period noncooperative game in which the utility functions are separable (as indicated in (2.1)), the object is to identify the noncooperative solutions to a system of m simultaneous n-period dynamic programs. In particular, individual i has a payoff function which can be defined recursively as

$$(2.2)\quad U_n^i(p_i, \gamma_i) = \max_{0 \le x_i \le p_i + \gamma_i} \{\varphi_i[K^i(x_1, \dots, x_m)] + \beta_i U_{n-1}^i(p_i, \gamma_i - x_i + p_i(\sum_{j=1}^{m} x_j)\},$$

where

$$K^i(x_1, \dots, x_m) = \begin{cases} \dfrac{x_i G}{\sum_{j=1}^{m} x_j}, & x_j > 0 \text{ for some } j, \\ 0, & x_1 = \dots = x_m = 0, \end{cases}$$

p_i is i's fraction of ownership, $0 < p_i < 1$,

$p_i + \gamma_i$ is i's fraction of money, $-p_i < \gamma_i < 1 - p_i$,

m is the number of players,

n is the number of periods remaining,

x_j is the amount of money j spends in the first of n periods,

φ_i is i's one-period utility function,

G is the total amount of the real good to be distributed,

and β_i is i's discount factor, $0 \leq \beta_i \leq 1$.

Of course, U_n^i depends not only on p_i and γ_i but also on all the other variables, especially x_j for $j \neq i$. Note that n indicates the number of periods remaining. The representation in (2.2) is fine as long as $n < \infty$ but would have to be altered with an infinite horizon. With discounting, we would then naturally count time forwards and obtain instead of (2.2)

$$(2.3) \qquad U^i(p_i, \gamma_i) = \max_{\{x_{ik}, k \geq 1\}} \sum_{k=1}^{\infty} \beta_i^{k-1} \varphi_i[K^i(x_{1k}, \ldots, x_{mk})]$$

such that

$$x_{ik} \leq p_i + \gamma_{ik}$$

$$\gamma_{i(k+1)} = \gamma_{ik} - x_{ik} + p_i \sum_{j=1}^{m} x_{jk} \ .$$

If we restrict attention to stationary strategies, then we consider the functional equation in (2.2) where both U_n^i and U_{n-1}^i are placed with U^i . Finally, if there is no discounting in the infinite horizon, then an entirely different criterion is needed, e.g., average return per period.

2.4.2. State Strategies

In terms of general game theoretic considerations, a strategy is a complete plan of play which might depend in detail upon all aspects of previous history. In particular, a strategy may depend delicately upon information conditions. In oligopolistic financial markets there is ample evidence that information conditions do play a vital role in determining strategies. However, in the models explored here a considerable limitation and simplification of information conditions is assumed. Furthermore, we limit ourselves to an extremely special set of overall strategies which can best be described as simple state strategies where an individual's behavior depends only upon the state and period that he is in and not upon the history of how he arrived in that state. In this particular model the restriction to state strategies does not appear to be particularly binding, but for us it remains an assumption. We start by assuming that the dynamic programming recursion is justified.

2.4.3. Competitive Equilibrium

More deeply rooted in economics is the concept of a competitive equilibrium. A competitive equilibrium is a set of prices (one price per period here) and a set of allocations (of real goods to each player each period) such that for all i the allocations to individual i are optimal for him alone subject to all budget constraints being met and the outcome being Pareto optimal. Pareto optimality means the players cannot all simultaneously do better by choosing different strategies.

In the customary definition of the competitive equilibrium, budget constraints (i.e., limits on an individual's purchasing power in terms of the net worth of an individual's ownership evaluating the goods at the given market prices) but not cash flow constraints are active. This is as though either perfect trust exists for the whole trading period or, equivalently, credit is freely available. We consider a modified competitive equilibrium where the individuals have money and its amount is fixed (there is no credit) and trade is carried out in money. In this model the cash flow constraints are of importance.

As before, neither the amount of money in the system nor the amount of real goods put on the market each period changes from period to period, so the prices must be chosen accordingly. The concept of competitive equilibrium is primarily intended for perfect competition where the influence of each individual relative to the market as a whole is negligible. In such a situation it is reasonable to consider the economic behavior in terms of many isolated or decentralized maximization problems taking place simultaneously with each individual ignoring his influence on the market. While the notion of competitive equilibrium is most meaningful when associated with perfect competition including no credit constraints, we can nevertheless identify the set of competitive equilibria given cash flow limitations. In the notation of (2.2) individual i is confronted with the optimization problem:

$$\max_{\{x_{ik}\}} \sum_{k=0}^{n-1} \beta^k \varphi_i(x_{ik}) \tag{2.4}$$

subject to: $x_{ik}a_k \leq M_k^i$, $1 \leq k \leq n$,

$$M_{k+1}^i = M_k^i - x_{ik}a_k + p_i G a_k \ , \quad 1 \leq k \leq n-1 \ ,$$

$$M_1^i = (p_i + \gamma_i)M \ ,$$

$$\sum_{i=1}^{m} M_k^i = M \ , \quad 1 \leq k \leq n \ ,$$

and

$$\sum_{i=1}^{m} x_{ik} = G \ , \quad 1 \leq k \leq n \ ,$$

where M_k^i is the money i has available in period k ,

a_k is the price in period k ,

and M is the total money in the system.

We shall be interested in comparing the set of (constrained) competitive equilibria with the set of noncooperative equilibria.

2.4.4. Other Solutions

There are many other solution concepts which we could consider but which we will not. Among these are the cooperative game theory solution concepts such as the core, value, nucleolus, and bargaining set. A rather different approach would be to use a behavioral model in which individuals are assumed to use heuristics or limited optimizations in order to make their decisions. For example, the individuals might respond adaptively over time or spend a random amount each period. Behavioral models differ from more general optimization problems in degree rather in kind, but they are usually characterized by having a relatively simple mechanism which produces decisions. In terms of the mathematical analysis, the problem is not one of complicated optimization to yield decisions, but one of describing the evolution of the system when the decision-generation mechanism is specified. An example of the behavioral approach to an optimization model is the technique of stationary analysis in inventory theory, cf. Part IV of [1]. Behavioral models have considerable appeal for representing actual human behavior because of limitations in human information processing ability. However, it is difficult to avoid ad hoc modelling. Thus, we will not consider this approach at this time.

3. Results

3.1. Noncooperative Game with Two Players and Finite Horizon

Consider the system of dynamic programs in (2.2) with the additional assumptions that $m = 2$, $1 \leq n < \infty$, and $\varphi_i(x) = x$, $x \geq 0$, $i = 1, 2$. Since the one-period payoff function is now homogeneous of degree zero, we can let $G = 1$ without loss of generality. For notational simplicity, we drop the subscripts on p_1 and γ_1 and use $1-p$ and $-\gamma$ for p_2 and γ_2. Furthermore, we stipulate that $\gamma \geq 0$ so that the first player always begins with at least as much money as ownership. It turns out that the money advantage γ rather than the initial money supply $p+\gamma$ determines the strategic character of a player. We thus refer to the first player as the strong player or just I and the second player as the weak player or just II.

To place our game in perspective, first note that it is not constant-sum. It almost is when $\beta_1 = \beta_2$, but it is not because $K(x_1, x_2) = 0$ for $x_1 = x_2 = 0$. If we let $K(0,0) = 1/2$, then we would have a constant-sum game when $\beta_1 = \beta_2$. Even in this special case it does not seem possible to deduce much directly from the existing game theory literature. However, we are able to draw some interesting conclusions from a straightforward approach.

Theorem 1. There exists a noncooperative equilibrium solution.

Theorem 2. There is only one solution whenever one of the following conditions holds:

(i) $\beta_2 \leq \beta_1$;

(ii) <u>there are only two periods</u>;

(iii) $\beta_1 \leq \frac{(1+p)(1-p-\gamma)}{(1-p)}$

<u>and</u> $\beta_2 \leq p+\gamma$;

(iv) $\beta_2 pA + \beta_1(1-p) \leq 1$, <u>where</u>

$$A = n^{-1} \sum_{j=0}^{n-1} (\beta_2/\beta_1)$$

<u>and there are</u> n <u>periods</u>.

<u>Remark</u>. More refined conditions for uniqueness when $\beta_2 > \beta_1$ are still needed. It is evident that uniqueness holds when $\beta_2 > \beta_1$ in many other situations besides the ones we mentioned. However, it does not always hold. We give a counterexample involving a three-period game in Case 4 of the proof of Theorem 2 in Section 4.

The following theorems and corollaries describe a (often the) non-cooperative equilibrium solution in more detail.

<u>Theorem 3</u>. <u>The second player always spends all his money each period</u>.

<u>Remark</u>. Theorem 3 depends critically on the discount factors β_1 and β_2 being less than or equal to one. It is rather contrary to intuition when $\beta_2 > \beta_1$. Evidently the relatively unfavorable initial money distributions still gives II a solution at his boundary. Note, however, that it is precisely when $\beta_2 > \beta_1$ that uniqueness can fail.

Theorem 4. *If* $A(n)$ *represents the first period in which* I *spends all his money, then*

$$(3.1)\quad A(n) = \begin{cases} \max\{k : 1 \le k \le n\ ,\ \beta_1^{\frac{k(k-1)}{2}} > c\}\ , & \beta_1 > c \text{ and } n \ge 2\ , \\ 1 & ,\ \beta_1 \le c \text{ or } n = 1\ , \end{cases}$$

where

$$(3.2)\qquad c = \frac{1-p-\gamma}{1-p}\ .$$

Corollary 5.

(a) *If* $\beta_1 = 1$ *and* $\gamma > 0$, *then* I *never spends all his money until the last period*.

(b) *If* $\beta_1 < 1$, *then* I *spends all before the last period if the horizon is sufficiently long*.

(c) *If* $\beta_1 < c$, *then both players spend all their money in the first period*.

(d) *If* $A(n) = k$, *then both players spend all their money each period from* k *to* n.

(e) *If* $A(n) = k < n$, *then* $A(n+m) = k$ *for all* $m \ge 0$.

Remark. As a consequence of Corollary 5, we refer to the state in which $\gamma = 0$ or, equivalently, the state in which both players spend all their money as the equilibrium or stationary state within the noncooperative equilibrium solution.

Theorem 6. If $A(n) = k$, then

(a) in the first period I spends

$$\hat{x}_{11} = c^{\frac{k-1}{k}} \beta_1^{-\frac{(k-1)}{2}} - (1-p-\gamma) ;$$

(b) in period j $(2 \leq j \leq k)$ the total money spent is

$$\hat{x}_{11} + \hat{x}_{21} = c^{\frac{k-j}{k}} \beta_1^{-(k-j)} ;$$

(c) the discounted value of the real good consumed by I in period j is

$$\frac{\beta_1^{j-1}\hat{x}_{1j}}{\hat{x}_{1j} + \hat{x}_{2j}} = \begin{cases} \beta_1^{j-1} - \beta_1^{\frac{k-1}{2}} (1-p)^{\frac{k-1}{k}} (1-p-\gamma)^{\frac{1}{k}} , & j \leq k \\ \beta_1^{j-1} p , & k < j \leq n , \end{cases}$$

(d) the discounted value of the real good consumed by II in period j is

$$\frac{\beta_2^{j-1}\hat{x}_{2j}}{\hat{x}_{1j} + \hat{x}_{2j}} = \begin{cases} \beta_2^{j-1} \beta_1^{\frac{k-2j+1}{2}} (1-p)^{\frac{k-1}{k}} (1-p-\gamma)^{\frac{1}{k}} , & j \leq k \\ \beta_2^{j-1} (1-p) , & k < j \leq n . \end{cases}$$

(e) The n-period utilities $U_n^1(p,\gamma)$ and $U_n^2(1-p, \gamma)$ are continuously differentiable strictly increasing functions of γ, which are convex in $(0, 1-p)$ and $(0,p)$ respectively.

Virtually all aspects of this solution are now easy to describe. We list a few additional properties.

Corollary 7. _If_ $A(n) < n$, _then the (n+m)-period solution coincides with the n-period solution during the first_ n _periods for all_ $m \geq 1$. _At the end of period_ n, $\gamma = 0$, _so that both players spend all thereafter_.

Corollary 8. _If_ $A(n) = k$, _then_ II's _utilities from consumption during the first_ k _periods increases or decreases according to whether_ $\beta_2 > \beta_1$ _or_ $\beta_2 < \beta_1$. _If_ $\beta_2 = \beta_1$, _then_ II's _utility from consumption in period_ j _is_

$$\frac{\beta_1^{(j-1)}\hat{x}_{2j}}{\hat{x}_{1j} + \hat{x}_{2j}} = \begin{cases} \beta_1^{\frac{k-1}{2}} (1-p)^{\frac{k-1}{k}} (1-p-\gamma)^{\frac{1}{k}}, & 1 \leq j \leq k, \\ \beta_1^{j-1}(1-p), & k+1 \leq j \leq n. \end{cases}$$

Remark. Corollary 8 gives an interesting characterization of the solution. If I spends all his money for the first time in period k, then his spending is such that II's discounted utility from consumption would be constant over the first k periods if he used I's discount factor β_1.

Corollary 9. _If_ $A(n) = k$ _and_ $\beta_1 < 1$, _then both_ I's _actual consumption and his utilities decrease from period to period during the first_ k _periods_. _If_ $\beta_1 = 1$, _then_ $A(n) = n$ _and_ I's _actual consumption as well as utility each period is_

$$\frac{\hat{x}_{1j}}{\hat{x}_{1j} + \hat{x}_{2j}} = 1 - (1-p)^{\frac{n-1}{n}} (1-p-\gamma)^{\frac{1}{n}}.$$

Let V_n represent the excess utility I receives beyond what he would receive if he received goods equal to his ownership every period. In other words, let

$$V_n^1(p,\gamma) = U_n^1(p,\gamma) - p\sum_{k=0}^{n-1}\beta_1^k . \tag{3.3}$$

Corollary 10.

(a) *If* $A(n) = k$, *then* $V_j = V_k$ *for* $j \geq k$ *and* $V_{j+1} > V_j$ *for* $1 \leq j \leq k-1$.

(b) *If* $\beta_1 = 1$, *then*

$$V_n(p,\gamma) = n(1-p)(1 - c^{1/n}) ,$$

$$\lim_{n\to\infty} V_n(p,\gamma) = -(1-p)\log\left(1 - \frac{\gamma}{1-p}\right) ,$$

and

$$\lim_{\gamma\uparrow(1-p)}\ \lim_{n\to\infty} V_n(p,\gamma) = \infty ,$$

where c *is defined in* (3.2).

Corollary 11. *Player* II's *amount of money is strictly increasing from* $1-p-\gamma$ *until* $(1-p)$ *is reached. If* $A(n) = k$, *then* II's *amount of money increases from period to period by a factor of* $c^{-1/k}\beta_1^{-1} \geq 1$ *for* $3 \leq j \leq k-1$.

Theorem 12. *If* $\beta_1 = \beta_2$, *then every noncooperative equilibrium solution is Pareto optimal, but if* $\beta_1 \neq \beta_2$, *then none is.*

3.2. Infinite Horizon

We now consider infinite-horizon versions of our two-person sequential game. We still assume that $\varphi_i(x) = x$, $x \geq 0$.

3.2.1. $\beta_i < 1$

If the discount factors are both less than one, then we can use the discounted utility functions in (2.3). If we restrict attention to stationary strategies, then we can use the functional equations in (2.2) with both U_n^i and U_{n-1}^i replaced by U^i , namely,

$$(3.4)\quad U^i(p_i,\gamma_i) = \sup_{0\leq x_i\leq p_i+\gamma_i} \{K^i(x_1,x_2)+\beta_i U^i(p_i,\, \gamma_i - x_i + p_i(x_1+x_2))\} .$$

We use supremum instead of maximum because the maximum might not be attained.

If $\beta_1 = \beta_2$, our game is almost, but not quite, strictly competitive or constant-sum. It is not constant sum because of the one-period distribution of goods at (0,0) spending. If we redefine the one-period distributions at (0,0) to be $K^1(0,0) = K^2(0,0) = 1/2$, then our game would be constant-sum. The resulting model is then similar to the deterministic version of Shapley's [9] stochastic game. It is not quite the same, though, because Shapley only considered finite sets of states and actions. However, it is known that the theory also applies to various infinite sets under additional assumptions. In particular, this is evident from the fundamental paper by Denardo [5]. Another treatment in the spirit of Blackwell [3] has recently been provided by Maitra and Parthasarathy [8]. Unfortunately, none of these papers appears to be directly applicable here, although the flexible framework provided by Denardo [5] is promising. In the more general context of nonconstant-sum sequential games, the recent paper by Sobel [13] is also related but again not directly applicable. The discontinuity of K in (2.2) at $(0,0)$ is the principal source of difficulty.

Although the game theory literature does not appear very helpful, we can easily apply our previous finite-horizon results to this infinite-horizon model. For this purpose, define $A(\infty)$ just as in (3.1). Since $\beta_i < 1$, $A(\infty) < \infty$. On the basis of Corollary 7, it is obvious that the infinite-horizon model has the same properties as the n-period problem for $n > A(\infty)$. Let

$$W_n^i(p_i,\ \gamma_i) = U_n^i(p_i,\ \gamma_i) + p_i \sum_{k=n}^{\infty} \beta_i^k . \tag{3.5}$$

Obviously W_n^i represents the infinite-horizon utility to individual i when both players follow the n-period solution for the first n periods and are in the steady-state thereafter.

<u>Corollary 13</u>. <u>If</u> $A(\infty) = k$ <u>and</u> $n \geq k$, <u>then</u> (W_n^1, W_n^2) <u>defined in</u> (3.5) <u>is a solution to the infinite-horizon functional equations in</u> (3.4). <u>Moreover, the allocations and spending agree with the n-period solution for the first</u> n <u>periods and are in the steady-state thereafter</u>.

It seems that the solution to (3.4) above should be the only one which is bounded below by 0 and above by $(1 - \beta_i)^{-1}$. It also seems that nothing would change if we allowed non-stationary strategies. However, we do not yet have proofs.

3.2.2. $\beta_1 = \beta_2 = 1$

When there is no discounting, the criterion is usually changed to average return per period, but we shall not consider this criterion here. Instead, we shall consider the infinite-horizon functional equations related to the excess utilities V_n^i in (3.3). Corollary 10(b) suggests such a system might have a finite solution. In particular, we shall consider the functional equations

$$(3.6)\quad V^i(p_i,\gamma_i) = \max_{0\le x_i \le p_i+\gamma_i} \{K^i(x_1,x_2) - p_i + V^i(p_i, \gamma - x_i + p(x_1+x_2))\} ,$$

for $i = 1, 2$ and K^i in (2.2).

<u>Theorem 14</u>. <u>The limit in Corollary 10(b) is a solution to</u> (3.6) <u>and the limit in Theorem 6(a) is the amount</u> I <u>spends initially</u>, <u>i.e.</u>,

$$V^1(p,\gamma) = -V^2(1-p-\gamma) = -(1-p)\log\left(1 - \frac{\gamma}{1-p}\right)$$

<u>and</u>

$$\hat{x}_{11} = c - K = \frac{p(1-p-\gamma)}{1-p} .$$

<u>Remarks</u>. Since the proof of Theorem 14 is easy, we omit it. It is easy to see that the same result holds for I if $\beta_2 < 1$. Then β_2 must be included in the functional equation involving V^2 . Of course, II spends all every period. Again we have not yet resolved the issue of uniqueness.

Corollary 15. *The solution above is not realized by* I *following the indicated strategy. Consumption each period equals ownership and* I *carries his extra money* γ *into the future.*

Since the optimal value cannot actually be attained, we look for strategies which can come arbitrarily close.

Corollary 16. *An* ϵ-*optimal (stationary) strategy for* I *can be obtained by using an appropriate optimal finite-horizon strategy.*

3.3. Many Players

We now consider our sequential money game with more than two players. In the beginning we still assume that $\varphi_i(x) = x$, $x \geq 0$, but later we show how this can be generalized.

When there are more than two players, each player can still think of himself being in a two-person game because he can lump all the other players together, but as the number of players increases, the number of cases and the complexity of the analysis increases. Even a two-period version with ten players would present a formidable task. The m-person game does reduce to a two-person game in some special cases however. In particular, this occurs if all the players or all but one player would be at the boundary (spending all their money in the first period) in the associated two-person game in which all other players are lumped together. The two-person results in Sections 3.1 and 3.2 thus immediately imply the following two corollaries.

Corollary 17. (No Big Strong Player) If $\gamma_i \leq 0$ or $\beta_i \leq (1-p_i-\gamma_i)/(1-p_i)$ for each individual i, then everyone spending all their money in every period yields a noncooperative equilibrium solution.

Corollary 18. (One Big Strong Player) If $\gamma_1 > 0$ and $\beta_1 > (1-p_1-\gamma_1)/(1-p_1)$ while either $\gamma_i \leq 0$ or $\beta_i \leq (1-p_i-\gamma_i)/(1-p_i)$ for all $i \geq 2$, then there exists a noncooperative equilibrium solution in which individual 1 follows the two-person strategy for player I in Sections 3.1 and 3.2 and the others spend all each period.

Remark. We have not yet successfully characterized uniqueness in Corollaries 17 and 18, but it is easy to verify for a finite horizon in special cases of interest as we illustrate below.

We now apply Corollary 17 to investigate large economies in a state approaching perfect competition. We obtain a "law of large numbers" comparable to the classical probability theorem with that name. Just as in the probabilistic setting, we need to require not only that the number of individuals be large but also that each one be asymptotically negligible in relation to the whole. Since p_i and $p_i+\gamma_i$ represent the fractions of total ownership and money respectively, it suffices for these to be small.

Theorem 19. (Perfect Competition) If $p_i \leq n^{-1}$, $\gamma_i \leq n^{-1}$, and $\beta_i \leq 1-\delta < 1$ for each individual i, then for sufficiently large n there exists an equilibrium solution in which each individual spends all his money every period. If the horizon is finite, then this is the only solution.

A similar result corresponding to Corollary 18 also holds.

Theorem 20. (One Fat Cat) *Let p_1 and $\gamma_1 > 0$ be independent of n. If $p_i \leq n^{-1}$, $\gamma_i \leq n^{-1}$, and $\beta_i \leq 1-\delta < 1$ for each individual $i \geq 2$, then for sufficiently large n there exists an equilibrium solution in which each individual i, for $i \geq 2$, spends all his money every period. If the horizon is finite, then this is the only solution.*

Remark. Theorems 19 and 20 go beyond Corollaries 17 and 18 by providing uniqueness. Unfortunately, our uniqueness proofs in Section 5 only apply to arbitrary finite horizons. It seems intuitively obvious that uniqueness should also hold for the model with an unbounded horizon but a proof eludes us. Since the actual horizon out there in the world appears to be unbounded, it is natural to question the value of our finite-horizon results. However, our finite-horizon results do have a natural interpretation in the infinite horizon. You can think of each player using a rolling strategy; that is, each period each player looks a specified finite number of periods into the future and selects his strategy assuming the world or his interest in the world terminates at the end of those periods. This process is repeated each period so that the players are always making their decisions based on the present plus a specified number of periods of the future. With the time discounts, each player knows, in our model at least, that the part of the future he is failing to consider is negligible. It is significant that in the setting of Theorems 19 and 20 the solution in succeeding periods after the first is the same (everyone spends all) using a rolling strategy or the second period strategy from a fixed finite-horizon strategy.

We also expect the right argument will yield uniqueness in the infinite horizon, which would imply that the rolling strategy solution coincides with the infinite-horizon solution.

<u>Remark</u>. The results in this section obviously hold for quite general one-period utility functions φ_i , cf. (2.2). When φ_i is changed, we must specify the total amount G of the good to be distributed. It is then natural to let G grow linearly in m as m , the number of players, increases. Suppose that φ_i is twice-differentiable with $\varphi_i' > 0$ and $\varphi_i'' < 0$. Then it is easy to see that Corollary 17 still holds if, for each i ,

$$\frac{\varphi_i'\left(\dfrac{Gx_i}{x_i + 1 - p_i - \gamma_i}\right)}{\varphi_i'(G[p_i + (1 - p_i)(p_i + \gamma_i - x_i)])} \geq \frac{\beta_i(1 - p_i)}{G(1 - p_i - \gamma_i)} \tag{3.7}$$

for $0 \leq x_i \leq p_i + \gamma_i$. To get (3.7), it suffices to have

$$\frac{\varphi_i'(G[p_i + \gamma_i])}{\varphi_i'(Gp_i)} \geq \frac{\beta_i(1 - p_i)}{G(1 - p_i - \gamma_i)} \quad , \tag{3.8}$$

where the right side is less than G^{-1} . The uniqueness in Theorems 19 and 20 also obviously carries over to more general φ_i , but we have no nice conditions.

3.4. Constrained Competitive Equilibria

For comparison, we now investigate the set of competitive equilibria. In particular, assume each player is confronted with the optimization problem in (2.4). We look for a set of prices and allocations such that the allocations are optimal for each individual at those prices, the constraints are met, and the overall solution is Pareto optimal.

Theorem 21. *If* $\beta_i = \beta \leq 1$ *for all* i , *then there is a unique competitive equilibrium solution. The price is* M/G *every period and the players spend all their money every period. If* $\beta_i \neq \beta_j$ *for some* i *and* j , *then Pareto optimality is lost.*

Corollary 22. *Under the conditions of Theorem 19, the set of noncooperative equilibrium solutions coincides with the set of competitive equilibrium solutions for sufficiently large* n .

Remark. Corollary 22 is strictly correct only when $\beta_i = \beta$ for all i , but is true more generally if we relax the requirement of Pareto optimality in the definition of a competitive equilibrium. As we have noted before, the current models without credit do not cope with uneven time preferences for goods. This has been illustrated here with different interest rates. It can also be illustrated by allowing ownership to change over time.

4. Proofs for the Two-Person Finite-Horizon Games

Proof of Theorem 1: Existence

Our proof will be by induction. We will thus want to combine our existence proof (Theorem 1) with the description proofs (Theorems 3, 4, and 6 plus associated corollaries). All the results are trivial for one period since each player is clearly motivated to spend all his money regardless of what the other does. Hence, we shall verify that the solution described in Section 3.1 is in fact a solution for $n+1$ periods assuming that it is for k periods for each k, $1 \leq k \leq n$. To simplify expressions, we shall let $K = (1-p-\gamma)$ and $c = K/(1-p)$.

Case 1: Player I's Optimization when $A(n+1) = k < n+1$

We first look at I, assuming that II spends all his money in the first period as well as every period thereafter. Suppose $A(n+1) < n+1$, where $A(n)$ is defined in (3.1). Player I's optimal value is thus

$$(4.1) \quad U^1_{n+1}(p,\gamma) = \max_{0 \leq x_1 \leq p+\gamma} \left\{ \frac{x_1}{x_1 + K} + \beta_1 U^1_n(p,\ (1-p)(p+\gamma-x_1)) \right\},$$

where $K = 1-p-\gamma$. We will drop the subscript on β_1 since the other discount β_2 will be of no concern to I. Using Theorem 4, Corollary 7, and the induction hypothesis, we have

$$(4.2) \quad U^1_n(p,\ (1-p)(p+\gamma-x_1)) = U^1_{n-1}(p,\ (1-p)(p+\gamma-x_1)) + p\beta^{n-1}$$

if $\beta^{\frac{n(n-1)}{2}} \leq c_n$, where we let $c_n = (1-p-\gamma_n)/(1-p)$ with γ_n denoting I's excess money at the beginning of the second period (with n of $n+1$

periods to go). Here

$$c_n = \frac{1 - p - (1-p)(p+\gamma - x_1)}{1-p} = 1 - p - \gamma + x_1 \ . \tag{4.3}$$

Hence, for $x_1 \geq \beta^{\frac{n(n-1)}{2}} - K$, we can substitute $U^1_{n-1} + p\beta^{n-1}$ for U^1_n inside (4.1) which means $U^1_{n+1}(p,\gamma) = U^1_n(p,\gamma) + \beta^n p$. Then, by Theorem 6(a),

$$\begin{aligned} \hat{x}_{11} &= c^{\frac{k-1}{k}} \beta^{-\frac{(k-1)}{2}} - K \\ &\geq \beta^{\frac{k(k-1)}{2}} - K , \end{aligned} \tag{4.4}$$

so that $\hat{x}_{11} \geq \beta^{\frac{n(n-1)}{2}} - K$ as required. If $k = n$, then we use the hypothesis that $A(n+1) < n+1$ to get $\beta^{\frac{n(n+1)}{2}} \leq c$, which implies that $\hat{x}_{11} \geq \beta^{\frac{n(n-1)}{2}} - K$.

It remains to show that I does not want x_1 for which $\beta^{\frac{n(n-1)}{2}} > c_n$ or, equivalently, for which $x_1 < \beta^{\frac{n(n-1)}{2}} - K$. If $\beta^{\frac{n(n-1)}{2}} > c_n$, then by Theorem 6(c),

$$\begin{aligned} U^1_n(p, \gamma_n) &= U^1_n(p, (1-p)(p+\gamma - x_1)) \\ &= -n\beta^{\frac{n-1}{2}} (1-p)^{\frac{n-1}{n}} (1 - p - \gamma_n)^{\frac{1}{n}} + \sum_{j=0}^{n-1} \beta^j \\ &= -n\beta^{\frac{n-1}{2}} (1-p)(1 - p - \gamma + x_1)^{\frac{1}{n}} + \sum_{j=0}^{n-1} \beta^j \ . \end{aligned} \tag{4.5}$$

Next we differentiate (4.1) using (4.5) to get

$$(4.6) \qquad f'(x_1) = \frac{df(x_1)}{dx_1} = \frac{K}{(x_1 + K)^2} - \beta^{\frac{n+1}{2}}(1-p)(x_1 + K)^{-\frac{(n-1)}{n}}$$

and

$$(4.7) \qquad f''(x_1) = \frac{-2K}{(x_1 + K)^3} + \beta^{\frac{n+1}{2}}(1-p)\left(\frac{n-1}{n}\right)(x_1 + K)^{-\frac{(2n-1)}{n}} ,$$

where $f(x)$ denotes the expression inside the maximum in (4.1) when (4.5) is used. If we set $f'(x_1) = 0$, then we get the unique solution

$$(4.8) \qquad \bar{x}_{11} = c^{\frac{n}{n+1}}\beta^{-\frac{n}{2}} - K .$$

We now must verify that $f''(\bar{x}_{11}) < 0$ to show that (4.8) gives a maximum. This is not immediately obvious because (4.7) contains one positive term as well as one negative term. However,

$$(4.9) \quad f''(\bar{x}_{11}) = (\bar{x}_{11} + K)^{-3}\left[-2K + (1-p)\beta^{\frac{n+1}{2}}\left(\frac{n-1}{n}\right)(\bar{x}_{11} + K)^{\frac{n+1}{n}}\right] ,$$

$$= -(\bar{x}_{11} + K)^{-3}(1-p)^{-1}\beta^{\frac{n+1}{2}}\left[\left(\frac{2n}{n-1}\right)c\beta^{-\frac{(n+1)}{2}} - (\bar{x}_{11} + K)^{\frac{n+1}{n}}\right] ,$$

where

$$\left(\frac{2n}{n-1}\right)c\beta^{-\frac{(n+1)}{2}} > c\beta^{\frac{-(n+1)}{2}} = \left(c^{\frac{n}{n+1}}\beta^{-\frac{n}{2}}\right)^{\frac{n+1}{n}} = (\bar{x}_{11} + K)^{\frac{n+1}{n}} .$$

Now note that $\bar{x}_{11}$ in (4.8) exceeds $\beta^{\frac{n(n-1)}{2}} - K$ because $\beta^{\frac{n(n+1)}{2}} \leq c$.

Hence, the maximum value possible for $x_1 \leq \beta^{\frac{n(n-1)}{2}} - K$ occurs at the boundary, i.e.,

$$\tilde{x}_1 = \beta^{\frac{n(n-1)}{2}} - K . \tag{4.10}$$

We now must verify that I prefers (4.4) to (4.10). Applying the induction hypothesis with (4.4), we have

$$\begin{aligned} f(\hat{x}_{11}) &\geq \sum_{j=0}^{n-1} \beta^j - n\beta^{\frac{n-1}{2}} (1-p)^{\frac{n-1}{n}} (1-p-\gamma)^{\frac{1}{n}} + p\beta^n \\ &= \sum_{j=0}^{n} \beta^j - n\beta^{\frac{n-1}{2}} (1-p)c^{\frac{1}{n}} - (1-p)\beta^n , \end{aligned} \tag{4.11}$$

where the inequality is due to our using the solution generated by $A(n) = n$. If $A(n) < n$, then (4.11) is a strict inequality. Corresponding to (4.10), we have

$$\begin{aligned} f(\tilde{x}_{11}) &= \frac{\beta^{\frac{n(n-1)}{2}} - K}{\beta^{\frac{n(n-1)}{2}}} + \beta\left[\sum_{j=0}^{n-1} \beta^j - n\beta^{\frac{n-1}{2}} (1-p)^{\frac{n-1}{n}} \left(1 - p - (1-p)\left(1 - \beta^{\frac{n(n-1)}{2}}\right)\right)\right] \\ &= \sum_{j=0}^{n} \beta^j - K\beta^{-\frac{n(n-1)}{2}} - n(1-p)\beta^n . \end{aligned} \tag{4.12}$$

Comparing (4.11) and (4.12), we see that

$$(1-p)^{-1}[f(\hat{x}_{11}) - f(\tilde{x}_{11})] \geq c\beta^{-\frac{n(n-1)}{2}} + n\beta^n - n\beta^{\frac{n-1}{2}} c^{\frac{1}{n}} - \beta^n$$

$$= \left(\frac{c^{\frac{1}{n}}}{\beta^{\frac{n-1}{2}}}\right)^n - \beta^n - n\beta^{n-1}\left(\frac{c^{\frac{1}{n}}}{\beta^{\frac{n-1}{2}}} - \beta\right)$$

$$(4.13) \qquad = (\alpha^n - \beta^n) - n\beta^{n-1}(\alpha-\beta)$$

$$= (\alpha-\beta)\left(\sum_{k=0}^{n-1} \alpha^k\beta^{n-k-1} - n\beta^{n-1}\right)$$

$$\geq 0 ,$$

where $K = 1-p-\gamma$ and $c = K(1-p)^{-1}$ as before, $\alpha = c^{\frac{1}{n}}\beta^{-\frac{(n-1)}{2}} \geq \beta$, and $\alpha^k\beta^{n-k-1} \geq \beta^n$. We have thus shown that if $A(n+1) < n+1$, then the $(n+1)$-period solution for I is the same as the n-period solution except I gets $\beta^n p$ more in the $(n+1)^{st}$ period. We have in fact shown a bit more, namely, that the prospective strategy in (4.10) is dominated by the strategy in (4.4) with $k = n$, which in turn is dominated by the optimal strategy.

Case 2: Players I's Optimization when $A(n+1) = n+1$

Now we assume that $\beta^{\frac{n(n+1)}{2}} > c$. As in Case 1, I will spend less than all his money each period until the last period, beginning in the second period, if $x_1 \leq \beta^{\frac{n(n-1)}{2}} - K$. Assuming this to be the case, we get the solution generated by $\bar{x}_{11}$ in (4.8). However, since $\beta^{\frac{n(n+1)}{2}} > c$

now, $\bar{x}_{11} < \beta^{\frac{n(n-1)}{2}} - K$, so $\bar{x}_{11}$ is the natural candidate for the initial spending which generates the optimal solution. The return using $\bar{x}_{11}$ in (4.8) is easily calculated:

$$f(\bar{x}_{11}) = \frac{\bar{x}_{11}}{\bar{x}_{11} + K} + \beta U_n^1\left(p,\ (1-p)\left(1 - c^{\frac{n}{n+1}}\beta^{-\frac{n}{2}}\right)\right)$$

$$= 1 - \frac{K}{c^{\frac{n}{n+1}}\beta^{-\frac{n}{2}}} + \beta\left[\sum_{j=0}^{n-1}\beta^j - n\beta^{\frac{n-1}{2}}(1-p)^{\frac{n-1}{n}}(1-p-\gamma_n)^{\frac{1}{n}}\right]$$

(4.14)

$$= \sum_{j=0}^{n}\beta^j - Kc^{-\frac{n}{(n+1)}}\beta^{\frac{n}{2}} - n\beta^{\frac{n+1}{2}}(1-p)^{\frac{n-1}{n}}\left[1-p-(1-p)\left(1 - c^{\frac{n}{n+1}}\beta^{-\frac{n}{2}}\right)\right]^{\frac{1}{n}}$$

$$= \sum_{j=0}^{n}\beta^j - (n+1)\beta^{\frac{n}{2}}(1-p)^{\frac{n}{n+1}}(1-p-\gamma)^{\frac{1}{n+1}} .$$

The second step in (4.14) is justified because

$$\beta^{\frac{n(n-1)}{2}} > c_n = \frac{1-p-\gamma_n}{1-p} = c^{\frac{n}{n+1}}\beta^{-\frac{n}{2}} \tag{4.15}$$

since $\beta^{\frac{n(n+1)}{2}} > c$ by assumption.

It remains to rule out $x_1 \geq \beta^{\frac{n(n-1)}{2}} - K$. As in (4.1)-(4.4) of Case 1, we would obtain $U_{n+1}^1(p,\gamma) = U_n^1(p,\gamma) + \beta^n p$ if x_1 is constrained to be bigger than $\beta^{\frac{n(n-1)}{2}} - K$. The prospective solution would then be generated by $\hat{x}_{11}$ in (4.4) for some $k \leq n$. If $\hat{x}_{11} \geq \beta^{\frac{n(n-1)}{2}} - K$, then $\hat{x}_{11}$ is a legitimate candidate. For this case it suffices to show that $f(\bar{x}_{11}) > f(\tilde{x}_{11})$ with $f(\bar{x}_{11})$ in (4.14) and

$$f(\hat{x}_{11}) = \sum_{j=0}^{n} \beta^j - k\beta^{\frac{k-1}{2}}(1-p)c^{\frac{1}{k}} - (1-p)\sum_{j=k}^{n} \beta^j , \tag{4.16}$$

for any $k \leq n$. In fact, it is easier to prove a stronger result. We shall show that (4.16) is strictly increasing in k . For $k \leq n$,

$$(1-p)^{-1}[f(\hat{x}_{11})_{k+1} - f(\hat{x}_{11})_k]$$

$$= k\beta^{\frac{k-1}{2}}c^{\frac{1}{k}} + \beta^k - (k+1)\beta^{\frac{k}{2}}c^{\frac{1}{k+1}}$$

$$= \left(\beta^k - \beta^{\frac{k}{2}}c^{\frac{1}{k+1}}\right) - k\left(\beta^{\frac{k}{2}}c^{\frac{1}{k+1}} - \beta^{\frac{k-1}{2}}c^{\frac{1}{k}}\right)$$

$$= \beta^{\frac{k}{2}}\left[\left(\beta^{\frac{1}{2}}\right)^k - \left(c^{\frac{1}{k(k+1)}}\right)^k\right] - k\beta^{\frac{k-1}{2}}c^{\frac{1}{k+1}}\left[\beta^{\frac{1}{2}} - c^{\frac{1}{k(k+1)}}\right] \tag{4.17}$$

$$= \beta^{\frac{k}{2}}\left(\beta^{\frac{1}{2}} - c^{\frac{1}{k(k+1)}}\right)\left[\sum_{j=0}^{k-1}\left(\beta^{\frac{1}{2}}\right)^{(k-j-1)}\left(c^{\frac{1}{k(k+1)}}\right)^j - k\beta^{-\frac{1}{2}}c^{\frac{1}{k+1}}\right]$$

$$= \beta^{\frac{k-1}{2}}\left(\beta^{\frac{1}{2}} - c^{\frac{1}{k(k+1)}}\right)\sum_{j=0}^{k-1}\left[\beta^{\frac{k-j}{2}}c^{\frac{j}{k(k+1)}} - c^{\frac{k}{k+1}}\right] ,$$

which is positive because

$$\beta^{\frac{1}{2}} - c^{\frac{1}{k(k+1)}} > 0$$

and

$$\beta^{\frac{k-j}{2}} c^{\frac{j}{k(k+1)}} - c^{\frac{k}{k(k+1)}} > 0$$

since

$$\beta^{\frac{k(k+1)}{2}} \geq \beta^{\frac{n(n+1)}{2}} > c \ .$$

Furthermore, the argument above shows that $f(\hat{x}_{11})_k = f(\hat{x}_{11})_{k+1}$ if $\beta^{\frac{k(k+1)}{2}} = c$.

There is still one point more to dispose of in this case. It could happen that $\hat{x}_{11} < \beta^{\frac{n(n-1)}{2}} - K$. Applying (4.7) and (4.9) for $k \leq n$, we see that $\hat{x}_{11}$ is a maximum and $f(x_1)$ decreases as x_1 moves away from $\hat{x}_{11}$. Therefore, the candidate to be considered is $\tilde{x}_{11}$ in (4.10) instead of $\hat{x}_{11}$ in (4.4). Thus, it remains to show that $f(\bar{x}_{11}) > f(\tilde{x}_{11})$ for $f(\bar{x}_{11})$ in (4.14) and $f(\tilde{x}_{11})$ in (4.12), but

$$\beta^{-\frac{n}{2}}(1-p)^{-1}[f(\bar{x}_{11}) - f(\tilde{x}_{11})]$$

$$= \beta^{-\frac{n}{2}}\left[c\beta^{-\frac{n(n-1)}{2}} + n\beta^{n} - (n+1)\beta^{\frac{n}{2}}c^{\frac{1}{n+1}}\right]$$

$$= n\left(\beta^{\frac{n}{2}} - c^{\frac{1}{n+1}}\right) - \left(c^{\frac{1}{n+1}} - c\beta^{\frac{-n^2}{2}}\right) \tag{4.18}$$

$$= \left(\beta^{\frac{n}{2}} - c^{\frac{1}{n+1}}\right)\left[n - c^{\frac{1}{n+1}}\beta^{\frac{-n^2}{2}}\left(\beta^{\frac{n^2}{2}} - c^{\frac{n}{n+1}}\right)\Big/\left(\beta^{\frac{n}{2}} - c^{\frac{n}{n+1}}\right)\right]$$

$$= \left(\beta^{\frac{n}{2}} - c^{\frac{1}{n+1}}\right)\left[n - c^{\frac{1}{n+1}}\beta^{\frac{-n^2}{2}}\sum_{j=0}^{n-1}\left(\beta^{\frac{n}{2}}\right)^{j}\left(c^{\frac{1}{n+1}}\right)^{n-j-1}\right],$$

which is positive because

$$\beta^{\frac{n}{2}} - c^{\frac{1}{n+1}} > 0$$

and

$$c^{\frac{1}{n+1}} \beta^{\frac{-n^2}{2}} \beta^{\frac{nj}{2}} c^{\frac{n-j-1}{n+1}} = c^{\frac{n-j}{n+1}} \beta^{-\frac{n(n-j)}{2}} < 1 ,$$

since

$$\beta^{\frac{n(n+1)}{2}} > c .$$

This completes the proof for I.

Case 3: A Possible Advantage for Player II

We now show that the weak player spends all his money every period. Continuing the induction proof, we assume II spends all his money every period in each k-period problem for $k \leq n$ and we use the strategies just verified for I.

We begin by assuming $\beta_1 < c$ so that I spends all his money in the first period. Then I loses his money advantage in the second period. The new advantage to I with n periods to go becomes

$$\begin{aligned} \gamma_n &= \gamma + px_2 - (1-p)x_1 \\ &= -p(1-p-\gamma-x_2) \leq 0 , \end{aligned} \tag{4.19}$$

where x_2 is II's initial spending. Hence, II's optimization problem is

$$U^2_{n+1}(1-p, -\gamma) = \max_{0 \leq x_2 \leq 1-p-\gamma} g(x_2) , \tag{4.20}$$

where

$$g(x_2) = \frac{x_2}{x_1 + x_2} + \beta_2 U_n^2(1\text{-}p, \ +\gamma_n(x_2))$$

(4.21)

$$= \frac{x_2}{x_2 + p + \gamma} + \beta_2 U_n^2(1\text{-}p, \ p(1 - p - \gamma - x_2)) \ .$$

Note that $U_n^2(1\text{-}p, +\gamma)$ in (4.21) corresponds to I's strategy in period n with p and 1-p switched, obtained via the induction hypothesis, because now II has the money advantage. We know the second term in (4.21) increases as x_2 decreases and as β_2 increases, so let $\beta_2 = 1$. This will only make $\hat{x}_{21}$, the initial spending for II, smaller. Then

$$(4.22) \quad U_n^2(1\text{-}p, +\gamma_n) = U_n^2(1\text{-}p, \ p(1 - p - \gamma - x_2)) = np^{\frac{n-1}{n}} (p - \gamma_n)^{\frac{1}{n}} = np(p + \gamma + x_2)^{\frac{1}{n}}$$

by virtue of (4.14) or Theorem 6(c). Then

$$(4.23) \qquad g'(x_2) = \frac{p+\gamma}{(x_2 + p + \gamma)^2} - \frac{p}{(x_2 + p + \gamma)^{\frac{n-1}{n}}}$$

and

$$(4.24) \qquad g''(x_2) = \frac{-2(p+\gamma)}{(x_2 + p + \gamma)^3} + \frac{p(n-1)}{n(x_2 + p + \gamma)^{\frac{2n-1}{n}}}$$

for g in (4.21). Setting $g'(x_2) = 0$, we get the unique solution

$$(4.25) \qquad \bar{x}_{21} = \left(\frac{p+\gamma}{p}\right)^{\frac{n}{n+1}} - (p+\gamma)$$

which is obviously strictly greater than $K = 1-p-\gamma$ for $\gamma > 0$. Also $g''(\bar{x}_{21}) > 0$ because

$$\frac{2(p+\gamma)}{(\bar{x}_{21}+p+\gamma)^3} > \frac{p(n-1)}{n(\bar{x}_{21}+p+\gamma)^{\frac{2n-1}{n}}} \tag{4.26}$$

or, equivalently,

$$\frac{2n}{n-1}\,\frac{p+\gamma}{p} > (\bar{x}_{21}+p+\gamma)^{\frac{n+1}{n}} = \frac{p+\gamma}{p} . \tag{4.27}$$

This means $\bar{x}_{21}$ is indeed a maximum. Since $\bar{x}_{21} > 1-p-\gamma$, II's initial spending should be

$$\hat{x}_{21} = 1-p-\gamma . \tag{4.28}$$

The present case demonstrates that II will not spend less to capitalize on a money advantage he could obtain over I. We have shown this for $x_1 = p+\gamma$, but it also applies to all other possible x_1 , i.e., $c - K = \frac{p(1-p-\gamma)}{1-p} \leq x_1 \leq p+\gamma$. Note from (4.22) that

$$\frac{dU_n^2(1-p,\ \gamma)}{d\gamma} = \left(\frac{p}{p-\gamma}\right)^{\frac{n-1}{n}} . \tag{4.29}$$

The largest money advantage II can obtain at the end of the first period occurs when I spends all his money in the first period. Hence,

$$\frac{dU_n^2(1-p,\ \gamma_n)}{d\gamma_n}\,\frac{d\gamma_n}{dx_2} \leq \left(\frac{p}{p+\gamma+x_2}\right)^{\frac{n-1}{n}} p \leq p . \tag{4.30}$$

On the other hand, the least marginal advantage II can obtain from spending in the first period occurs at the point where he spends all his money. This rate is

$$(4.31) \qquad \frac{x_1}{(x_1 + 1 - p - \gamma)^2} = 1 - (1-p-\gamma)^{1-a}(1-p)^a \geq p \ ,$$

for $x_1 = c^a - K$, $0 \leq a \leq 1$, which covers all x_1 , $c-K \leq x_1 \leq p+\gamma$. Comparing (4.30) and (4.31), we see that the conclusion in this case applies to all x_1 which I might select as optimal following the strategies previously determined for him.

Case 4: The Possibility of II Reducing I's Advantage

The critical difference between this case and the one before is that now II is considering spending less so that I will have a smaller money advantage in the second period instead of spending less so that II can achieve an actual money advantage for himself. We shall see that II still spends all his money.

We have treated $A(n+1) = 1$ in the last case. Now suppose $A(n+1) = k \geq 2$ with I's advantage

$$(4.32) \qquad \gamma_n = \gamma + px_2 - (1-p)\left[c^{\frac{k-1}{k}} \beta_1^{-\frac{(k-1)}{2}} - K \right] \geq 0 \ .$$

Then, instead of (4.21), we have

$$(4.33) \qquad g(x_2) = \frac{x_2}{x_1 + x_2} + \beta_2 U_n^2(1-p, -\gamma_n(x)) \ .$$

If $x_2 = (1-p-\gamma)$, then

$$(4.34)\quad \begin{aligned} g(x_2) &= \frac{1-p-\gamma}{c^{\frac{k-1}{k}}\beta_1^{-\frac{(k-1)}{2}}} + \beta_2\beta_1^{\frac{k-2}{2}}(1-p)^{\frac{k-2}{k-1}}(1-p-\gamma_n)^{\frac{1}{k-1}}\sum_{j=0}^{k-2}(\beta_2/\beta_1)^j \\ &= \frac{K}{c^{\frac{k-1}{k}}\beta_1^{-\frac{(k-1)}{2}}} + \beta_2\beta_1^{\frac{k-2}{2}}(1-p)^{\frac{k-2}{k-1}}\left[(1-p)c^{\frac{k-1}{k}}\beta_1^{-\frac{(k-1)}{2}}\right]^{\frac{1}{k-1}}\sum_{j=0}^{k-2}(\beta_2/\beta_1)^j , \end{aligned}$$

by virtue of the induction hypothesis. If $\beta_1 = \beta_2$, then $U_n^2(1-p, -\gamma_n(x))$ $= \sum_{j=0}^{n-1}\beta_1^j - U_n^1(p, \gamma_n(x))$, but if $\beta_1 \neq \beta_2$, then we must include the last sum in (4.34).

It is evident that both terms of the derivative in (4.33) are decreasing in x_2 . Hence, it suffices to consider the derivative evaluated at the largest possible value of x_2 . In other words,

$$(4.35)\quad g'(x_2) \geq g'(1-p-\gamma) = \begin{cases} \dfrac{z-K}{z^2} - p\beta_2 & , k = 2 , \\ \dfrac{z-K}{z^2} - p\beta_2\beta_1^{\frac{k-2}{2}}(k-1)^{-1}z^{-\frac{(k-2)}{k-1}}\displaystyle\sum_{j=0}^{k-2}(\beta_2/\beta_1)^j & , k \geq 3 , \end{cases}$$

where

$$(4.36)\quad z = c^{\frac{k-1}{k}}\beta^{-\frac{(k-1)}{2}} .$$

If $A(n+1) = 2$, then (4.35) reduces to

$$g'(1-p-\gamma) = \frac{c^{\frac{1}{2}}\beta_1^{-\frac{1}{2}} - K}{c\beta_1^{-1}} - p\beta_2$$

(4.37)
$$= c^{-\frac{1}{2}}\beta_1^{\frac{1}{2}} - [\beta_1(1-p) + p\beta_2]$$

$$\geq 1 - 1 = 0$$

because $\beta_1 > c$. Hence, II will spend all his money every period if $A(n+1) = 2$.

For $A(n+1) = k \geq 3$, it suffices to let $\beta_2 = 1$. This can only make the negative term in (4.35) larger in absolute value. Then

$$g'(1-p-\gamma) \geq \frac{z-K}{z^2} - \frac{p\beta_1^{-\frac{(k-2)}{2}}}{z^{(k-2)/(k-1)}}$$

$$= z^{-\frac{(k-2)}{(k-1)}}\left[\frac{1}{z^{1/(k-1)}} - \frac{(1-p)c}{z^{k/(k-1)}} - p\beta_1^{-\frac{(k-2)}{2}}\right]$$

$$\geq \beta_1^{-\frac{(k-2)}{2}} - (1-p) - p\beta_1^{-\frac{(k-2)}{2}}$$

$$\geq (1-p)\left(\beta_1^{-\frac{(k-2)}{2}} - 1\right) \geq 0 .$$

Hence, the weaker player always spends all his money.

Related Theorems and Corollaries

The proof of Theorem 1 has employed Theorems 3, 4, and 6. Therefore, to properly complete the proof of Theorem 1, we must verify that these descriptions prevail in the (n+1)-period problem. The only parts remaining are (b)-(e) in Theorem 6. Theorem 6(b) is easy to verify directly for $j = 2$ and by induction for $j > 2$. Using (b), it is easy to compute the money each player has in the beginning of period j. Then the next consumption is easy to determine, from which (c) and (d) follow easily. Part (e) is evident for a given strategy. The argument in (4.17) shows there is no difficulty at the transition points.

The remaining corollaries in Section 3.1 follow easily from Theorems 1, 3, 4, and 6. In Corollary 10(b) the first limit can be obtained by applying Taylor's Theorem to $c^{n^{-1}} = e^{n^{-1}\log c}$.

Proof of Theorem 2: Uniqueness

We now consider whether the solution just obtained is the only solution. We show that it is if either $\beta_2 \le \beta_1$ or $\beta_1 \le c_n$, where

$$c_n = \frac{1-p-\gamma_n}{1-p} = \frac{1-p-\gamma-px_2+(1-p)x_1}{1-p}.$$

However, we do not obtain uniqueness when $\beta_1 > c_n$, that is, when the strong player is not motivated to spend all his money in the second period. After proving uniqueness where it holds, we give a counterexample to uniqueness when $\beta_2 > \beta_1$ and $\beta_1 > c_n$.

Again our proof will be by induction. Just as with the existence, the uniqueness is obvious for one period. Therefore, we shall show it is true for $n+1$ periods given that is true for k periods for each $k \leq n$.

Case 1: Boundary Values

Let (x_1, x_2) denote the initial spending by I and II for a prospective second solution with $n+1$ periods. First, note that $0 < x_1 < p+\gamma$ and $0 < x_2 < 1-p-\gamma$. If $x_2 = 1-p-\gamma$, then the previous solution would be obtained. Furthermore, we have seen in Case 3 of the last proof that II spends all whenever I does. Hence, $x_1 < p+\gamma$. Finally, zero spending for either player is obviously not an equilibrium point. The other player is then motivated to select an arbitrarily small positive initial spending in order to get all the goods in the first period without significantly jeopardizing his position in the second period. Of course, this does not yield a well-defined strategy, but even if it did, the player who had planned to spend nothing would then himself be motivated to spend a small positive value. In other words, we can begin by considering solutions in the interior of the possible spending intervals.

Case 2: $\beta_1 \geq \beta_2$

In the interior of the possible spending intervals all possible solutions are solutions to the pair of equations obtained by taking derivatives of the return functions. We allow arbitrary spending in the first period but we use the previous solution thereafter. In other words, we apply the induction hypothesis. If I's money advantage in the second period is nonnegative, that is, if

$$\gamma_n = \gamma + px_2 - (1-p)x_1 \geq 0 , \tag{4.39}$$

and if $A_2(n) = k \geq 2$, where $A_2(n)$ is the number of the period beginning with the second when I first spends all his money, then the two equations are:

$$\begin{aligned} \frac{x_2}{(x_1 + x_2)^2} &= \frac{\beta_1 (1-p)(1-p)^{\frac{k-1}{k}} \beta_1^{\frac{k-1}{2}}}{(1 - p - \gamma - px_2 + (1-p)x_1)^{\frac{k-1}{k}}} \\ \frac{x_1}{(x_1 + x_2)^2} &= \frac{\beta_2 pA(1-p)^{\frac{k-1}{k}} \beta_1^{\frac{k-1}{2}}}{(1 - p - \gamma - px_2 + (1-p)x_1)^{\frac{k-1}{k}}} , \end{aligned} \tag{4.40}$$

where

$$A = k^{-1} \sum_{j=0}^{k-1} (\beta_2/\beta_1)^j . \tag{4.41}$$

The equations in (4.40) come from (c) and (d) of Theorem 6 or (4.14) and (4.34). As an immediate consequence, we get a relationship between x_1 and x_2 , namely,

$$x_1 = \frac{\beta_2 pA}{\beta_1 (1-p)} x_2 . \tag{4.42}$$

Substituting (4.42) into (4.39), we get

$$\gamma_n = \gamma + (1-p)x_1\left[\frac{\beta_1}{\beta_2 A} - 1\right] \quad , \tag{4.43}$$

so that $\gamma_n \geq \gamma$ if and only if $\beta_2 \leq \beta_1$. If $\beta_2 \leq \beta_1$, then

$$x_2 = \frac{\beta_1(1-p)\left[1 - p - \gamma - px_2\left(1 - \frac{A\beta_2}{\beta_1}\right)\right]^{\frac{k-1}{k}}}{[\beta_2 pA + \beta_1(1-p)]^2(1-p)^{\frac{k-1}{k}}\beta_1^{\frac{k-1}{2}}}$$

$$\geq \frac{(1-p)^{\frac{1}{k}}\left[(1-p-\gamma)\left(1 - p\left(1 - \frac{A\beta_2}{\beta_1}\right)\right)\right]^{\frac{k-1}{k}}}{[\beta_2 pA + \beta_1(1-p)]^2\beta_1^{\frac{k-3}{2}}} \tag{4.44}$$

$$= \frac{(1-p)^{\frac{1}{k}}(1-p-\gamma)^{\frac{k-1}{k}}}{[\beta_2 pA + \beta_1(1-p)]^{\frac{k+1}{k}}\beta_1^{\left(\frac{k-3}{2} + \frac{k-1}{k}\right)}}$$

$$\geq (1-p)^{\frac{1}{k}}(1-p-\gamma)^{\frac{k-1}{k}}$$

$$\geq 1-p-\gamma \ .$$

Hence, we have a contradiction with our requirement that $0 < x_2 < 1-p-\gamma$. The only solution with $\beta_1 \geq \beta_2$ is the one previously determined. In the second step of (4.44) we have made the right side smaller by replacing x_2 with $(1-p-\gamma)$.

Case 3: $A_2(n) = 1$

Suppose now that $\beta_2 > \beta_1$. If

$$\beta_1 \leq \frac{(1+p)(1-p-\gamma)}{(1-p)}$$

and

$$\beta_2 \leq p+\gamma ,$$

then, for all x_1 and x_2 , $\beta_1 \leq c_n$ if $\gamma_n \geq 0$ and $\beta_2 \leq \bar{c}_n$ if $\gamma_n \leq 0$, where

$$\bar{c}_n = \frac{p - (-\gamma_n)}{p} = \frac{p + \gamma_n}{p} ,$$

cf. Theorem 2(iii). Note that this is always true for two periods, cf. Theorem 2(ii). In this case the equations in (4.40) or (4.56) reduce to

$$\frac{x_2}{(x_1 + x_2)^2} = \beta_1(1-p) \tag{4.45}$$

and

$$\frac{x_1}{(x_1 + x_2)^2} = \beta_2 p .$$

We again get (4.42) but with $A = 1$. Substituting (4.42) into (4.45), we get

$$x_2 = \frac{\beta_1(1-p)}{[\beta_1(1-p)+\beta_2 p]^2}$$

(4.46)

and $$x_1 + x_2 = [\beta_1(1-p)+\beta_2 p]^{-1} \ ,$$

so that $x_1 + x_2 > 1$ unless $\beta_1 = \beta_2 = 1$. However, if $\beta_1 = \beta_2 = 1$, then $x_2 = 1-p \geq 1-p-\gamma$. Hence, there is no other solution for $k = 1$. This case completes the uniqueness proof for the two-period problem.

Case 4: A Counterexample to Uniqueness if $\beta_2 > \beta_1$

Suppose $\beta_2 > \beta_1$, $A_2(n) > 1$, and $\gamma_n > 0$. Then we still have (4.40) and the first equation in (4.44). If

(4.47) $$\beta_2 pA + \beta_1(1-p) \leq 1 \ ,$$

which is not necessarily to be expected now, then

(4.48)
$$\begin{aligned} x_2 &\geq \beta_1(1-p)c_n^{\frac{k-1}{k}}\beta_1^{-\frac{k-1}{2}} \\ &\geq (1-p)c_n^{\frac{k-1}{k}} \\ &\geq (1-p)c^{\frac{k-1}{k}} = (1-p)^{\frac{1}{k}}(1-p-\gamma)^{\frac{k-1}{k}} \\ &\geq 1-p-\gamma \ . \end{aligned}$$

However, without (4.47), uniqueness can be lost. We demonstrate this for $A_2(n) = 2$, which corresponds to a three-period game. We then have

$$(4.49) \qquad x_2 = \frac{\beta_1^{\frac{1}{2}}(1-p)^{\frac{1}{2}}\left[1-p-\gamma+px_2\left(\frac{A\beta_2}{\beta_1}-1\right)\right]^{\frac{1}{2}}}{[\beta_2 pA+\beta_2(1-p)]^2}$$

or

$$(4.50) \qquad ax_2^2 - bx_2 - K = 0 ,$$

where $a > 1$, $b > 0$, and $K = 1-p-\gamma$. In particular,

$$(4.51) \qquad a = \frac{[\beta_2 pA + \beta_1(1-p)]^4}{\beta_1(1-p)}$$

and

$$b = p\left(\frac{A\beta_2}{\beta_1} - 1\right) .$$

It is evident that (4.50) has exactly one positive real root, namely,

$$(4.52) \qquad x_2 = \frac{b + (b^2 + 4aK)^{1/2}}{2a} .$$

Now we show that it is possible to select a , b , and K appropriately so that $0 < x_1 < p+\gamma = 1-K$ and $0 < x_2 < 1-p-\gamma = K$. It turns out that haphazard selections of p , γ , β_1 , and β_2 will not do. It is important to make A and β_1^{-1} very large. For example, think of β_1^{-1} as 10^{100} although this may be a bit bigger than necessary; we shall just let $\beta_1^{-1} = N$ with the understanding that N is big. Let $p = 1-p = 1/2$; let $\beta_2 = 1$; and let $K = 1/2$, which means that $\gamma = 1/4$. We shall express x_2 approximately using N and generic constants c_i , $i \geq 1$.

We get $A = c_1 N$, $a = c_2 N^5$, $b = c_3 N^2$, and

$$x_2 = \frac{c_3 N^2 + (c_3^2 N^4 + 4Kc_2 N^5)^{1/2}}{2c_2 N^5} \tag{4.53}$$

$$= c_4 N^{-5/2} < K .$$

Moreover,

$$x_1 = \frac{\beta_2 pA}{\beta_1 (1-p)} x_2 = c_5 N^2 x_2 = c_6 N^{-1/2} < 1-K \tag{4.54}$$

and

$$\gamma_n = \gamma - px_2\left(\frac{A\beta_2}{\beta_1} - 1\right) \tag{4.55}$$

$$= \gamma - pc_4 N^{-4} c_7 N^2$$

$$= \gamma - c_8 N^{-2} \geq 0 .$$

Case 5. $\gamma_n \leq 0$

We have not discussed the case in which $\gamma_n \leq 0$, that is, when the advantage shifts to II. Instead of (4.40), we have

$$\frac{x_2}{(x_1 + x_2)^2} = \frac{\beta_1 (1-p) p^{\frac{k-1}{k}} \beta_2^{\frac{k-1}{2}}}{(p + \gamma + px_2 - (1-p)x_1)^{\frac{k-1}{k}}} \tag{4.56}$$

and

$$\frac{x_1}{(x_1 + x_2)^2} = \frac{\beta_2 pAp^{\frac{k-1}{k}} \beta_2^{\frac{k-1}{2}}}{(p + \gamma + px_2 - (1-p)x_1)^{\frac{k-1}{k}}} .$$

Again we obtain (4.42). Substituting it into the first equation of (4.56), we get

$$
\begin{aligned}
x_2 &= \frac{\beta_1(1-p)\left[p+\gamma+(1-p)x_1\left(\dfrac{\beta_1}{A\beta_2}-1\right)\right]^{\frac{k-1}{k}}}{[\beta_1(1-p)+\beta_2Ap]^2\,p^{\frac{k-1}{k}}\,\beta_2^{\frac{k-1}{2}}} \\
&\geq \frac{\beta_1(1-p)\left[p+\gamma+(1-p)(p+\gamma)\left(\dfrac{\beta_1}{A\beta_2}-1\right)\right]^{\frac{k-1}{k}}}{[\beta_1(1-p)+\beta_2Ap]^2\,p^{\frac{k-1}{k}}\,\beta_2^{\frac{k-1}{2}}} \\
&= \frac{\beta_1(1-p)(p+\gamma)^{\frac{k-1}{k}}\,[\beta_1(1-p)+\beta_2Ap]^{\frac{k-1}{k}}\,\beta_2^{-\frac{(k-1)}{k}}}{[\beta_1(1-p)+\beta_2Ap]^2\,p^{\frac{k-1}{k}}\,\beta_2^{\frac{k-1}{2}}} \qquad (4.57) \\
&= \beta_1(1-p)\left(\frac{p+\gamma}{p}\right)^{\frac{k-1}{k}}[\beta_1(1-p)+\beta_2Ap]^{-\frac{(k+1)}{k}}\,\beta_2^{-\frac{(k-1)(k+2)}{2}},
\end{aligned}
$$

where the second step involves replacing x_1 by $p+\gamma$ on the right. Since β_1 can be arbitrarily small, (4.57) is rather difficult to work with directly. If we assume (4.47), then we can apply (4.42) plus (4.57) to obtain

$$x_1 + x_2 \geq [\beta_1(1-p) + \beta_2 Ap]\left(\frac{p+\gamma}{p}\right)^{\frac{k-1}{k}}[\beta_1(1-p) + \beta_2 Ap]^{-\left(\frac{k+1}{k}\right)}\beta_2^{-\frac{(k-1)(k+1)}{2}}$$

$$(4.58) \quad = \left(\frac{p+\gamma}{p}\right)^{\frac{k-1}{k}}[\beta_1(1-p) + \beta_2 Ap]^{-\frac{1}{k}}\beta_2^{-\frac{(k-1)(k+1)}{2}}$$

$$> 1 \ ,$$

which contradicts the basic spending constraints: $x_1 \leq p+\gamma$ and $x_2 \leq 1-p-\gamma$. However, (4.58) is not possible without (4.47). A counterexample for $A_2(n) = 2$ is easily constructed here just as in (4.49)-(4.55). This completes our discussion of uniqueness. In particular, we shall neither attempt to provide more detailed conditions for uniqueness when $\beta_2 > \beta_1$ nor attempt to describe the other noncooperative equilibrium solutions at this time.

Proof of Theorem 12: Pareto Optimality

If $\beta_1 = \beta_2$, then our game is constant-sum except for the case in which both players spend nothing. Since such spending never occurs in any solution, cf. Case 1 of the Proof of Theorem 2, any solution is Pareto optimal when $\beta_1 = \beta_2$. However, if $\beta_1 \neq \beta_2$, then both players could simultaneously do better by judiciously (cooperatively) spending less initially. Since both players spend strictly positive amounts in any solution, both players are always free to spend less. The returns to I and II using positive x_1 and x_2 initially and the optimal strategies thereafter are

$$R^1_{n+1}(p,\gamma) = \frac{x_1}{x_1+x_2} + \beta_1 U^1_n(p,\ \gamma + px_2 - (1-p)x_1)$$

(4.59)

and $$R^2_{n+1}(1-p,\ -\gamma) = \frac{x_2}{x_1+x_2} + \beta_2 U^2_n(1-p,\ \gamma + px_2 - (1-p)x_1)\ .$$

For $\beta_1 = \beta_2$, it is easy to see that

$$-\frac{dU^2_n(1-p,\ -\gamma)}{d\gamma} = \frac{dU^1_n(p,\gamma)}{d\gamma} \geq \frac{dU^1_n(p,0)}{d\gamma} = 1\ . \tag{4.60}$$

If $\beta_1 > \beta_2$, then it is easy to verify that

$$-\frac{dU^2_n(1-p,\ -\gamma)}{d\gamma} < \frac{dU^1_n(p,\gamma)}{d\gamma}\ . \tag{4.61}$$

Hence, $R^1_{n+1}(p,\gamma)$ can be kept constant by decreasing both x_1 and x_2 a small amount so that I's first period loss equals his future gain. This can be done in many ways. Then II's first period gain coincides with I's first period loss, but II's second period loss is less than I's second period gain because $\beta_2 < \beta_1$. Hence, II is better off while I is indifferent. A parallel argument applies to $\beta_1 < \beta_2$.

5. Proofs for Many Players

Proof of Theorem 19: Perfect Competition

The bounds on p_i and γ_i imply that

$$(5.1) \qquad c_i = \frac{1 - p_i - \gamma_i}{1 - p_i} \geq \frac{n-2}{n} \ ,$$

so that $\beta_i \leq 1-\delta < c_i$ for sufficiently large n. Hence, the condition of Corollary 17 is satisfied. We now investigate uniqueness beginning with two periods. Suppose some player, say I, does not spend all his available money in the first period. This means that I, faced with the problem

$$(5.2) \qquad \max_{0 \leq x_1 \leq p_1 + \gamma_1} \left\{ \frac{x_1}{x_1 + y} + \beta_1 (p_1 + \gamma_1 - x_1 + p(x_1 + y)) \right\}$$

where $y = \sum_{j=1}^{m} x_j - x_1$, elects to spend

$$(5.3) \qquad \hat{x}_{11} = \left[\frac{y}{\beta_1 (1-p)} \right]^{1/2} - y < p_1 + \gamma_1 \leq 2n^{-1} \ ,$$

which he obtains by differentiating in (5.2) as in Section 4. It is easy to see from (5.3) that we must have $y < n^{-1}$. First, $\hat{x}_{11}$ is increasing in y for $0 \leq y \leq [4\beta(1-p)]^{-1}$ and decreasing in y for $[4\beta(1-p)]^{-1} \leq y \leq 1 - p_1 - \gamma_1$. If $y = 1 - p_1 - \gamma_1$, then $\hat{x}_{11} > p_1 + \gamma_1$ since $\beta_1 \leq c_1$. If $y = n^{-1}$, then

$$(5.4) \qquad \hat{x}_{11} = \beta_1^{-1/2} (1-p)^{-1/2} n^{-1/2} - n^{-1} > (n^{1/2} - 1) n^{-1} \ .$$

Since $y < n^{-1}$, one of the remaining players must spend less than $[n(n-1)]^{-1}$ because $y = \sum_{j=1}^{m} x_j - x_1$ and $m \geq n$. These two arguments can be repeated to show that $y < (n-1)^{-k}$ and $\hat{x}_{11} < (n-1)^{-(k+1)}$ for all $k \geq 1$, where of course the distinguished player changes each time. This argument, easily made precise by induction, demonstrates that there is no spending at all in the first period, which is a contradiction because zero spending is obviously not an equilibrium solution. Hence, there is no second solution with some players spending less than all their money in the first period. This proof was for two periods. It is extended to any finite number of periods by induction. For sufficiently large n, the position in the second of several periods will correspond to the initial position with two periods, which we have just analyzed in detail. For example, player i's money supply in the second period can be no greater than $2p_i + \gamma_i$. Hence, $c_i \geq (n-3)/n$ in the second period and the spending is bounded above by $3n^{-1}$. Thus, by virtue of the induction hypothesis, I is again faced with (5.2) and the same argument applies.

Proof of Theorem 20: One Fat Cat

The proof above applies to show that none of the small players will spend less than all their money each period. This guarantees uniqueness.

6. Constrained Competitive Equilibrium

Proof of Theorem 21

First, consider the one-period problem. Everyone is clearly motivated to spend all available money at any price. Since $\sum_{i=1}^{m} x_i = G$, the price must be M/G. Next, assume the theorem to be true for n periods and consider the $(n+1)$-period optimization problem in (2.4). By virtue of the induction hypothesis, it reduces to the following two period problem for each individual:

$$\max_{\{x_i\}} \{x_1 + \beta x_2\} \tag{6.1}$$

subject to
$$x_1 a_1 \leq (p_1 + \gamma_1)M$$

$$x_2 \frac{M}{G} = (p_1 + \gamma_1)M - x_1 a_1 + p_1 G a_1 .$$

If $M/G < a_1\beta$, then each player wants x_2 instead of x_1. This would lead to zero spending in the first period. If $M/G = a_1\beta$, then each player is indifferent between x_1 and x_2. If $\beta < 1$, then both cases can be ruled out because the price in the first period must be less than or equal to M/G since no more than M will be spent for the total goods G. If $\beta = 1$, then $a_1 = M/G$ is only possible if each player spends all his money in the first period. If $M/G > a_1\beta$, then each player prefers x_1 to x_2. Hence, each player will spend all his available money in the first period. The associated price is then M/G.

If $\beta_i = \beta$ for all i, then the allocations are constant-sum and thus obviously Pareto optimal. If $\beta_i > \beta_j$ for some i, j, then both players could simultaneously do better if j gave i some goods in the future in exchange for some goods in the present.

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