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A THEORY OF MONEY AND FINANCIAL INSTITUTIONS

PART IX

COMPETITIVE AND CONTROLLED PRICE ECONOMIES: THE ARROW-DEBREU MODEL REVISITED

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COMPETITIVE AND CONTROLLED PRICE ECONOMIES: THE ARROW-DEBREU MODEL REVISITED

by

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I. INTRODUCTION

In an article entitled "A Social Equilibrium Existence Theorem," by Gerard Debreu [1] and in a subsequent article entitled "Existence of an Equilibrium for a Competitive Economy" by Kenneth Arrow and Gerard Debreu [2], the existence of a competitive equilibrium in a closed economy was examined in terms of the existence of a noncooperative equilibrium point in the sense of Nash [3] in a game related to the closed market model. The apparent motivation of the authors for the use of the noncooperative game model was the mathematical convenience provided by the existence proof for a noncooperative equilibrium point which in this instance could be shown to also be a "competitive equilibrium" point in the economic model they constructed.

In his book Debreu [4] subsequently abandoned the artifact of using the noncooperative game to establish the existence of a price system in a closed economy and provided an elegant direct proof.

It is the purpose of this article to point out an interpretation of the Arrow-Debreu noncooperative game model as a model of a centralized price governed economy. Furthermore it is noted that there are some difficulties concerning the meaning of payoffs and the implications of the Arrow-Debreu definition of strategy which was not fully discussed in the original formulation but which must be cleared up before a satisfactory economic interpretation of their model is to be given. Finally their model is contrasted with a different noncooperative game model in which all traders are treated symmetrically [5]. It is argued that this model in contrast with the Arrow-Debreu model stresses decentralization and the emergence of a competitive price as a phenomenon associated with increasing numbers of traders rather than price as a given supplied by a centralized agency.

In a previous paper [6] the paradox of the role of the price system in a centralized and in a competitive system was noted generally. Here we may examine the specific differences in the specification of the strategy spaces.

2. THE ARROW-DEBREU MODEL

In an extremely ingeneous paper Arrow and Debreu establish the existence of a competitive equilibrium for a closed economy by defining an "abstract economy" which is presented as a generalization of that of a game [7] and then establishing that the noncooperative equilibrium point of this game is the competitive equilibrium of the closed economy.

The criticism offered here is not directed against the elegant mathematical argument of the authors but it is directed towards some critical features in the modeling of the economy and in interpreting its meaning.

2.1 The Players and Payoffs

The abstract economy considered by Arrow and Debreu has $m + n + 1$ players [8]. These are divided into three classes:

m consumption units
n production units and
1 market participant.

Each consumer i strives to maximize his payoff or welfare which is given by a utility function $u_i(x_i)$ where x_i is a consumption chosen by i that is feasible given his budget constraint.

Each producer j strives to maximize his profits which are given by py_j where y_j represents his production plan.

The market participant attempts to maximize a quantity pz where $z = x - y - \xi$ can be interpreted as having as its components the excess of demand over supply for the various commodities. The symbols x, y and ξ can be interpreted respectively as the vectors representing demand, supply from production and supply from the initial holdings of the consumers [9]. The strategy of the market player is to name a vector of prices p which has as its components a price for each commodity in the economy [10]. All prices are constrained to be nonnegative.

The budget constraint of any consumer includes not only the market value of his initial bundle of resources but also a share in the profits made by the production units. Consumer i obtains a percentage α_{ij} of the profits of producer j.

A consumer unfortunately cannot know if a strategy he wishes to employ is feasible without being given his budget constraint which depends upon the choices of the other players. The game is not completely defined unless we specify the information conditions and the order of moves. We must also specify what the payoffs are when the system is not in equilibrium? Who gets paid what if a consumer names a nonfeasible consumption? Is there a rationing scheme or an iterative process which corrects his error?

Arrow and Debreu were able to sidestep the difficulties associated with the details noted above. They were interested in establishing the existence of the competitive equilibrium as a noncooperative equilibrium, and there are many models with the added details specified for which their theorem is true. Our interest is in defining and interpreting some of these models.

2.2 The Extensive Form and the Strategy Spaces

A game in strategic form played noncooperatively is played by all players selecting their strategies simultaneously. After this has been done the payoffs are calculated and paid to all the players. An example of the extensive form [11] that appears to be associated with the strategic model of Arrow and Debreu is given in Figure 1. It is drawn for the case

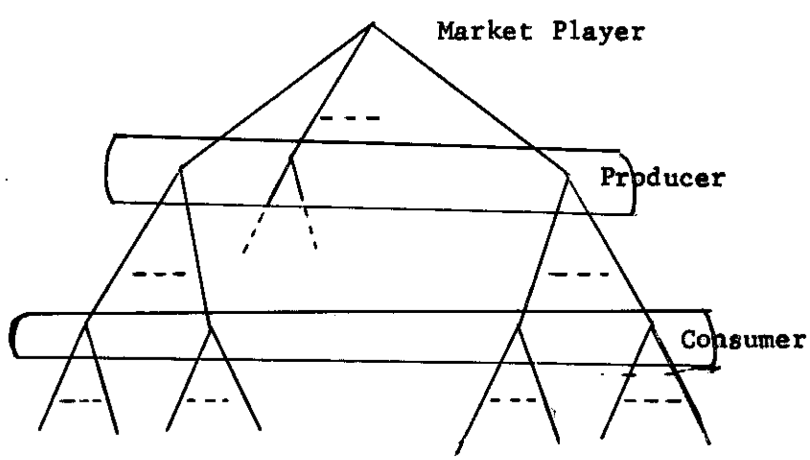

Figure 1

where $m=1$, $n=1$ and there is one market player. In this representation the market player moves first. The producer does not know what the market player has chosen as the price system. He then moves in ignorance. At the level of the consumer the ignorance is compounded. He knows neither what the "price board" nor the producer has done. The lack of information suffered by the producers and consumers is indicated at each level by the single information set encircling all choice points at that level. Thus the producer and the consumer are unable to distinguish among any of the states in which they may be.

Because the consumer is unable to check to see if a particular consumption plan is feasible without knowing his income we must define his strategy in such a way that given his action a feasible outcome is attainable. One way of doing this is by having the consumer announce a plan which covers the contingencies in such a way that some set of actions is always feasible. For example he might specify that his choice is $x_i = f_i(y, p)$. He does not know in advance what y and p are but all possibilities have been taken into account. Alternatively instead of having the consumer select the function f_i we may specify as part of the rules that there is some mechanism or transformation which permits him to adjust an intended consumption vector x_i so that it is transformed into a feasible consumption once the referee has been informed of everyones' strategies. This is essentially the approach adopted by using an "abstract economy".

"An abstract economy, then, may be characterized as a generalization of a game in which the choice of an action by one agent affects both the pay-off and the domain of actions of other agents" [12]

The abstract economy provides us with a way of looking at a whole class of adjustment methods which select an attainable outcome for every combination of strategies by the players.

In particular it is worth noting that if we considered a related non-cooperative game in which the market player not only moved first, but announced his prices and the producer moved second and announced his production this game would have the competitive equilibrium outcome as a result of the players being in noncooperative equilibrium. The extensive form is shown in Figure 2.

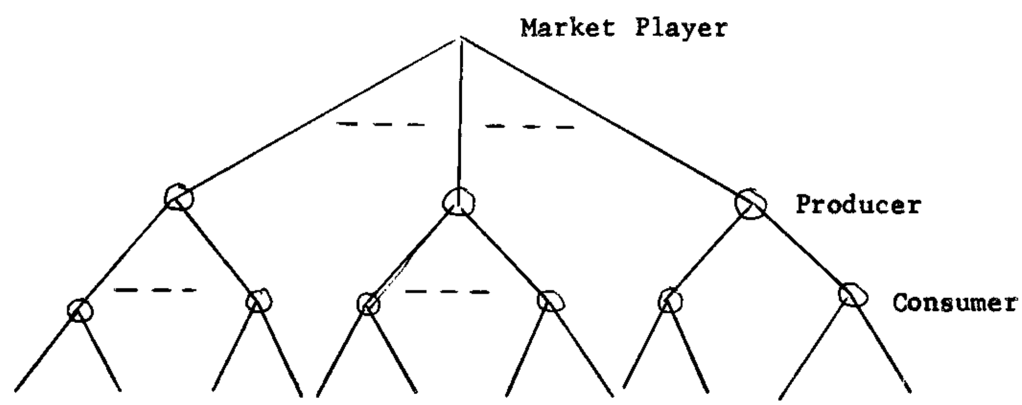

Figure 2

An interpretation of this model can be made as a price controlled economy where a central planning board broadcasts prices to producers and consumers who are constrained to take these prices as given.

2.3 Rationing and Money

Suppose that the market player were to announce a set of prices that are distinctly nonoptimal, what happens? We know that in this instance

$pz < 0$, i.e. he makes a loss. Furthermore there may be excess demand or supply in some markets. Some method must be selected for rationing goods in short supply. Many methods of rationing could easily cause losses to producers. For example in a centralized economy if there is a misallocation of a joint input although everything else may be present production may be reduced to zero and large losses will result.

We must also account for excess supply. What happens to the profits of producers who cannot sell their product? How are sales rationed among overproducing firms?

If we do not impose conservation conditions on the economy the above problems can be taken care of easily. We may assume that excess demand is taken care of by "imports" and that excess supply is wasted. We must also assume that there is a reserve of some sort of money in order for those who have made losses to settle up their debts.

At equilibrium all markets balance in terms of the supply of and demand for physical resources. Furthermore the accounts balance. Away from equilibrium neither of these conditions hold. However in order to completely define the model as a game the state of the system away from equilibrium must be completely specified.

Given that the Arrow-Debreu model is that of a well-defined game, what does it appear to be a model of? One interpretation is that it could be regarded as a model of a planned economy where a central board uses its power to announce prices as a means for achieving a certain amount of decentralization*

*I am grateful to Herbert Scarf for pointing out this interpretation.

Furthermore when the board blunders it can modify the effect of its actions by using its monetary reserves and engaging in external trade. This is by no means the same as a closed competitive economy. This is discussed further in 3.2.

2.4 Replication; Many Market Players?

It is a commonplace assumption in economic theorizing that the number of competitors in a market is related in some manner to the emergence of a competitive price system. It is argued that as the number of traders increases the power of the individual relative to the market as a whole attenuates.

One way in which the increase in traders can be studied is by investigating a sequence of replicated games where, for example, if the original game contains m consumers and n producers we consider a new game with km consumers and kn producers where each individual has been replaced with k exactly like him in all respects. We may study the behavior of a solution such as the noncooperative equilibrium for $k = 1, 2, 3, \ldots$. Examples of the study of the behavior of the core of a market game and its relationship to the price system have been given by Edgeworth [13], Shubik [14], Debreu and Scarf [15] and many others subsequently.

When a game is replicated care must be taken to see that each trader still has the same set of strategies. It is possible that in a sufficiently large market the strategic worth or power of an individual attenuates. But this by no means implies a change in his available moves. For example an individual may be able to offer up to 100 tons of wheat to the market.

If there is only one other trader with 100 tons, his power is extremely different from the market in which there are ten thousand like him. In both instances his set of available moves are the same.

When there is more than one type of player we may wish to consider several different ways of constructing larger but associated games. For example suppose that we have a game with three types of players with their original numbers being m_1, m_2 and m_3 . The k^{th} full replication would be a game with km_1, km_2 and km_3 players respectively. We might under some circumstances want to study the behavior when the size of the different groups is varied.

The original Arrow-Debreu market has three types of players with their numbers being m, n and 1 . A full replication would have us consider a market with km consumers, kn producers and k market players.

It is hard to attach much meaning to their model with more than one market player. Formally there is no difficulty in defining the model. However if more than one group is announcing all prices we must specify the mechanism whereby the same product is sold at different prices. This problem was recognized by Edgeworth [16] and is central to the theory of rationing. Levitan [17] has considered this in terms of oligopolistic competition.

The Arrow-Debreu market player was introduced by them merely as a formal device so that a certain mathematical apparatus could be utilized for establishing the existence of an efficient price system [18] hence the observation that when one considers a replication of their game increasing the number of market players does not seem to be reasonable is not a criticism of the paper

per se but an indication that there may be a noncooperative game model closely related to their's which provides some insight into the running of an efficient price-controlled economy.

3. Two Noncooperative Market Models

In a previous paper a closed model of an economy viewed as a noncooperative game with one type of player was presented [19]. In this model the emergence of an efficient price system was dependent upon the number of traders and the availability of a commodity money. A brief description of this game is given in 3.1 and this is contrasted with a game with a market player in 3.2.

3.1 The Competitive Market

Let there be n traders and $m+1$ commodities in an economy. Each trader i has a utility function $\varphi_i(x_1^i, x_2^i, \ldots, x_{m+1}^i)$ where $i = 1, 2, \ldots, n$.

A trader i has an initial endowment of $A^i = (A_1^i, A_2^i, \ldots A_{m+1}^i)$. His strategy is a vector of $2m$ components consisting of decisions to buy or sell (but not both simultaneously) in the first m markets.

The $m+1^{st}$ commodity, although it is not distinguished in the utility functions is distinguished in the strategies of the traders. It is used as a commodity money. All purchases must be paid for in this money. We may fix its price at $p_{m+1} = 1$.

If a trader i decides to purchase commodity j he offers an amount d_j^i of commodity $m+1$ for it. If he decides to sell commodity k he offers an amount q_k^i to the market. The price of commodity p_j where $j=1, 2, \ldots m$ is given by*:

$$p_j = \sum_{k=1}^{n} d_j^k \Big/ \sum_{s=1}^{n} q_j^s \quad . \tag{1}$$

A trader's purchases are constrained by his supply of commodity money

$$\sum_{j=1}^{m} d_j^i \leq A_{m+1}^i \quad . \tag{2}$$

If an individual i bids d_j^i for commodity j he obtains:

$$x_j^i = d_j^i \,/\, p_j \quad . \tag{3}$$

Suppose that:

$$\delta_j^i = \begin{cases} 0 & \text{if } d_j^i \geq 0 \\ 1 & \text{if } q_j^i > 0 \ . \end{cases}$$

then the payoff to a player can be written as:

$$\varphi_i(\{A_1^i - \delta_1^i q_1^i + (1 - \delta_1^i)x_1^i\}, \ldots, A_{m+1}^i - \sum_{j=1}^{m} d_j^i + \sum_{j=1}^{m} p_j q_j^i) \ . \tag{4}$$

The prices are selected in such a manner that markets always clear. No credit is granted in this economy. The income obtained from the sale of goods is not available for purchasing during the same period.

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*A special definition is needed if $\sum_{s=1}^{n} q_j^s = 0$.

3.2 The Price Control Market

Let there be n traders and $m+1$ commodities in an economy. Each trader i has a utility function $\varphi_i(x_1^i, x_2^i, \ldots, x_{m+1}^i)$ where $i = 1, 2, \ldots, n$. A trader i has an initial endowment of $A^i = (A_1^i, A_2^i, \ldots, A_{m+1}^i)$.

In this model no commodities are distinguished in the utility functions. Furthermore no commodity is used as a money. The $m + 1^{st}$ commodity is used as numeraire hence $p_{m+1} = 1$.

A differentiated $n + 1^{st}$ player is introduced. He may be called "the price-planning agency". His strategy is to name a vector of prices $p = (p_1, p_2, \ldots, p_m)$.

The information conditions in this game are such that the player P, the price-planning agency moves first and announces the prices to all. Then all traders move simultaneously. A move by a trader i is a vector x^i where $x^i = (x_1^i, x_2^i, \ldots, x_{m+1}^i)$ which is a feasible consumption plan in the sense that:

(5) $$\sum_{j=1}^{m+1} p_j x_j^i \leq \sum_{j=1}^{m+1} p_j A_j^i .$$

A strategy by a trader i is more complicated than his move. It is a set of contingency plans to pick out a specific consumption plan as a function of any prices that may be announced by the player P. The extensive form of this game is shown in Figure 3.

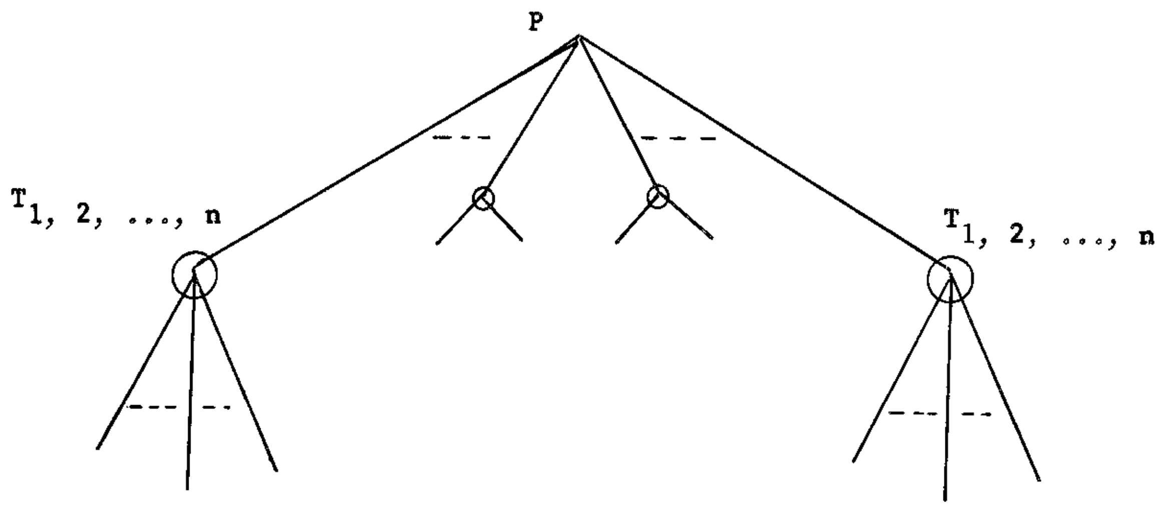

Figure 3

Figure 3 must be interpreted as only indicative because it is not possible to use this tree diagram if the price strategies are unbounded. The symbol P indicates that the price agency moves first. The symbol $T_{1,\ 2,\ \ldots,\ n}$ indicates that all the traders move simultaneously. The one element information sets indicate that they are completely informed.

All trade is monetized in this model. One way in which this can be done easily is to imagine that the government runs the market system. All of the initial endowments of the traders sold via the government markets. They extend a line of credit to each trader equal to his full income. As prices are known this calculation is feasible.

It is assumed that the markets stand to buy and sell in any quantities asked of them (or more precisely as the first act is to buy everything, the markets stand to sell). If demand is less than supply we may assume that the residual is carried on the books at some value no more than cost. If demand is greater than supply the excess may be assumed to be supplied from inventories or bought in international trade at a price no less than cost.

We may assume that the price agency is constrained to maximize its overall profits. This is equivalent to maximizing the market value of excess demand or:

$$z = \sum_{j=1}^{m+1} p_j(x_j - A_j) \tag{6}$$

This model is merely a special case of the model of Arrow and Debreu and hence it has a noncooperative equilibrium which immediately yields the prices and allocations of an efficient price system.

In order to completely define the payoffs associated with all strategies it was necessary to relax conditions calling for the conservation of goods in the system. The accounting money expenditures among the $n+1$ players always balance. But away from equilibrium excess demands or supplies may exist. Unless we wish to specify a rationing system the planning group must fill demands.

The result is independent of the number of traders and requires only one price-naming agency. The equilibrium in 3.1 depended on numbers and had no price agency.

4. ON MODELS OF PRICE CONTROLLED ECONOMIES

4.1 The Goals of the Controllers

Is it reasonable to have the price planning agency and government markets try to maximize profits or the value of excess demand? If this is their only purpose would not a competitive system be better? If some of the types of traders were few in number and the price planning board were honest the solution to the game in 3.2 is certainly better than to the game in 3.1.

The comparison of the two systems boils down to specifying more features of the models than has yet been done. What are the tradeoffs between losses due to olipolistic behavior in the economy described in 3.1 and the bureaucratic costs of planning, information gathering, cheating and bureaucratic self-serving in the system described in 3.2?

The system in 3.2 has a government player introduced explicitly. In a planned economy the goals of the board that fixes prices (or their masters) are invariably more than merely providing orderly markets for consumer goods. They include the directing of resources into public goods (and bureaucratic pockets). In order to study the relative merits of a competitive price system or a price planned system (or a mixed system) considerably more detail would have to be modeled. It is a fairly safe guess that the answers are extremely sensitive to details and can go either way.

Government bureaucracies are usually sufficiently large relative to the size of a modern society that it is doubtful if a model which treats the bureaucracy as an automaton leaving out the goals of the class of bureaucrats is adequate to answer most interesting questions.

4.2 Planning, Dynamics and Price Adjustment

Market processes have an extremely high institutional content. The economist is caught between a desire for generality and a need to be certain that his models do in fact reflect the major relevant features of classes of processes.

Unfortunately the general equilibrium analysis is extremely inadequate for handling situations where lack of information, information handling and gathering and patterns of communication play an important role. The major virtue of the price system in existence is that it has relatively low information requirements for the individual. He needs to know his own preferences, prices and his own endowments. These however are conditions at equilibrium and not in getting to equilibrium. The tattonement processes in different markets vary considerably in speed stability and efficiency. Examples are provided by the stock market auction complete with specialists and short selling, Dutch auctions for tulips, sealed bidding for government contracts, the corn-hog cycle, commodity futures markets, wage contracts pegged for several years or money exchange markets changing every moment.

The analysis of Radner [20] and an elementary consideration of the effect of varying the information structure on a game in extensive form show that economic processes can be extremely insensitive or sensitive to slight changes in information patterns. For example do I get to see the price board's new price list before it is announced to the economy?

It is perhaps premature to expect economic analysis to cope at any depth with the fine structure of information. It is not however premature to suggest that a tattonement or adjustment process is not like a virtual displacement in mechanics. It takes place in too many dimensions and is too big. Even if the farmers in a hog-corn cycle are heading to an equilibrium they may starve to death before they get there. The study of price adjustment mechanisms belongs to the study of the economics of organization. In order to compare different organizations not only does more institutional detail appear to be necessary but at least we can start to examine the sensitivity of the processes to the sequencing of moves and the leaking of information by spelling out the game in extensive form or by being explicit in showing why it is not necessary to do so.

In particular the understanding of the differences between a price planning board controlled economy and a market economy and their relative efficiencies depends upon a clear specification of the price formation mechanism*

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*A good test for completeness and consistencyof a model is to check to see if it can actually be played as a game or run as a computer program.

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