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# VOTING, PREFERENCE ORDERINGS AND COOPERATIVE GAMES

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by

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## 1. Introduction

This article is devoted to the examination of voting and social choice viewed as a cooperative game. Some extremely weak assumptions are made, essentially those used by Arrow.[1] However stress is laid upon defining the power of coalitions and in discussing the concept of solution. Thus the approaches of Shapley,[2] Wilson,[3] Riker,[4] and Aumann and Maschler[5] are considered.

In particular it is noted that logrolling, or exchanging votes does not in general improve the outcomes from a decision process based upon voting unless strong assumptions are made about the items being voted upon and the rights of miniorities.

## 2. The Social Choice Problem

Suppose that there are $n$ individuals in a society and that they have to choose among $m$ outcomes or states.

Each individual has a complete preference ordering over all $m$ outcomes. The ordering is said to be <u>weak</u> if indifference between outcomes is considered. It is <u>strong</u> if only strict preference of one outcome over another is considered.

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If we considered only strong complete preference orderings then the number of different preference structures is $(m!)^n$ for a society with $n$ individuals and $m$ outcomes. When weak orderings are considered the number is far larger.

As many of the interesting examples appear when $n = 3$ and $m = 3$, the orderings are displayed for an individual. There are 6 strong and 13 weak orderings. These are shown as follows for the outcomes A , B , C .

TABLE 1a

| Strong Orders | | | | | | |
|---|---|---|---|---|---|---|
| | A | A | B | B | C | C |
| | B | C | A | C | A | B |
| | C | B | C | A | B | A |

TABLE 1b

| Weak Orders | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | A | A | B | B | C | C | (AB) | (AC) | (BC) | C | B | A | (ABC) |
| | B | C | A | C | A | B | C | B | A | (AB) | (AC) | (BC) | |
| | C | B | C | A | B | A | | | | | | | |

The preference order is given by the positions of the outcomes in the columns; thus the first column indicates that A is preferred to B and B to C . Parentheses indicate indifference, thus (AB) indicates that an individual is indifferent between A and B .

For 3 people with 3 outcomes and strong preference orderings $6^3 = 216$ preference structures can be considered. With weak orders $(13)^3 = 2,197$ preference structures exist.

For any single individual $i$ we could make a list of all the preference orderings he might have. If we consider weak orderings over 3 objects this list consists of the 13 items shown above. Call the $j^{th}$ ordering of the $i^{th}$ individual $R_j^i$ . Then we may represent the individual preference orderings of all members of society by an array $(R_{j_1}^1, R_{j_2}^2, \ldots, R_{j_i}^i, \ldots, R_{j_n}^n)$ where $j_i$ is some number in the list of preference orderings of the $i^{th}$ individual.

The <u>social aggregation</u> problem as posed by Arrow is, does there exist a group or social complete ordering which is derived from any array of individual orderings and which satisfies certain requirements which are deemed by some to be reasonable.

Call the $j^{th}$ ordering of society as a whole $R_j$ . Can we find an aggregation rule such that for any:

$$(R_{j_1}^1, R_{j_2}^2, \ldots, R_{j_i}^i, \ldots, R_{j_n}^n) \rightarrow R_j .$$

The number of aggregation rules is enormous. For only 3 individuals considering 3 outcomes there are $13^{2197}$ ways in which any set of the three individual weak orderings can be aggregated to determine social choice. Several of the rules however have been selected as being of particular interest and have been given special names. Examples have been called "dictatorships, simple majority vote and unanimous vote."

In each case a simple three person three outcome example serves to illustrate the rule. Referring to Table 1b assume that the preference orderings of the first, second and third individuals are respectively the first, fifth and fourth shown. This can be written as:

TABLE 2

| | | | | |
|---|---|---|---|---|
| $(R^1_1,\ R^2_5,\ R^3_4)$ | or | A | C | B |
| | | B | A | C |
| | | C | B | A |

Dictatorship. Under dictatorship only the preferences of the dictator determine the social choice. Thus in the example above if the first individual is dictator.

$$(R^1_1,\ R^2_5,\ R^3_4) \rightarrow R_1 \ .$$

Furthermore if the preferences of the second or third individuals change this will not influence the social choice.

Simple Majority Vote. If an outcome A is preferred over B by a simple majority or more this rule states that it is preferred by the group as a whole. Applying this to the example above we observe the well known Condoret voting paradox A is preferred to B by 2:1, B is preferred to C and C is preferred to A .

The simple majority rule for aggregating individual preferences in this case produces a societal preference structure which is intransitive, $A > B > C > A$ .

It is worth noting that the simple majority rule is not completely defined if the number of individuals is even, or if choice is not made in a pairwise fashion. The first objection may be overcome by the rule that any two outcomes for which the "vote" is equal are judged to be indifferent by society. If, as is frequently the case in elections the choice is among three or more candidates there may be no majority preferring any outcome over the others. In particular in the example above there would be one individual preferring A over B and C , one preferring B and one C . Hence a further specification of the rule is called for.

The "unanimous vote" rule says A is preferred to B by society if it is preferred by every member of society.* Applying this rule to the example no outcome is preferred over any other.

## 3. On Voting, Social Choice and Group Preferences

The Arrow result shows that given certain conditions on the aggregation of individual preferences into a social preference that no aggregation method which satisfies all of the conditions and produces a completely ordered social or group preference exists.[6]

Our concern here is not with this aggregation problem and the details of the special conditions suggested by Arrow;** but with voting and social choice. It is easy to misinterpret Arrow's discussion in terms voting and for this reason alone it is important to understand the important differences between the Arrow preference aggregation problem and voting or social choice. Arrow himself begins to make this distinction when he rules out all strategic behavior for his voters.[7] They are treated as dummies with preference orderings. A so-called simple majority vote rule for m outcomes consists of checking $m(m-1)/2$ pairwise comparisons of the positions of two items in n preference orderings.

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*This statement can be modified to include some members being indifferent.

**There are reasons to doubt the robustness of Arrow's result as it stands. However a critique of his result is not relevant here and is given elsewhere (see Shapley and Shubik).[8]

With adminable abstractness no restriction concerning the nature of the outcomes is made, i.e. no properties whatsoever of the distribution space are used. Furthermore no specification of process is given by Arrow. It is especially this last point which indicates that Arrow's work in fact does not deal with voting. In clear contrast Farquharson's work[9] does.

The political process called simple majority voting even at the most abstract level does not consist of a psychometric experiment where $n$ individuals have to make $m(m-1)/2$ pairwise comparisons which are then fed through some formula to determine a social preference ordering. The phenomenon called voting frequently involves choice among more than two items at a time. Furthermore the sequence of voting must be specified and may be of importance.

In general not all possible outcomes are brought to the vote. Most of the outcomes are sifted through in the preparation of the agenda. Even at the most abstract level, if we wish to study voting there are several natural ways to restrict the set of outcomes to be considered. There is a direct analogy which can be made both with economic theory and game theory. We may assume that the outcome will be Pareto optimal hence we restrict the domain of outcomes considered to those which are Pareto optimal. This is done in economic models of the price system and the core. Alternatively we can adopt a noncooperative approach assuming that the outcome chosen may not necessarily be Pareto optimal. If we do this, then the strategic model of the "game" or the voting process must be specified in detail. In economics this is done in the discussion of oligopoly.[10] In the study of voting Farquharson has adopted the noncooperative approach[11] and Riker a cooperative model.[12]

An example from Farquharson shows the way he specifies the voting process. Suppose a group has to vote on acquital or condemnation of an individual. If he is found guilty he may be exiled or put to death. There are three outcomes:

A - acquital

B - guilty and death

C - guilty and banish.

The process calls most naturally for two binary votes. This is shown in Figure 1a. Less naturally, but feasibly all three outcomes could be voted on simultaneously as is shown in Figure 1b.

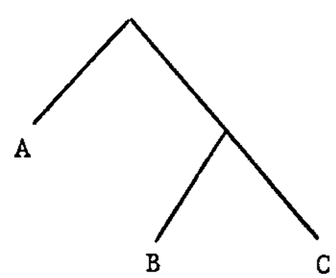



FIGURE 1a

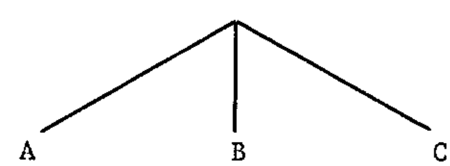



FIGURE 1b

In the remainder of this paper we adopt a cooperative approach, i.e. we limit our concern to Pareto optimal outcomes. This obscures some of the fine structure of process[13] but leaves in enough to capture some of the flavor of voting.

## 4. Pareto Optimal Outcomes, Imputations and Partially Ordered Social Preference

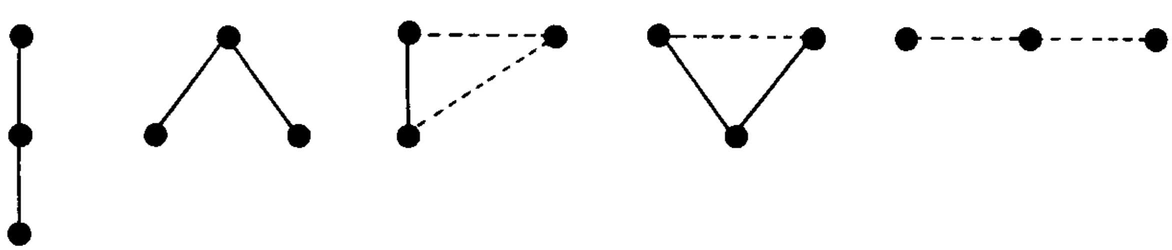
Suppose for example that we have a polity consisting of 3 individuals who must choose among 3 outcomes. For ease of calculation suppose each has a complete strong preference ordering. We can represent the preference ordering of any individual by a utility function such that any other utility function that can be obtained by an order preserving transformation will do equally well.

We only use utility functions because they afford us an easy notation and diagramatic representation. Let $u_j^i$ be the utility number attached by individual $i$ to outcome $j$. Call the three outcomes 1, 2, 3 .

An outcome is preferred by society over another outcome if it is preferred by every member. Thus in utility notation if outcome $j$ is preferred to $k$ :

$$u_j^i > u_k^i \quad \text{for} \quad i = 1, 2, 3 .$$

If this condition does not hold then we say that society as a whole cannot express a group preference between $j$ and $k$ . As has been noted in Section 2 there are 216 different preference structures that 3 individuals can have over three outcomes. They can be broken down into the group partial orderings shown in Figure 2.



6 cases 18 cases 12 cases 6 cases 174 cases

FIGURE 2

The first 6 cases give a complete ordering for society's preferences when all members are alike. The next 18 cases are almost as good in the sense that all individuals are similar in what they like best. Thus for 24 cases there is only one Pareto optimal outcome. For 18 more cases there are two Pareto optimal outcomes and for 174 there are three.

We call an outcome that is Pareto optimal an <u>imputation</u>. If we assume that society as a whole is cooperative to the extent that any outcome that is not Pareto optimal will be ruled out in the selecting of the agenda or in bargaining* then we may concentrate on the important point of how to select among those outcomes which are optimal.

If there is only one Pareto optimal point then "society speaks with one voice" and there is no allocation problem. If one individual is a dictator in the sense that he always gets to pick the imputation there is by definition no strategic problem. He picks the imputation which is the best in his preference ordering.

If there are two or more points on the Pareto optimal surface, i.e. if there are two or more imputations and if more than one individual has some control over the outcome then the problem of political choice becomes critical.

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*This is a large assumption and wipes out much of the analysis of process and political manoeuvre. In order to avoid this assumption we could extend the Farquharson analysis and concentrate on a noncooperative model with stress laid on the extensive form representation of a game and on dynamics.[14] Undoubtedly process has to be dealt with in detail for a full understanding of political choice, however as a first step we consider political choice as a cooperative game.

Before turning to the formulation of political choice and various types of voting as a cooperative game we may make use of diagramatic representations with utility scales in order to indicate the Pareto optimal set or set of imputations and to indicate the problems in defining sidepayments or transfers among individuals.

## 4.1. Utility Scales, Imputations and Transfers

Consider the preference orderings of 3 individuals for 3 outcomes shown in Table 2 and reproduced below in Table 3a. We may use the following utility values which reflect the orderings

$$u_1^1 = u_3^2 = u_2^3 = 2 \ ; \quad u_2^1 = u_1^2 = u_3^3 = 1 \ ; \quad u_3^1 = u_2^2 = u_1^3 = 0 \ .$$

Thus the outcomes A, B, and C can each be represented by a point in a three dimensional utility space as shown in Figure 3a. Their coordinates are given in Table 3b.

TABLE 3a

| | | |
|---|---|---|
| A | C | B |
| B | A | C |
| C | B | A |

TABLE 3b

| | |
|---|---|
| A | (2,1,0) |
| B | (1,0,2) |
| C | (0,2,1) |

Any order preserving transformation on the three utility scales will not distort the representation of the preference structure. Thus if all three individuals preferred A over B and B over C the coordinates A (2,2,2), B (1,1,1) and C (0,0,0) or A (3,8,27), B (2,4,9) and C (1,2,3) are equally valid representations. The first is shown in Figure 3b.

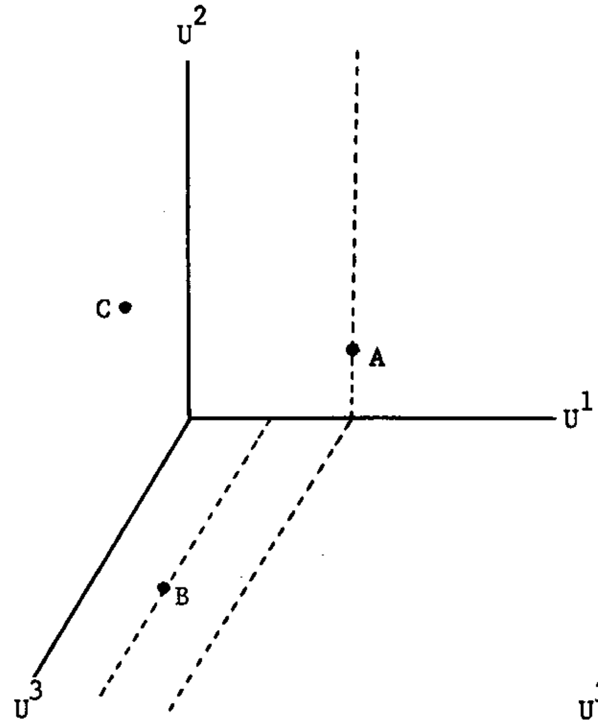


FIGURE 3a

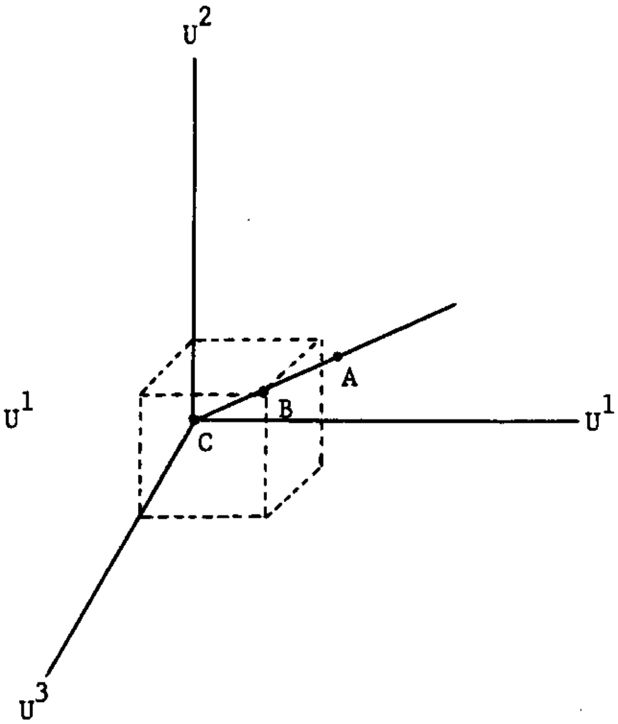


FIGURE 3b

If our model of political choice really has only a finite set of outcomes m then if there are n individuals the outcomes can be represented as m points in a space of n dimensions. The Pareto optimal outcomes or the imputation set is a subset of these m points. In the example in Figure 3a is consists of all three points A , B , C . In Figure 3b it consists only of A .

Frequently we talk about compensation, taxes and subsidies. In welfare economics elaborate schemes for compensation have been suggested.[15] On occasion it is suggested that the payoff be in money.

Once money or other payoffs are introduced a new set of modelling problems must be considered. Do we consider unlimited divisibility of money, continuously increasing preference for more money and enough money available for transfer to compensate or buy off any individual in any situation? Furthermore who holds the money initially?

If we make all of the assumptions above and specify an initial ownership of money and the existence of markets using money then the imputation set may be converted into a continuous surface. For example suppose that in the example in Table 3a outcomes A , B , and C stand for various states of society under which each individual has a million dollars then we may define an infinite set of outcomes which differ from A , B and C only in the distribution of $3,000,000. These sets of outcomes may give rise to a continuous Pareto optimal surface as shown in Figure 4.*

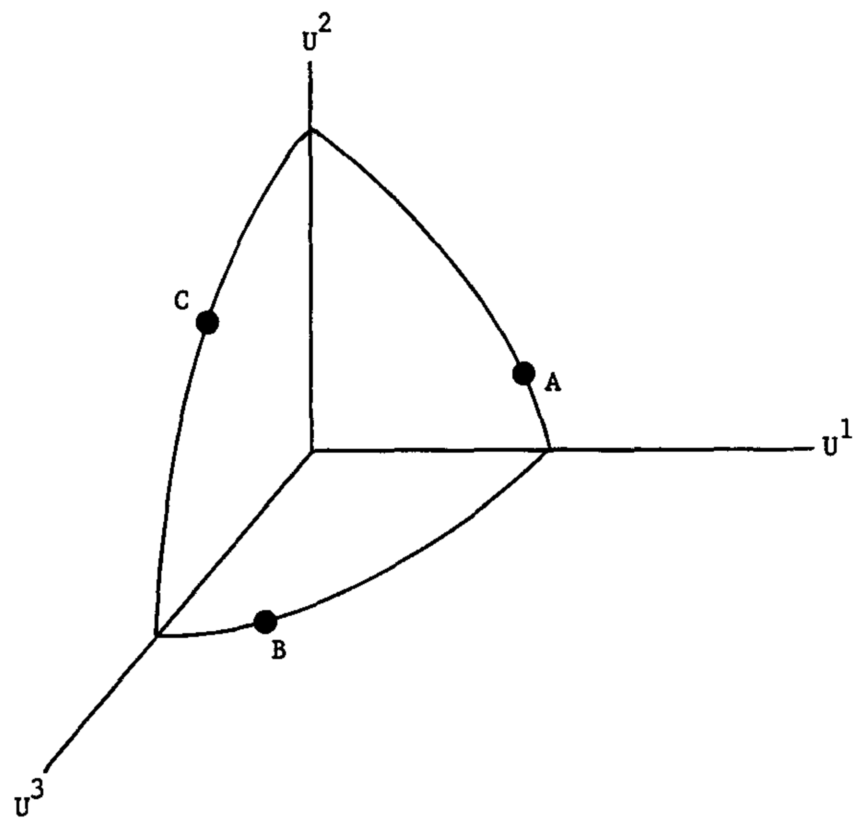


FIGURE 4

*This surface is not necessarily concave as it has been drawn.

In this paper only the finite sets of outcomes are considered not because they are believed to be more realistic, but because I wish to divide difficulties and to avoid confounding issues. The meaning of sidepayments and transfers and the role of money is subtle, and the careful modeling of political choice where sidepayments can be made requires detailed treatment* which is not given here. Instead here we confine ourselves to seeing how far we can go without assuming any form of sidepayments.

## 4.2. Political Choice, Economic Choice and the Number of Alternatives

If we regard the conscious decision to buy or sell some item as an economic choice then most people are engaged in many economic acts continuously. Furthermore even given a reasonable aggregation of economic goods and services into broad classes there are still many classes for many different economic purposes. Economic decisions may even involve formal procedures (such as linear programming and other planning methods) to search among the many alternatives.

The price system which is used for large parts of every economy in the world is used not because it produces a complete social ordering out of individual orderings but because it appears to be administratively highly attractive and produces a Pareto optimal outcome if certain conditions hold.[17]

In fact not only does an efficient price system not generate a complete social ordering, it is not necessarily unique. It is easy to construct markets with more than one competitive equilibrium. However if society were asked to vote among the outcomes obtained by a price mechanism a simple majority vote would further select among them.

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*See Shapley and Shubik.[16]

In society, voting is used for the election of candidates for office. The choice is almost always among 2 to 10 or 15 candidates at most. Voting is used for the election of boards of directors, trustees and other fiduciaries.

Voting is used to decide bond issues, and to approve or reject a usually small number of items (almost always in the range of 1 to 10) proposed to stockholders of corporations or members of clubs.

Bodies such as Congress, legislative assemblies and international committees may regularly process bills and other businesses by voting. However the myriads of possible outcomes are sifted down and modified so that very few come to the vote no matter what the voting procedure happens to be. An assembly in the course of a session will in general not exceed a few votes a day and a few hundred to thousand issues will be settled at the most in the whole session. For each issue the vote will be either binary (yes or no) or if amendments are included at most a few dozen outcomes can be considered.

In an assembly where a sequence of bills is being voted upon it may pay voters to exchange votes or "logroll." This possibility is considered in 6.3. It must be stressed here however that in spite of the attractive analogy between a market for goods and votes these are conceptually <u>fundamentally different</u> phenomena. The first by definition has a great deal to do with the properties of the commodities being traded. For a price system to exist, the economy as has been noted above,[18] must have many special properties. Logrolling can be defined, and is so defined in 6.3 in such a way that the phenomenon in no way need depend upon the specific physical properties or structure of the outcomes.

In summary the price system deals with a considerable part of economic choice. Voting deals with part of political choice, but the vote itself is almost always only a fraction of the process of an intermixture of political and economic decisionmaking. Neither the price system nor voting depends upon there existing a complete ordering of social preferences. In general the interesting question to ask of any economic or political choice process is: does it select a set of outcomes that are imputations; and does it select a unique imputation that is in some sense socially optimal or does it at least narrow the selection to only a few imputations?

## 5. On Voting Games

Limiting ourselves to that small subset of political decisionmaking involving a set of $n$ individuals using some type of voting mechanism to decide among a set of $m$ outcomes there are at least six qualitatively different types of models involving voting which can be considered. They are as follows:

(1) $n$ small and $m$ small and fixed in advance.

(2) $n$ small, $m$ small but somewhat enlargeable by the introduction of amendments according to well defined rules.

(3) $n$ small and $m$ relatively small but with a special structure defined. For example there are a series of $k$ votes to be taken in sequence hence $m = m_1 \times m_2 \times \dots \times m_k$. This may give rise to the possibility of logrolling.

(4) $n$ small and $m$ small but selected from a larger set by an individual or group empowered to fix the agenda.

(5) n with a special structure formally specified with subsets of varying size and compound voting process on m outcomes (usually few in number). The President, Senate and House considered as a compound voting unit serve as an example.*

(6) n large and m is small and fixed in advance. An example of this is a presidential or senate election after the candidates have finished compaigning.

These six categories are not meant to be exhaustive or even formally fully defined. They are merely suggestive of several important and different problems if we wish to even start to analyze models that can be recognized as reflecting pertinent features of that which is called voting.

Once more it must be stressed that this listing leaves out the consideration of an intermixture of economic and political problems involving various types of sidepayments such as noted by Shapley and Shubik[22] and Klevorick and Kramer.[23]

## 5.1. The Extensive, Strategic and Coalition Representations

Even though we limit our investigation to extremely formal models with n voters and m outcomes three highly different representations of a voting game must be considered. They can be illustrated by an example which serves at the same time to illustrate logrolling.

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*See Shapley's discussion of compound simple games, for example[19] or Luce and Rogow,[20] Shubik[21] and other writings where party structure or other formal organizational structure is introduced.

Consider a committee of three individuals with two items of business (I and II). The members are required to vote yes or no on each of the items. No amendments are to be entertained. There are four outcomes or final states. They can be denoted as A , B , C and D . These can be interpreted as follows:

A both items are voted for,

B the first item is voted for, the second against,

C the first item is voted against, the second for,

D both items are voted against.

Limiting the voting on each item to a simultaneous vote there are two extensive representations of the voting depending upon whether the sequence is I, II or II, I. The first is shown in Figure 5. The symbol

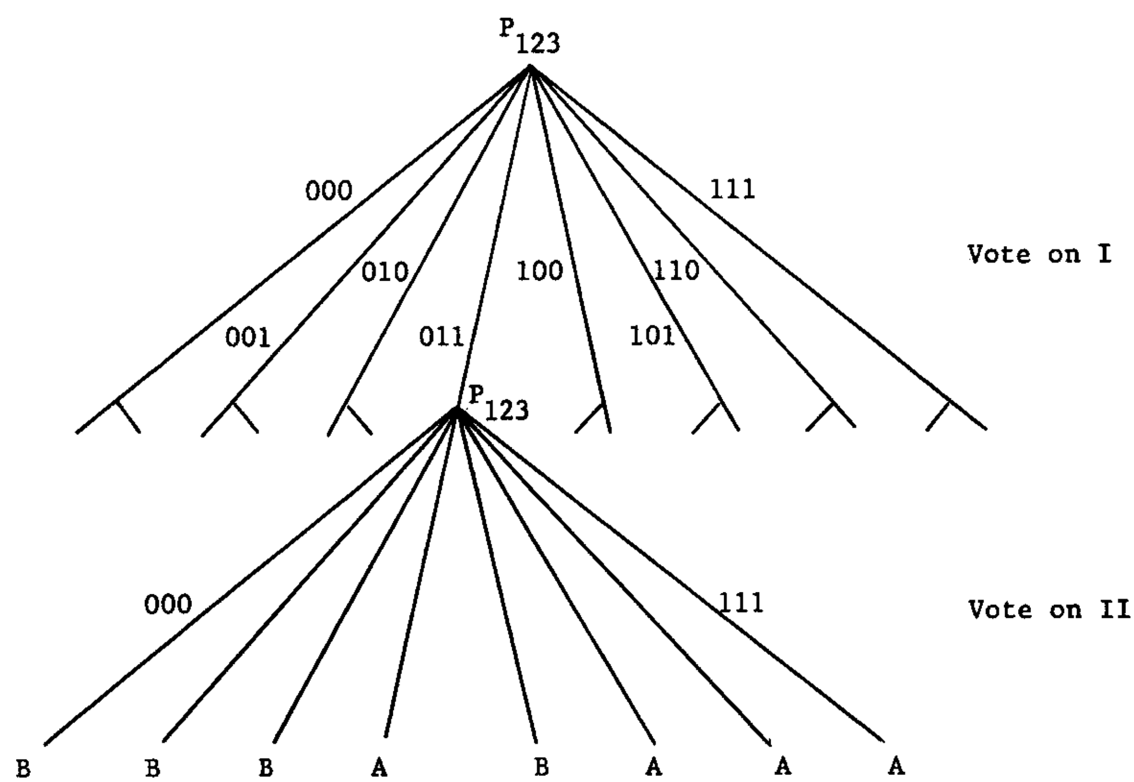


FIGURE 5

$P_{123}$ indicates that all three voters are required to vote simultaneously at the decision point with this label. The eight branches indicate all possible votes on item I. The label 000 indicates three votes against, 001 indicates that the first two voters are against and the third for. Only the branches emanating from the choice point reached by a vote of 011 on item I are shown in the second stage of Figure 5. The letters at the end of these branches indicate the outcomes if we assumed that the method of vote were simple majority. For example 011 followed by 000 means I passes and II fails, this yields outcome B. There are 64 end points. Associated with each is one of the outcomes A, B, C or D.

The _extensive form_ of a game shows its fine structure.* For many purposes of analysis we may not be interested in this level of detail. If, for example, we wish to plan a strategy for an individual all we are concerned with is the strategies which can be adopted by each and the outcome resulting from employing them.

The _strategic form_ of a game is concerned only with strategies and outcomes. In the example above each individual has 32 strategies. He can vote for or against item I then depending upon what the others have done he may vote for or against item II.

A strategy may be regarded as a complete set of instructions that could be left in advance and given to the chairman telling how you will vote under all circumstances. An example of one of the 32 strategies for Voter 1 is given:

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*In this discussion the details concerning the many different patterns of information which might be present have been supressed. See Shapley and Shubik.[24]

"Vote against I if the vote is 000 vote against II
if the vote is 001 vote against II
if the vote is 010 vote against II
if the vote is 011 vote for II
Otherwise vote for I."

Call the $i^{th}$ strategy of Voter 1 $r_i$ $i = 1, \ldots, 32$

Call the $j^{th}$ strategy of Voter 2 $s_j$ $j = 1, \ldots, 32$

Call the $k^{th}$ strategy of Voter 3 $t_k$ $k = 1, \ldots, 32$ .

A voting game in <u>strategic form</u> can be described by an <u>outcome function</u> which specifies the outcome resulting from the use of any set of $n$ strategies, one from each voter. In the example given let $w(r_i, s_j, t_k)$ be the outcome function. Any triple of strategies will produce one of the 4 outcomes A , B , C , D .

We could represent the outcome function in matrix form by a 3-dimensional matrix or cube with $(32)^3 = 32,768$ cells.

Instead of using an outcome function we could represent the game in strategic form by three <u>payoff functions</u>. Each function indicates the worth or value of any outcome resulting from the employment of any set of strategies.

Suppose that the preference orderings of the 3 voters for the 4 outcomes can be represented as in Table 4a. In Table 4b utility scale values

<u>TABLE 4a</u>

| | | |
|---|---|---|
| D | B | C |
| A | A | B |
| C | C | A |
| B | D | D |

<u>TABLE 4b</u>

| | |
|---|---|
| A | (2,2,1) |
| B | (0,3,2) |
| C | (1,1,3) |
| D | (3,0,0) |

have been assigned to the preferences of each voter for the outcomes. Thus we can represent the voting game by three payoff functions $P_1(s_i, r_j, t_k)$, $P_2(s_i, r_j, t_k)$ and $P_3(s_i, r_j, t_k)$ each of which selects a number 0, 1, 2 or 3.

Suppose that instead of voting on the two items in sequence each member of the committee had to submit his vote on each item at the same time. Thus a strategy for a voter no longer involves any contingencies. Each voter has only 4 strategies. All of the strategies for the first voter are shown below:

| | | | |
|---|---|---|---|
| $s_1$ | Vote for | I and for | II | 11 |
| $s_2$ | Vote for | I and against | II | 10 |
| $s_3$ | Vote against | I and for | II | 01 |
| $s_4$ | Vote against | I and against | II | 00 |

The extensive form is indicated in Figure 6. Here only 4 of the 64 branches have been drawn. The notation 00 00 00 indicates that each voter has selected as his strategy a vote against I and II. The outcome is indicated by D and the worth of the outcome to each voter is given by (3,0,0).

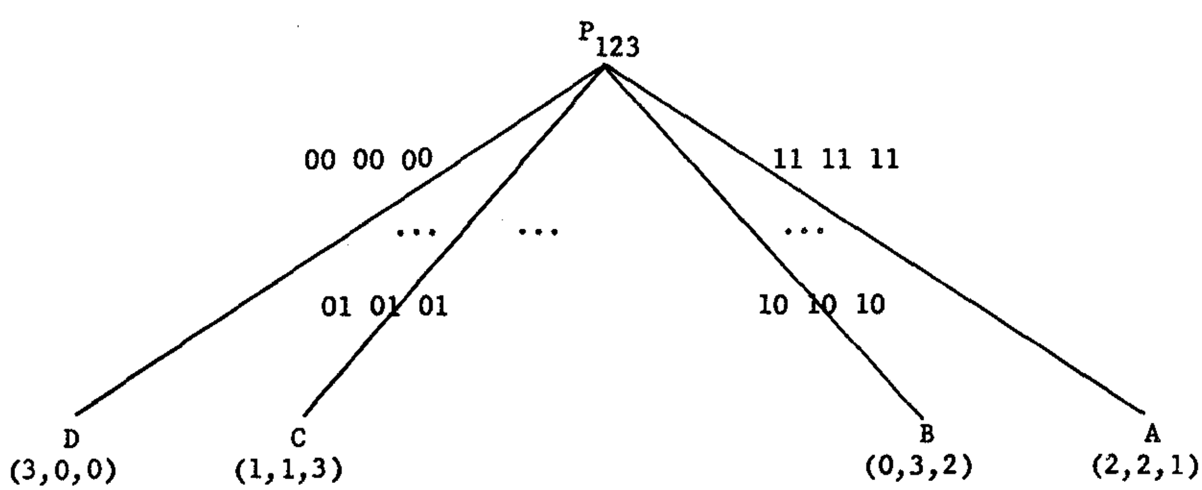


FIGURE 6

Comparing Figures 5 and 6 and the different methods of voting they represent we observe that the number of votes cast is the same, the number of outcomes is the same, the payoffs are the same, but the number of strategies is different and the information conditions are different. The voters know the outcome of the first vote before they vote on the second item in one case but not the other.

The analysis of Farquharson[25] makes use of the strategic form of a game and is directed towards finding sets of strategies which are in equilibrium. The major part of his analysis can be described as noncooperative and coalitions and "deals" are not considered.

The reader interested in the noncooperative strategic* analysis of voting should refer directly to Farquharson's book. The remainder of this article is devoted to cooperative solutions, coalitions, logrolling and deals.

Suppose that individuals can coordinate their strategies. If we wish to study the worth of coalitions we need a notation which enables us to describe that which can be achieved by each coalition or grouping of voters. For $n$ voters there are $2^n - 1$ nonempty coalitions which can be formed.

We denote a coalition consisting of voters $i$ and $j$ by $\{i,j\}$. Two functions, the first called the <u>characteristic function</u> and the second the <u>characterizing function</u> are introduced. The first is used to describe what individuals can obtain in games with sidepayments or transfers and the second in games without sidepayments. Although our concern is with

*Actually Farquharson does not consider all strategies, but limits his investigations to state strategies.[26]

the latter both are illustrated as the characteristic function provides some easy examples both here and when discussing solutions in Section 6.

Consider the following game. A referee tells 3 individuals that he will give them collectively \$30 if they tell him how they have agreed to split the money. Suppose, for the sake of simplicity that it is possible to make interpersonal comparisons and that \$1 has the same worth to each. Furthermore over the range of \$30 the marginal value of money to each is constant. In this case the distribution of welfare or "utility" and the distribution of money coincide. The Pareto optimal surface is the plane as indicated in Figure 7. Suppose further that the referee states that

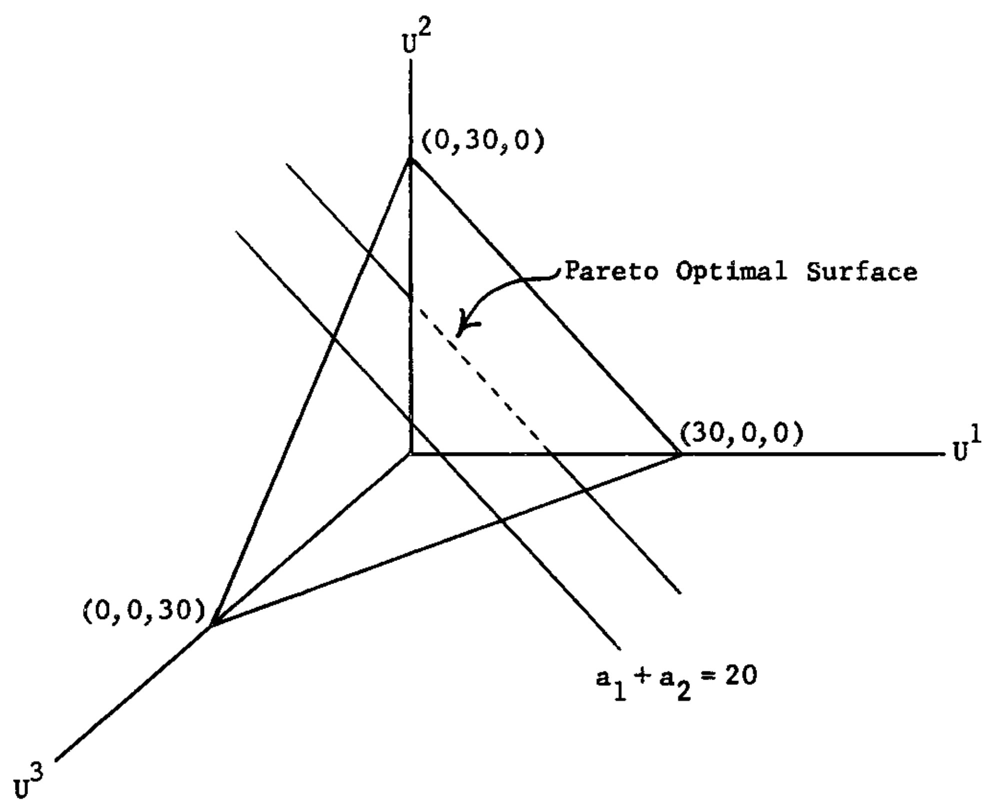


FIGURE 7

if all three cannot agree he will give \$20 to 1 and 2 if only they can agree how to split it; \$10 to 2 and 3 and \$10 to 1 and 3. An individual by himself will obtain nothing. The characteristic function for this game is shown in Table 5. As the money is transferable among the three players any triad of (nonnegative) numbers $(a_1, a_2, a_3)$ which add to 30, i.e. $a_1 + a_2 + a_3 = 30$, is an imputation.

TABLE 5

$$v(\{1\}) = 0 \qquad v(\{2\}) = 0 \qquad v(\{3\}) = 0$$

$$v(\{1,2\}) = 20 \qquad v(\{1,3\}) = 10 \qquad v(\{2,3\}) = 10$$

$$v(\{1,2,3\}) = 30$$

The characteristic function shows how much a group of players can guarantee for itself. In the example with transferable comparable payoffs this is a single number.

Reverting to our example in Table 4a we can construct a game without sidepayments. This game is <u>not</u> a voting game. The characterizing function of this game is shown in Table 6a and that for the simple majority voting game with the same preference structure is shown in Table 6b.

The new game is as follows. If all 3 players can agree they can achieve any of the outcomes A, B, C and D. If only 1 and 2 agree the referee rules that C is the outcome. For 2 and 3 the outcome is C; for 1 and 3 the outcome is B. If no one agrees the outcome is D.

TABLE 6a

$$V(\{1\}) = \{(1,1,3)\}, \quad V(\{2\}) = \{(3,0,0)\}, \quad V(\{3\}) = \{(3,0,0)\}$$
$$V(\{1,2\}) = \{(1,1,3)\}, \quad V(\{2,3\}) = \{(1,1,3)\}, \quad V(\{1,3\}) = \{(0,3,2)\}$$
$$V(\{1,2,3\}) = \{(2,2,1),\ (0,3,2),\ (1,1,3),\ (3,0,0)\}$$

The characterizing function contains a list of the payoffs at the outcomes which every group can obtain. Thus for example, the coalition of all can obtain any outcome, hence all 4 sets of payoffs are noted.

It is important to stress that there is a difference between what an individual or group can guarantee for itself and what it may reasonably expect to get. For example the third player cannot guarantee more than 0 by himself. He cannot force 1 and 2 into a coalition, however it is likely that they may form a coalition in which case he may obtain 3.

TABLE 6b

$$V(\{1\}) = \{(0,3,2)\}, \quad V(\{2\}) = \{(3,0,0)\}, \quad V(\{3\}) = \{(3,0,0)\}$$
$$V(\{1,2\}) = V(\{1,3\}) = V(\{2,3\}) = V(\{1,2,3\}) = \{(2,2,1),\ (0,3,2),\ (1,1,3),\ (3,0,0)$$

Table 6b displays the characterizing function for the simple majority voting game. We note that the coalitions split into "controlling" and "noncontrolling" in the sense that any group with enough members to control the vote can achieve any outcome. The individuals alone cannot guarantee for themselves more than their least favored outcome.

Summarizing we observe that the characteristic function is a set function (defined on all the coalitions or sets of players) which indicates the amount that any coalition can obtain. This amount is a single number and can be thought of as an "ideal money" or transferable utility which can be distributed among the members of a coalition in any way they see fit.

The characterizing function is similar to the characteristic function except that as no sidepayments are permitted instead of giving a single value for each set of players it indicates the set of achievable outcomes and their values.

## 6. Cooperative Solutions to Ordinal Voting Games

When the number of voters is relatively few and especially where the voters know each other, such as in the Senate it is reasonable to consider cooperation and the formation of coalitions prior to voting. Thus cooperative solution concepts appear to be reasonable, i.e. methods for predicting outcomes based on coalition formation (and implicitly high levels of communication). When the number of voters is extremely large and scattered the cooperative solutions may not be as reasonable. A noncooperative approach (Farquharson) or possibly a mechanistic approach stressing the properties of large numbers and low communication and information conditions appears to be called for.

### 6.1. Pareto Optimality and the Characterizing Function as Solutions

A solution to a voting game will be regarded as a prescription (normative) or prediction (descriptive or behavioral) which limits the outcomes to a set hopefully far smaller than the set of all possible outcomes.

Intuitively we may regard the strength of a solution to be inversely related to the size of the number of outcomes in the solution. Furthermore our faith or belief in a solution may be related to its sensitivity. If the model is changed only slightly we would hope that the solution set should not change violently.

Pareto optimality is an extremely weak solution nevertheless even this condition cuts down considerably on the possible outcomes. For example in Figure 4 the outcomes are a three dimensional set, the Pareto optimal points or imputations are only part of its surface.

If we wish to limit the possibilities still further without introducing further structure on the nature of the outcomes we can consider the power of coalitions. The ability to evaluate what coalitions can obtain adds considerably to the analysis of social choice.

The <u>characterizing function</u> provides the information concerning the effectiveness of coalitions. It refines the set of outcomes into subsets achievable by coalitions. The obtaining of the characterizing function involves two types of difficulties. They are definitional and computational. We do not discuss the computational problems, but the definitional ones are vital to the understanding of voting, minority rights and the political system.

There are several different ways in which a characterizing function can be defined each treats the threats between groups differently. Two ways are noted and a third is referred to.

<u>C1</u>: The maximum Characterizing Function

Tables 6a and 6b provide examples of the evaluation of what a group of individuals can obtain on the basis that individuals not in the coalition will band together to inflict upon them as much damage as possible. This is an extremely pessimistic assumption and in many instances is unreasonable, as was noted in evaluating $V(\{3\})$.

*C2*: "Inner-Directed" Characterizing Functions

In a voting game, if we assume that a controlling coalition is concerned with its own welfare and does not wish as a matter of policy to devote its energies to punishing those individuals not in the coalition we may be able to select among the attainable outcomes a smaller set representing the ones they may choose. An example will help to illustrate this. Table 7 is obtained from Table 4b and shows the choice problem facing each controlling coalition in the simple majority voting game.

TABLE 7

| Coalitions | | {1,2,3} | {1,2} | {1,3} | {2,3} |
|---|---|---|---|---|---|
| Outcomes | A | (2,2,1) | (2,2,-) | (2,-,1) | (-,2,1) |
| | B | (0,3,2) | (0,3,-) | *(0,-,2) | (-,3,2) |
| | C | (1,1,3) | *(1,1,-) | (1,-,3) | (-,1,3) |
| | D | (3,0,0) | (3,0,-) | (3,-,0) | *(-,0,0) |

(a) Subgroup Optimality

Looking at Table 7 we see that there are several ways in which we can limit the outcomes selected by a coalition. The weakest and most natural calls for the application of Pareto optimality to the subgroup. As is shown in Table 7 for {1,2} this cuts out C for {1,3} B and for {2,3} D . The characterizing function that replaces 6b will have the outcome sets for the two person coalitions modified.

(b) Subgroup Equity or Symmetry

A rule which radically cuts down the alternatives is that any group must limit itself to those outcomes for which the ranking of the least preferred outcome for any coalition member is maximized.+ This reflects a central tendency or equal rewards policy to coalition members. Applying this to Table 7 this policy selects A or C for $\{1,2,3\}$, A for $\{1,2\}$, C for $\{1,3\}$ and B for $\{2,3\}$. The complete characterizing function is shown in Table 8.

TABLE 8

$$V(\{1\}) = \{(0,3,2)\}, \quad V(\{2\}) = \{(1,1,3)\}, \quad V(\{3\}) = \{(2,2,1)\}$$
$$V(\{1,2\}) = \{(2,2,1)\}, \quad V(\{1,3\}) = \{(1,1,3)\}, \quad V(\{2,3\}) = \{(0,3,2)\}$$
$$V(\{1,2,3\}) = \{(2,2,1), (1,1,3)\}$$

A third way of defining the characterizing function and accounting for threats* has been suggested by Harsanyi,[27] however, his method applies to situations in which individuals are assumed to have utility scales determined up to a linear transformation.

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*It is of interest to note in attempting to evaluate threats in calculating the characterizing function there are four natural divisions of cases to be considered. Let $s$ be the coalition and $\bar{s}$ set of players not in the coalition. Once the coalitions have been formed then: (1) the acts of neither $s$ nor $\bar{s}$ influence the payoff to the other; (2) the acts of $s$ influence $\bar{s}$'s payoffs but not vice versa; (2) the acts of $\bar{s}$ influence $s$'s payoffs but not vice versa; (4) the acts of $s$ and $\bar{s}$ influence each other's payoffs. Market games[28] which feature prominently in economic applications of game theory fall into Case 1. Voting games in Cases 2 and 3 and game theoretic models of external economies and diseconomies into Case 4.

+This implies a willingness to make interpersonal comparisons of rankings.

## 6.2. Cooperative Solutions: The Core

We may take the characterizing function as the end of our analysis or use it as a basis for further calculations. Two solutions based on the characteristic function are the core[29] and the bargaining set.[30] The first is discussed here.

The core can be immediately illustrated from the example in Table 5 and Figure 7. It consists of that set of imputations that reward the individuals in such a manner that no coalition can form and guarantee all of its members more than they obtain at an imputation in the core.

When sidepayments can be made and the characteristic function can be used the definition of the core is that set of imputations which are undominated by any subset of players.

In the three person game shown in Table 5 an imputation is a triplet of numbers $(a_1, a_2, a_3)$ such that $a_1 + a_2 + a_3 = 30$. Table 9a expresses the core conditions for the specific game and 9b shows them for the general three person game. It is easy to see that an imputation such as (10,11,9) is

TABLE 9a

$$a_1 \geq 0, \quad a_2 \geq 0, \quad a_3 \geq 0$$
$$a_1 + a_2 \geq 20$$
$$a_2 + a_3 \geq 10$$
$$a_1 + a_3 \geq 10$$
$$a_1 + a_2 + a_3 = 30$$

TABLE 9b

$$a_1 \geq v(\{1\}), \quad a_2 \geq v(\{2\}), \quad a_3 \geq v(\{3\})$$
$$a_1 + a_2 \geq v(\{1,2\})$$
$$a_2 + a_3 \geq v(\{2,3\})$$
$$a_1 + a_3 \geq v(\{1,3\})$$
$$a_1 + a_2 + a_3 = v(\{1,2,3\})$$

in the core, as is (6,16,8) . In Figure 7 the line $a_1 + a_2 = 20$ is drawn on the imputation set. Any imputation to the left of it has $a_1 + a_2 < 20$ hence will be dominated by the set $\{1,2\}$ and cannot be in the core. In Figure 8 the imputation triangle shown in Figure 7 in three dimensions is drawn again in two dimensions. The core consists of the imputations in the cross hatched area.

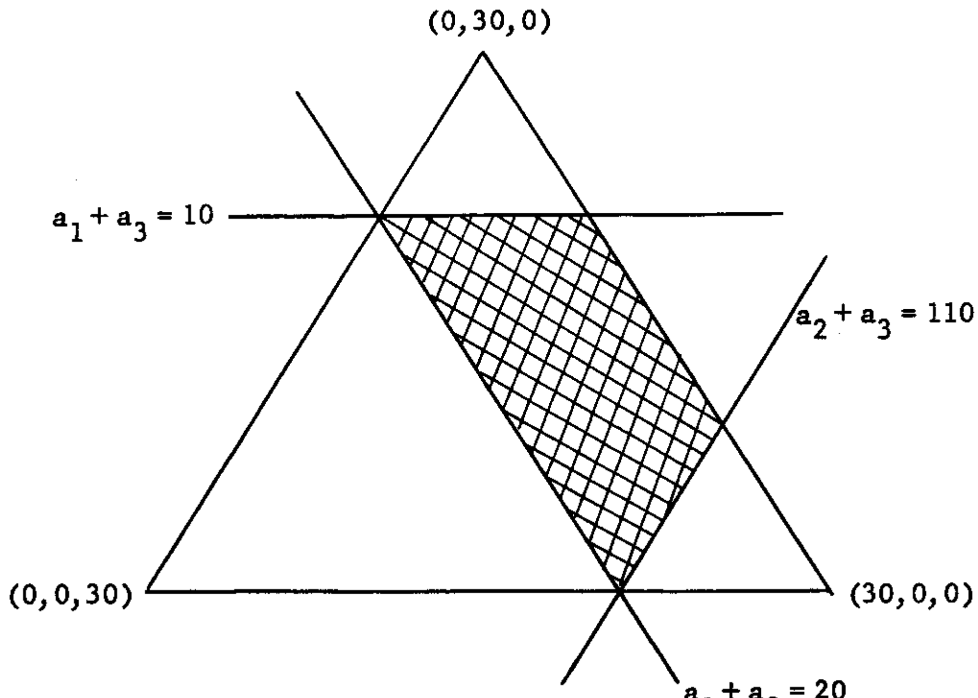


FIGURE 8

The definition of the core for games without sidepayments is a direct extension of the definition for games with sidepayments.*

As the core consists of those outcomes which are socially optimal and which cannot be effectively challenged by any coalition even though it may contain many outcomes it offers a compelling way to rule out many imputations.

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*See Shapley and Shubik.[31]

In economic game theory it can be shown that a necessary condition for the existence of a price system is that the economy formulated as a game has a core.[32] A game without a core cannot have a price system.

## 6.3. Voting Games, Minority Rights, Logrolling and the Core

Two distinctions can be made in the use of voting as a method of social choice. Voting may be used either to select a specific outcome from a set of outcomes or it may be used to decide who will have the power to select the outcomes. The vote on a bond issue is closer to the first while vote for a representative is closer to the second.

The simplest type of voting game is known as a simple game.[33] The individuals are assumed to vote for or against a single item. All coalitions can be divided into "winning" and "losing" coalitions. Those which have enough votes to control the outcome and those which do not.

If games with sidepayments are considered an extremely simple form of characteristic function can be defined for the single vote. This is shown in Table 10. For the simple majority vote $s$ is losing if the number

TABLE 10

$$v(s) = 0 \text{ if } s \text{ is losing}$$

$$v(s) = 1 \text{ if } s \text{ is winning}$$

of members in $s$, $|s| < n/2$ and is winning* if $|s| > n/2$. The characteristic function can be illustrated by the diagram shown in Figure 9a. As the game is symmetric with "one man one vote" any coalition with the same number

*A special rule must be given to resolve possible ties such as those that might arise when a simple majority is required and $n$ is an even number.

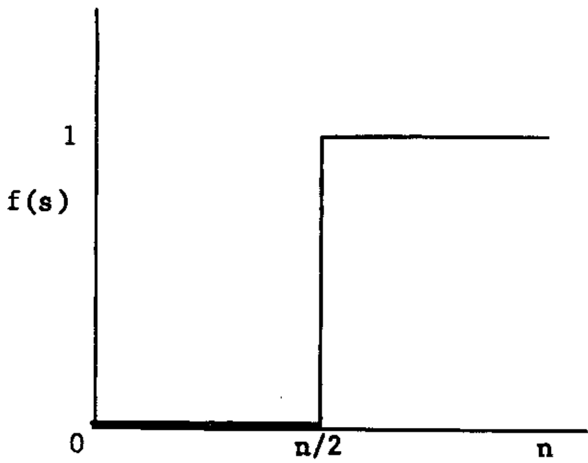


FIGURE 9a

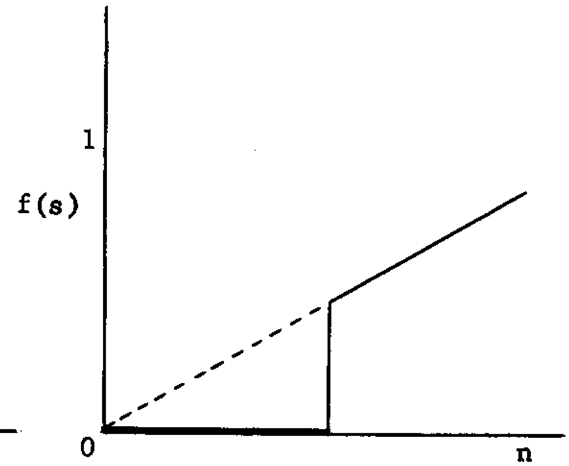


FIGURE 9b

obtains the same amount hence the number f(s) stands for the amount obtained by s individuals regardless of who they are.

This simple crude model has the winners take all. There is no protection of the minority. Losers lose all. Any simple game with 3 or more voters and without individuals with veto power has <u>no imputations</u> in the core. Furthermore (as has been discussed in its political context by Riker)[34] the winning coalitions of most concern are the <u>minimal winning coalitions</u>. These coalitions have no members who are not vital to victory, hence they contain the fewest among whom the spoils must be divided.

When sidepayments or adjustments among voters is not possible the tyranny of the majority becomes even more pronounced. Given n voters and only two outcomes there will be $2^n$ preference orderings possible and hence $2^n$ different simple majority games. If for example the two outcomes are valued (1,1, ..., 1,0,0,0, ..., 0) and (0,0, ..., 0,1,1,1, ..., 1) where slightly more than half favor the first outcome, that is what will be voted for. Without sidepayments or the possibility of adjusting outcomes it may not be possible to protect a minority. In the example given above if a member of the minority were given a veto this would then amount to tyranny of the minority.

When sidepayments are possible it is possible to protect minority rights. A clear practical example of this is illustrated by the characteristic function of the game shown in Figure 9b. Here a majority vote gives those individuals control over the corporation however if they try to pay out the proceeds which they control they must pay all stockholders in proportion. Thus a minimal winning coalition of say $50 + \epsilon\%$ may control the corporation but it can only take $50 + \epsilon\%$ of the proceeds.*+

If there are many outcomes and a voting game with three or more voters is used to decide what group controls choice then the resultant game may not have a core unless there are veto players.

The unanimity rule has every voter as a veto player and the core is the whole set of imputations.

Care must be taken to specify exactly what is meant by a veto player and to differentiate him from a dictator. A veto player is one who must be included in every winning coalition. If he forms a winning coalition by himself then he is a dictator.

If n-1 players are veto players then the remaining player is either a veto player or a dummy.

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*At least in theory it cannot take more. In practice there are many ways a control group can "rob the till" by using the camouflage of the accounting system, by paying themselves wages for services of dubious worth and so forth.

+The important feature of this example was that it was possible to add rules that converted a voting game without a core into one with a core.

On Logrolling

Suppose that a group of $n \geq 2$ voters none of whom is a dummy (i.e. each has a vote that counts in at least one situation) must vote in sequence on m bills and each vote is "accept" or "reject." Furthermore suppose that no player has a veto and the vote is taken in the same manner on each bill.

Allowing for logrolling, i.e. for agreements in which players exchange votes on different bills, in general no market or price system for votes can exist.

This game is strategically equivalent to a game with n voters voting on $2^m$ final outcomes. Any set of coalitions which are controlling for a single bill are also controlling for all m bills. The compound game will rarely contain a core* and hence cannot have a price system.

The existence of a core is a necessary condition for the existence of a price system. It is not sufficient. For a game to be a market game it must be "totally balanced," i.e. not only must the n person game have a core but so must all subgames which can be formed from the game. An economic example serves to illustrate. Suppose 4 traders each with an initial endowment of goods form a market. This can be characterized by a 4 person game. If any one trader is excluded we can describe 4 3-person games which are related to the 4-person game. In total there are $2^n - 1$ subgames of an n person game. Each of these must have a core for the game to be totally balanced.

*The existence of a core depends heavily upon how the power of coalitions is modelled. This depends upon the voting rules and the "threats" available to the players.

For a voting game with 3 or more individuals to be a market game every subgame must have at least one veto player.

## 7. Concluding Remarks

It is easy to make casual analogies between "the market for votes" and the market for goods. It is tempting to cast problems of voting and political process in economic terms.

Unfortunately a scrutiny of the basic structure of individual trading of individually owned items and of political decisionmaking show fundamental differences. The economic models are characterized by games with cores and by games which can be naturally described by a characteristic or characterizing function ("C-games").[35] The political models involving voting generally do not have a core unless veto players exist or minority rights are appropriately protected. Furthermore they are sensitive to the threat structure of the voting process (i.e. they are not necessarily C-games). This is another manifestation of the basic political problem of minority protection.

In order to obtain a market for votes considerable structure must be placed on the system being considered. An example of the type of structure is shown in Figure 9b.

It is possible to define voting games making use of only individual preference orderings for outcomes. Thus when there are $m$ outcomes and no sidepayments an <u>ordinal simple game</u> has a characterizing function which contrasts with a simple game as is shown in Table 11.

TABLE 11

| | | |
|---|---|---|
| V(s) = | The set of payoffs associated with all outcomes | if s is winning |
| = | The set of minimal payoffs* for each member of s | if s is losing |

The core and bargaining set can be considered for these games. Thus without making further assumptions concerning the measurability of utility or sidepayments some further analysis is possible.

The phrases winning or losing used in the description of coalitions in a simple game are best replaced by controlling and noncontrolling in an ordinal simple game. This helps to make the distinction between the ability to control the outcomes without being able to adjust payoffs to members of the coalition.

The value of a game[36] is a solution that has been defined in several different ways, but which requires at least the measurability of utility. An ordinal criterion for equity might be considered in terms of minimizing the difference in rank ordering of individuals for any outcome. However at this time no fully satisfactory ordinal version of the value solution exists.

The Arrow analysis, although it did not deal with voting or group decisionmaking, nevertheless had an important message for the study of these topics. That is that very little can be said if virtually no structure is given to the problem being analyzed. Rabbit stew requires at least some rabbit!

*See 6.1 for alternative definitions.

Generality can easily be specious generality. And even though Arrow himself did not suggest it, it is easy to be misled into believing that he somehow presented the right model for studying voting or group decision-making.

In particular chance is a fact of life and should be considered. With the introduction of chance (or other psychological considerations) the postulation of the measurability of utility is reasonable.

The special role of money needs to be recognized and the settlement and sidepayment structure must be specified.

In even an elementary attempt to consider the strategic aspects of voting and other political decisions the need for an adequate specification of threat conditions and protection of minority rights emerges.

The role of the effect of numbers in a polity needs to be taken into direct consideration. The definition of limiting behavior of solutions to models of voting formulated as n-person games provides a means to do this.

Finally concerning the political process of which voting is a part it may well be that the emphasis of our analysis might be directed more fruitfully towards the process that selects and modifies the alternatives which are voted on, than voting itself.

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