COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut

COWLES FOUNDATION DISCUSSION PAPER NO. 324

A THEORY OF MONEY AND FINANCIAL INSTITUTIONS

PART III

THE MISSING DEGREE OF FREEDOM:

COMMODITY MONEY AND OLIGOPOLY IN A GENERAL EQUILIBRIUM MODEL

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December 17, 1971

ABSTRACT

It is suggested that an extra degree of freedom is needed to construct a symmetric noncooperative price game in a market with n monopolists trading in n goods. This calls for the introduction of an n+1 good which can be interpreted as a commodity money.

When there are n monopolists using a commodity money in common a symmetric price or quantity noncooperative game can be constructed. The quantity game is examined. Necessary conditions are shown for the replicated game to have its noncooperative equilibria approach the competitive equilibria of the replicated market. It is demonstrated that unless there is "enough" commodity money convergence may not take place. There will be a "money shortage" and this will be reflected in a price for the commodity money higher than its utilitarian worth. This reflects the addition of a "shadow price" for the worth of relaxing the monetary capacity constraint.

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1. THE PROBLEM

General equilibrium theory presents a description of the functioning of the price system that is essentially both static and nonstrategic. Oligopoly theory has been presented as strategic, but with few exceptions models of oligopolistic behavior have been open or one-sided. Most frequently the stress has been upon the oligopolistic competitors as firms with the customers as passive price taking automata, best represented by demand functions.

From Cournot, through Chamberlin the predominant solution concept used has been that of the noncooperative equilibrium point [1]. The noncooperative solution has been employed primarily with quantity offered or with price as a strategic variable used by the firms.

Although much of the verbal treatment of the noncooperative models has been phrased in dynamic or quasidynamic terms, the mathematical formulation of the noncooperative equilibrium can be given as a static "one-shot" system to be solved simultaneously. In other words, the market may be represented by a game in strategic form where all players simultaneously announce their strategies.

There is little doubt that an adequate model of oligopoly calls for an explicitly dynamic treatment, however before attempting to develop dynamics we may consider a problem that arises in statics even though the model that must be built is not "realistic."

Why is it that it is so easy to formulate a static symmetric noncooperative game as a representation of an open oligopolistic system; but it does not appear to be either easy or "natural" to formulate a static symmetric noncooperative game as a representation of a closed economic system?

In the remainder of this paper it is shown that the answer to this question is closely related to the role of money in an economy, and this in turn provides a link from statics to dynamics

2. ON CLOSED AND OPEN SYSTEMS

Before starting to construct an explicit model, the distinction between open and closed models is made.

The Edgeworth bilateral model of trade is a closed nonmonetary system. Goods flow within the system and after trade the change in the amount of goods in the system is zero. This is shown in Figure 1. The Walrasian system may be regarded as this type of system.

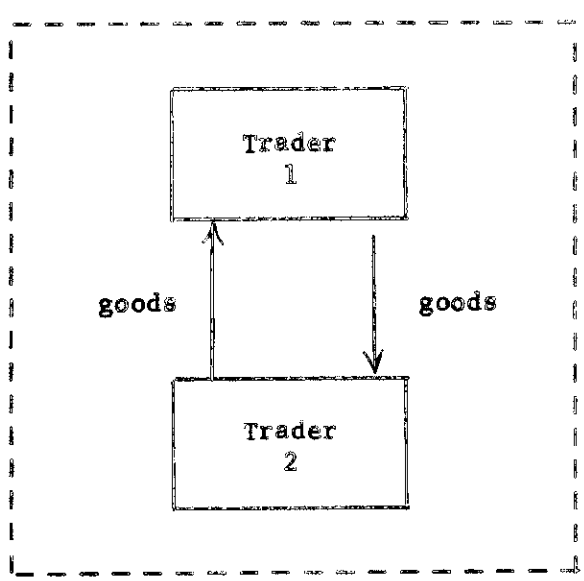

FIGURE 1

The Cournot duopoly model is an open market model involving money. After trade has taken place neither the amount of goods, nor the amount of money in the system is conserved. Goods flow out into "the market" and money flows in from the market. This is shown in Figure 2. Virtually all models of oligopoly are of this type.

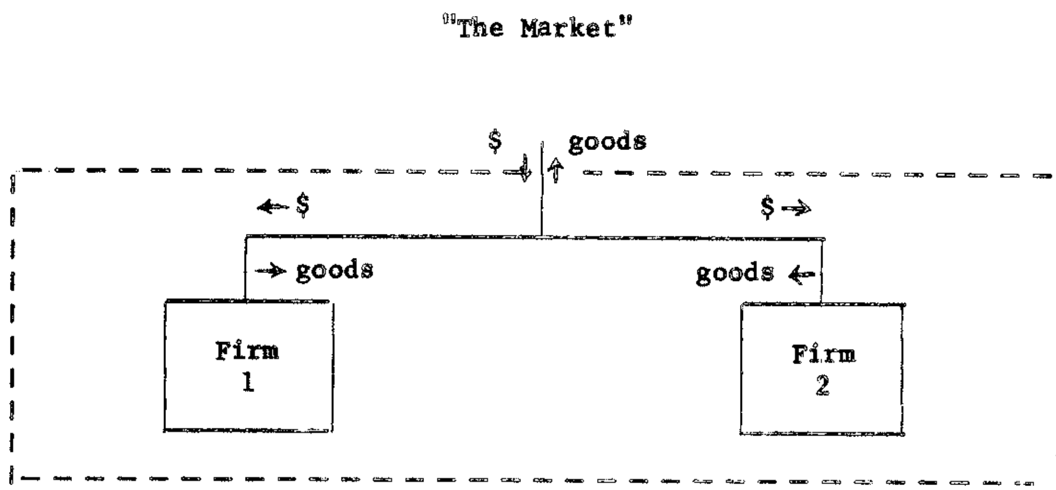

FIGURE 2

A general equilibrium system which trades in money must conserve both goods and money. Thus the flow diagram for a two trader market becomes that shown in Figure 3. We may also wish to consider systems which are "open" or "closed" in time. This involves specifying interperiod balances.

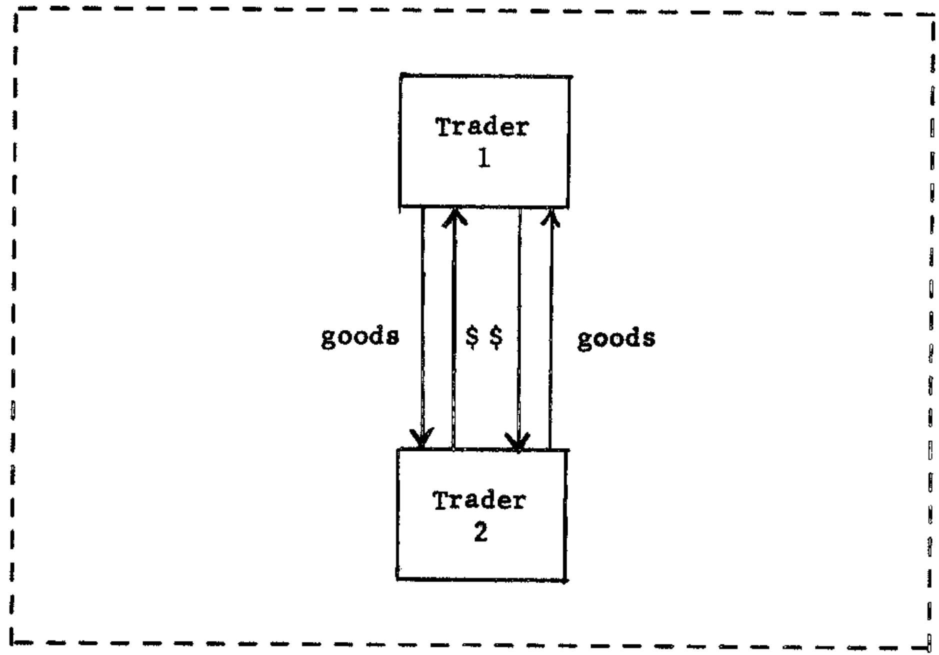

FIGURE 3

3. THE DIFFICULTIES WITH THE EDGEWORTH BOX

The simplest and most attractive device for illustrating the properties of the price system is the Edgeworth box. Given two traders with initial endowments of $(A_1, 0)$ and $(0, A_2)$ and preferences which can be represented by utility functions $\varphi(x_1, y_1)$ and $\Psi(x_2, y_2)$. The familiar Edgeworth box diagram is drawn in Figure 4. The line OP indicates the price system. P represents a point on the Pareto optimal surface that trading will reach when the price system is used. This simple model illustrating the price system for two traders, trading in two goods can be generalized for n traders trading in m goods [2]. In the

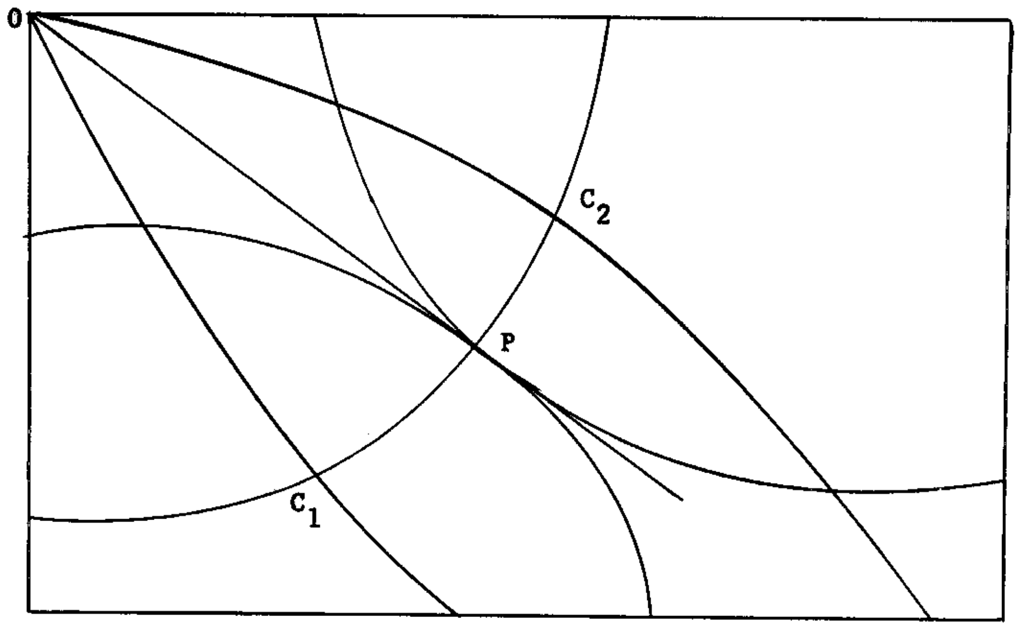

FIGURE 4

general case it can be shown that m prices $(p_1, p_2, \ldots, p_m)$ exist such that if each trader is constrained to maximize his welfare taking these prices as given the resultant trades will all balance and the outcome will be Pareto optimal (as is illustrated in the simpler case by P). It is well known that the price system is homogeneous of order 0 . As the prices in the general equilibrium model are essentially exchange ratios among the goods we could use one of the goods, say the m^{th} as a numeraire, fixing its price at 1. The price system becomes $(p_1/p_m, p_2/p_m, \ldots, p_{m-1}/p_m, 1)$ where the m-1 prices are independent.

The general equilibrium closed competitive economic system is modeled nonstrategically in the sense that each individual is <u>constrained</u> to maximize his welfare as though prices were fixed and that he has no market influence.

In Figure 1 the curve joining C_1C_2 is the contract curve. This set of imputations suggested by Edgeworth as a solution to the bilateral monopoly bargaining problem also coincides with the cooperative game theory solution known as the core. Game theory solutions involve making the assumption that the individuals have strategic freedom and influence.

Cooperative solutions, such as the core can be illustrated on the Edgeworth box because the strategic freedom, the dynamics, haggling and bargaining lie outside of the formal economic model. They are only implicit in the formal mathematical description. In particular no description is given of the communications or the nature of the discussion that takes place among the traders. It is assumed that after they have finished bargaining they then act jointly or "cooperatively" in selecting some outcome on curve C_1C_2.

The noncooperative solution is explicitly strategic, this is in contrast with the cooperative solutions (implicitly strategic) and the competitive equilibrium (nonstrategic). Duopoly models, which are open economic models (the customers being represented by a demand mechanism) have been studied using the noncooperative equilibrium solution where either price of quantity have been used as the strategic variable employed by the firms [3]. Given that the noncooperative solution to duopoly is well known and has been in economic literature since the time of Cournot, a natural question to ask is can the noncooperative equilibrium solution be applied to a bilateral monopoly? Furthermore do there exist two versions, one with quantity and the other with price as the strategic variables?

Another important question concerns the role of symmetry. The duopoly model is inherently symmetric in the roles of the two firms. The symmetry is reflected in the resultant noncooperative games. The bilateral monopoly model is also inherently symmetric. Can we construct a two-person symmetric noncooperative game that is a reasonable model of a closed bilateral monopoly?

One further question must be asked. Although the noncooperative equilibrium solutions to duopoly can be given a quasi-dynamic interpretation and in some instances some formal dynamic processes may be specified which lead to the noncooperative equilibrium, they can equally well be discussed in a static context. The existence of a noncooperative equilibrium point in a Cournot duopoly can be examined in much the same way as the existence of the competitive equilibrium can be examined. Can we obtain a noncooperative equilibrium model of a bilateral monopoly which is reasonably well portrayed in a static context?

We discover immediately that we are unable to formulate a satisfactory bilateral monopoly model where the traders are trading in only two commodities, as a symmetric noncooperative game with price as the strategic variable for each trader. In the closed model, prices can be regarded as ratios. If we select one commodity as the numeraire we destroy the symmetry of the model and deprive the monopolist in that commodity of his strategic freedom. If we do not pick a numeraire the resulting price game amounts to the game of "pick the biggest number" because there is really only one price in the market which is determined by the ratio of the two numbers named. No matter what the other trader has named it pays the remaining trader to name a higher number.

We are not able to formulate a satisfactory symmetric bilateral monopoly model using quantity as the strategic variable. If each names a quantity and the market is cleared the only equilibrium is $(0,0)$. Although, as is discussed below it is possible to formulate a satisfactory symmetric two commodity bilateral duopoly model as a symmetric noncooperative quantity strategy game _if_ one of the commodities is held in an appropriately abundant supply by all traders and is used as a commodity money, i.e., offers to buy anything else in the economy are made in terms of quantities of the commodity money.

The noncooperative game formulation for a closed symmetric general equilibrium trading model requires the existence of a commodity money. The important property of this commodity money for the noncooperative game is that it is used "to balance the books" after all traders have exerted their strategic control over the market. Because _all_ are required to be strategically neutral with respect to the commodity money a symmetric noncooperative price game in which each trader is permitted to name a price for some commodity requires, for n traders, that there exist at least $n+1$ commodities.

If the noncooperative game has quantities offered as the strategic variables it is possible to construct a symmetric game for as few as two commodities provided that one commodity is designated as a "money" and all traders possess a supply of money.

There are several difficulties encountered in defining the price noncooperative game and in defining and studying limiting behavior as numbers of traders are increased.*

We wish to study the limiting behavior of the noncooperative equilibria in order to investigate their relationship to competitive equilibrium points. This can be done in a straightforward manner with the quantity strategy noncooperative game. In Section 4 the quantity strategy noncooperative game is defined and studied.

4. A WORLD OF OLIGOPOLIES

The nature of the noncooperative game and equilibria can be most easily seen if we begin with an economy consisting of n monopolists each of whom possesses the sole supply of some commodity and has a supply of the commodity money.

Let there be n traders, each a monopolist in one commodity. There are $n+1$ commodities in the economy.

Trader i has an initial endowment given by $(0, 0, \ldots, 1, \ldots, 0, M^i)$ and has a utility function of $\varphi_i(x_1^i, x_2^i, \ldots, x_{n+1}^i)$.

A strategy by trader i is to offer a quantity $q_i^i \leq 1$ to the market and to offer quantities of "money" d_j^i $(j \neq i)$ for the remaining goods, where $\sum_{j \neq i} d_j^i \leq M^i$.

The price of a commodity j is given by:

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*An immediate problem with the price game when a limiting process is examined is to decide if additional players sell identical or differentiated products. A complicated rationing method for the description of demand is also called for [4].

(1) $$p_j = \sum_{i \neq j} d_j^i / q_j^j \quad \text{for*} \quad j = 1, \ldots, n \; .$$

(2) $$p_{n+1} = 1 \; .$$

The quantity of the j^{th} commodity obtained by trader i is:

(3) $$x_j^i = d_j^i q_j^j / \sum_{h \neq j} d_j^h \quad \text{for*} \quad j = 1, \ldots, n \; , \quad \text{and} \quad j \neq i \; .$$

(4) $$x_i^i = 1 - q_i^i \quad \text{and}$$

(5) $$x_{n+1}^i = M^i + p_i q_i^i - \sum_{j \neq i} d_j^i \; .$$

The goal of Trader i in the noncooperative game is to select $(d_1^i, d_2^i, \ldots, d_{i-1}^i, q_i^i, d_{i+1}^i, \ldots, d_n^i)$ to maximize

$$\varphi_i(x_1^i, x_2^i, \ldots, x_{n+1}^i) \; .$$

The φ_i are assumed to be twice differentiable with bounded first derivatives. We may write down the n^2 first order conditions for individual maximization in the noncooperative game. Although they can be written down formally, reflection on the nature of the model when the market consists of monopolists indicates that there will exist only one highly unsatisfactory noncooperative equilibrium point where no trade

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*In both of these instances if $q_j^j = 0$ or $\sum_{h \neq j} d_j^h = 0$ we must define values for p_j and x_j^i. These conditions are not treated in this paper. They will cause difficulties with a general investigation of the mathematical properties of this model. There is always a (highly unstable) equilibrium point at $(0, \ldots, 0)$.

takes place. If only one individual offers a commodity and obtains all of the money that the others have offered, regardless of the amount he offers, then his optimal strategy is to offer nothing. This paradoxical behavior associated with infinite price and a zero offering occurs for "the world of monopolies." When we consider that there are two or more traders of each type this paradoxical result remains, for if all but one of the traders offer none of their goods for sale the remaining one will also offer nothing. However a new and more reasonable equilibrium also appears.

Suppose that there are s traders of each type in the market where $s \geq 2$. We may express the first order conditions for the existence of a noncooperative equilibrium by:

$$(6) \quad \frac{\partial \varphi_i}{\partial d_j^{i,k}} = 0 \quad \text{for} \quad k = 1, \ldots, s \quad \text{and} \quad i, j = 1, 2, \ldots, n \quad \text{but} \quad j \neq i .$$

and

$$(7) \qquad \frac{\partial \varphi_i}{\partial q_i^{i,k}} = 0 \quad \text{for} \quad k = 1, \ldots, s \quad \text{and} \quad i = 1, \ldots, n .$$

These may be written out more fully for a trader of type 1 as:

$$(8) \quad \frac{\partial \varphi_1}{\partial q_1^{1,k}} = \frac{-\partial \varphi_1}{\partial x_1} + p_1 \frac{\partial \varphi_1}{\partial x_{n+1}} - q_1^{1,k} \frac{p_1}{\sum_\ell q_1^{1,\ell}} \frac{\partial \varphi_1}{\partial x_{n+1}} = 0 , \quad k = 1, \ldots, s$$

(9) $$\frac{\partial\omega_1}{\partial d_2^{1,k}} = \frac{\partial\omega_1}{\partial x_2}\left\{\frac{\sum_{k=1}^{s} q_2^{2,k}}{\sum_{i\neq 2}\sum_{\ell=1}^{s} d_2^{i,\ell}} - \frac{d_2^{1,k}\sum_{k=1}^{s} q_2^{2,\ell}}{\left(\sum_{i\neq 2}^{n}\sum_{\ell=1}^{s} d_2^{i,\ell}\right)^2}\right\} - \frac{\partial\omega_1}{\partial x_{n+1}} = 0$$

and similarly for the others. These give us two equation types for when an individual sells one of the first n commodities or when he buys. They can be expressed as:

(10) $$p_i = \frac{\partial\omega_i}{\partial x_i}\Bigg/\frac{\partial\omega_i}{\partial x_{n+1}}\left(1 - \frac{q_i^{i,k}}{\sum_{\ell} q_i^{i,k}}\right)$$

(11) $$\left\{\frac{1}{p_j} - \frac{d_j^{i,k}\sum_{\ell=1}^{s} q_j^{j,\ell}}{\left(\sum_{h\neq j}^{n}\sum_{\ell=1}^{s} d_j^{h,\ell}\right)^2}\right\} = \frac{\partial\omega_i}{\partial x_{n+1}}\Bigg/\frac{\partial\omega_i}{\partial x_j} \quad \text{for } i \neq j .$$

Given the symmetry among the traders of a specific type we may replace the summation over a type by s times the action of an individual of that type. Thus equation (10) becomes

(12) $$p_i = \frac{\partial\omega_i}{\partial x_i}\Bigg/\frac{\partial\omega_i}{\partial x_{n+1}}\left(1 - \frac{1}{s}\right)$$

and (11) becomes

(13) $$p_j = \left(1 - \frac{d_j^i}{s\sum_{h\neq j} d_j^h}\right)\frac{\partial\omega_i}{\partial x_j}\Bigg/\frac{\partial\omega_i}{\partial x_{n+1}} .$$

We know that $0 \leq d_j^i \leq \sum_{j\neq i} d_j^i$ hence

$$(14) \qquad 1 \geq \left(1 - \frac{d_j^i}{s \sum_{h \neq j} d_j^h}\right) \geq \left(1 - \frac{1}{s}\right) .$$

Thus as s increases in size, (12) and (13) approach the form of

$$(15) \qquad p_j = \frac{\partial \varphi_i}{\partial x_j} \Big/ \frac{\partial \varphi_i}{\partial x_{n+1}} \quad \text{for} \quad j = 1, 2, \ldots, n .$$

These together with $p_{n+1} = 1$ can be written as:

$$(16) \qquad \frac{\dfrac{\partial \varphi_i}{\partial x_1^{i,k}}}{p_1} = \frac{\dfrac{\partial \varphi_i}{\partial x_2^{i,k}}}{p_2} = \ldots = \frac{\dfrac{\partial \varphi_i}{\partial x_{n+1}^{i,k}}}{1} \qquad i = 1, \ldots, n .$$

But the equations (16) are similar in form to the conditions at a competitive equilibrium point in the $n+1$ commodity economy. This is suggestive that there may exist a class of noncooperative oligopoly games in a closed economy for which the limiting (in the sense of replication) symmetric noncooperative equilibrium points coincide with the competitive equilibrium points (see Appendix).

The noncooperative equilibria will be influenced by the amount of commodity money in existence. Furthermore there must be "enough money" held by each trader or a boundary solution may result. The meaning of enough is illustrated and discussed in the example in Section 6.

The indication of the relationship between the noncooperative equilibrium and the competitive equilibrium given above for a world of one product oligopolists appears to hold far more generally. Instead of limiting the model to individuals holding one product and "money" we may consider a market with n individuals and $m+1$ goods $(m \geq 1)$ where each individual i has as his initial holdings amounts $(A_1^i, A_2^i, \ldots, A_{m+1}^i)$. In this game a strategy is an m-dimensional vector where an individual may decide either to buy or sell commodity j. If he sells, he offers an amount $q_j^i \leq A_j^i$. If he buys he offers an amount of money d_j^i where $\sum_j d_j^i \leq A_{m+1}^i$.

5. DISCUSSION AND INTERPRETATION

The commodity money serves as an imperfect but adequate strategic decoupling device. It enables each trader to announce his strategy without coordination with the others. The properties required to make it work well should be that it is of some value; it is in sufficient supply that it is reasonable for all to be able to keep a store of it. It should be durable and the closer it has a relatively constant marginal utility the more stable the economy becomes with respect to injections of new money. In various societies, coconuts, salt bars, dried fish, bricks of tea, etc. have come reasonably close to fulfilling these requirements [5].

The noncooperative market described used quantity as the strategic variable. In considering the behavior of a large economy and in trying to make the connection between statics and dynamics, it appears reasonable to consider that the amount of goods offered and the amount of "money" chasing them are the determinants of prices rather than vice versa.

We note that the noncooperative equilibrium, although it can be given a static interpretation is an interlink to a dynamic model. The budget constraint conditions reflect this in the "cash or commodity money flow" account, in the sense that the items sold do not provide spendable funds for this period. They build up the individual's supply of money for the next trading period thus it is most appropriate to consider the utility of the end supply of commodity money as not only having intrinsic individual consumption utility, but also containing a socially determined utility derived from including the "value" of the trader's position in the next game as determined by his supply of commodity money.

Essentially the worth of the special commodity has two components, a direct "Robinson Crusoe" utility as a consumer good and a value as the "money" or store of wealth to be used in the next period of trade. This second component may be regarded as the derived positional value determined by the state the system enters in the next period. Provided that all individuals are willing to accept "blue chips" or pieces of paper a commodity money may be replaced by a fiat money which will have the second value property, but not the first. The discussion of this is not pursued in this article.

The problem of convergence of the noncooperative equilibrium to a specific competitive equilibrium appears to rest upon the full investigation of the conditions under which the equation systems (12) and (13) yield a solution identical with (16). It is conjectured that this will depend critically upon which commodity is chosen as the commodity money. We know that at least such a commodity must be in sufficient supply to all or the system will not work.

6. AN EXAMPLE

In this section a simple example is calculated to illustrate the nature (and non-optimality) of the noncooperative equilibrium and to illustrate how "not enough money" can influence trade.

Suppose that there are four traders in toto; two traders of two types. In order to keep the example as simple as possible let each trader have the same utility function. A type is distinguished by differences in initial endowments. Let the utility function of each individual be

$$U = \log xyz \tag{17}$$

where the third commodity is the commodity money.

Let the initial resources of traders of Type 1 be $(A, 0, M_1)$ and of Type 2 $(0, B, M_2)$.

Denote the two traders of the first type as 1 and 2 and of the second type as 3 and 4.

A strategy by a trader of the first type is to offer an amount q_i of the first commodity for sale and offer an amount of money d_i for the second commodity, where $i = 1, 2$.

A strategy by a trader of the second type is to offer an amount of money w_i for the first commodity and an amount r_i of the second, where $i = 3, 4$.

Let x_i be the amount of the first good held by trader i and let y_i be the amount of the second good held by trader i.

$$p_1 = \frac{w_3 + w_4}{q_1 + q_2}, \tag{18}$$

$$p_2 = \frac{d_1 + d_2}{r_3 + r_4} \tag{19}$$

where $0 \leq d_i \leq M_1$, $0 \leq w_i \leq M_2$, and $0 \leq q_i \leq A$, $0 \leq r_i \leq B$.

We may write:

$$\begin{cases} x_1 = A - q_1 & y_1 = d_1\left\{\dfrac{r_3 + r_4}{d_1 + d_2}\right\} \\ x_2 = A - q_2 & y_2 = d_2\left\{\dfrac{r_3 + r_4}{d_1 + d_2}\right\} \\ x_3 = w_3\left\{\dfrac{q_1 + q_2}{w_3 + w_4}\right\} & y_3 = B - r_3 \\ x_4 = w_4\left\{\dfrac{q_1 + q_2}{w_3 + w_4}\right\} & y_4 = B - r_4 \end{cases} \tag{20}$$

This enables us to describe the noncooperative game as:

$$\underset{q_1,\ d_1}{\text{Maximize}} \log(A - q_1)\left[d_1\left\{\frac{r_3 + r_4}{d_1 + d_2}\right\}\right](M_1 + p_1q_1 - d_1) \tag{21}$$

$$\underset{w_3,\ r_3}{\text{Maximize}} \log\left[w_3\left\{\frac{q_1 + q_2}{w_3 + w_4}\right\}\right](B - r_3)(M_2 + p_2r_3 - w_3). \tag{22}$$

There are two similar equations for the second and fourth traders.

From (21) we obtain, taking derivatives first on q_1 and then d_1

(23) $$\frac{1}{M_1 + p_1q_1 - d_1}\left\{\frac{w_3 + w_4}{q_1 + q_2} - q_1 \frac{(w_3 + w_4)}{(q_1 + q_2)^2}\right\} = \frac{1}{A - q_1}$$

and

(24) $$\frac{1}{d_1} - \frac{1}{d_1 + d_2} = \frac{1}{M_1 + p_1q_1 - d_1} .$$

Similarly from (22) we obtain:

(25) $$\frac{1}{M_2 + p_2r_3 - d_3}\left\{\frac{d_1 + d_2}{r_3 + r_4} - \frac{r_3(d_1 + d_2)}{(r_3 + r_4)^2}\right\} = \frac{1}{B - r_3}$$

and

(26) $$\frac{1}{w_3} - \frac{1}{w_3 + w_4} = \frac{1}{M_2 + p_2r_3 - d_3} .$$

Solving for the symmetric equilibria we may assume that identical traders are treated identically. Hence we may replace $r_3 + r_4$ by $2r$ and similarly elsewhere. More generally, instead of considering 2 traders of each type we might consider s traders and we may replace any sum such as $d_1 + d_2 + \dots + d_3$ by sd in the more general form of equations (23)-(26). This gives us from (18) and (19)

(27) $$p_1 = w/q$$

and

(28) $$p_2 = d/r .$$

From (23) and (24) we obtain:

(29) $$\frac{1}{A - q} = \frac{1}{M_1 + w - d}\left(1 - \frac{1}{s}\right)\frac{w}{q}$$

and

$$\frac{1}{d}\left(1 - \frac{1}{s}\right) = \frac{1}{M_1 + w - d} .$$

Similarly from (25) and (26):

(30) $$\frac{1}{B - r} = \frac{1}{M_2 + d - w}\left(1 - \frac{1}{s}\right)\frac{d}{r} \quad \text{and} \quad \frac{1}{w}\left(1 - \frac{1}{s}\right) = \frac{1}{M_2 + d - w} .$$

Solving the four equations for d , w , q and r , setting:

$$G = \left(1 - \frac{1}{s}\right)$$

we obtain:

(31) $$\frac{1}{A - q} = \frac{G^2 w}{dq} \quad \text{from (29)}$$

(32) $$\frac{d}{G} = M_1 + w - d \quad \text{from (30) and similarly}$$

(33) $$\frac{w}{G} = M_2 + d - w .$$

With further manipulation we obtain

$$d = \frac{G}{2}\left(M_1 + M_2 + \frac{M_1 - M_2}{1 + 2G}\right)$$

or

(34) $$d = \frac{G(M_1 + G(M_1 + M_2))}{1 + 2G}$$

(35) $$w = \frac{G(M_2 + G(M_1 + M_2))}{1 + 2G}$$

and

(36) $$\frac{q}{A} = \frac{GM_1 + (1+G)M_2}{(1 + G + G^3)M_1 + (G + G^2 + G^3)M_2},$$

and similarly for r

(37) $$p_1 = \frac{w}{q} = \frac{(1 + G + G^3)M_1 + G(1 + G + G^2)M_2}{(1 + 2G)A}$$

(38) $$p_2 = \frac{d}{r} = \frac{(1 + G + G^3)M_2 + G(1 + G + G^2)M_1}{(1 + 2G)A} .$$

The mass market aspect of the model can be studied by letting $G \to 1$. This gives:

(39) $$q \to \frac{M_1 + 2M_2}{3M_1 + 3M_2} A , \qquad d \to \frac{2M_1 + M_2}{3}$$

and

(40) $$r \to \frac{M_2 + 2M_1}{3M_1 + 3M_2} B , \qquad w \to \frac{M_1 + 2M_2}{3} ,$$

hence

(41) $$p_1 = (M_1 + M_2)/A \quad \text{and} \quad p_2 = (M_1 + M_2)/B \; .$$

Setting $A = B$ and $M_1 = M_2 = M$ we compare the markets for $s = 2, 3$ and $s \to \infty$.

s	q = r	d = w	$p_1 = p_2$
2	4A/5	M/2	5M/8A
3	9A/13	2M/3	26M/27A
∞	A/2	M	2M/A

Owing to the special nature of the utility function, in the table above it can be seen that any amount of this commodity money is "enough." This would not be so for other utility functions.

The above comment only holds true in the highly symmetric case. For this case, as can be seen for $s \to \infty$ the competitive equilibrium is reached.

If we look at the monetary constraints in general $d \leq M_1$ becomes:

(42) $$GM_1 + G^2M_1 + G^2M_2 \leq M_1 + 2GM_1 \; .$$

Similarly using $w \leq M_2$ and substituting for G we obtain

(43) $$\frac{s^2 - 2s + 1}{s^2 + s - 1} \leq \frac{M_1}{M_2} \leq \frac{s^2 + s - 1}{s^2 - 2s + 1} \; .$$

We observe that from (43) the distribution of money holdings must be equal to all if the noncooperative mass market is to approach the competitive equilibrium.

6.1. Another Simple Example: A "Linearly Valued" Commodity Money

If there is a commodity which enters the utility functions of all traders as a separate and linear term, i.e., the utility functions are of the form:

$$\phi(x_1, x_2, \ldots, x_n, m) = \varphi(x_1, x_2, \ldots, x_n) + bm \tag{44}$$

then if there is "sufficient money" the noncooperative equilibrium with active trade appears to converge to the competitive equilibrium regardless of the supply of commodity above the minimum needed by each. Consider, as an example, two types of players, each with a utility function of the form:

$$\log xy + z \tag{45}$$

and initial endowments $(A, 0, M)$ and $(0, B, M_2)$. It is easy to establish that:

$$q = A\left[\frac{(s-1)^2}{s^2 + (s-1)^2}\right] \quad \text{and} \quad r = B\left[\frac{(s-1)^2}{s^2 + (s-1)^2}\right] . \tag{46}$$

These hold for any amounts of initial goods and commodity money above a minimum needed for trade. As $s \to \infty$ these always approach the competitive equilibrium.

7. CONCLUDING REMARKS

In the models above commodity money plays several important roles as a medium of exchange, a numeraire, a store of value and a strategic decoupling device which enables each trader to name a strategy in a symmetric noncooperative game.

As traders do not buy or sell the commodity money directly as they do their stock-in-trade their roles are symmetric with respect to this money. As their numbers increase they become strategically weaker in the markets. This attenuation of the worth of strategic freedom, or individual power (but not the attenuation of strategic freedom) appears, under the appropriate circumstances, to lead to noncooperative equilibrium to approach the competitive equilibrium.

The "ideal commodity money" from the viewpoint of helping to guide a large market of noncooperative individuals would appear as a linear separate term in the utility functions of the individuals, or at least it should be sufficiently weakly interlinked with the other commodities that a shape such as that in (44) is a reasonable approximation in the neighborhood of the individual's usual expenditures.

Money, both commodity and fiat, is a large group phenomenon. It is conjectured that a fruitful approach to the understanding of many of the basic properties of fiat money is similar to the approach noted above: (1) The model must be strategic; (2) The players must occupy symmetric roles with respect to all commodities, but their strategic actions should not be directly applied to money (a theory of banking may call for nonsymmetric players, one class of which directly manipulates the supply of money [6]); (3) The properties of the system should emerge as the limiting behavior of an sk-person noncooperative game as $s \to \infty$ (where k is the number of types of players).

The key role of a money in a noncooperative game is as a strategic decoupling device which naturally interlinks a static economy with a dynamic one. Given the size of mass markets and the anonymity of individuals dealing in them a money needs also to be a carrier of value so that "the trust is in the cash." A commodity money has this property but the more closely it is interlinked in the consumption patterns of the individuals the poorer a decoupling device it becomes.

The ideal decoupling device should be as in (44). Where not only does the commodity money act as a direct store of wealth it also achieves its maximum flexibility and added worth as a state variable serving to relax constraints in a dymamic maximization problem. If people were willing or forced to trust fiat money or paper or "blue chips" (i.e. were willing to accept it in trade) then a fiat money could be introduced into the noncooperative game instead of a commodity. The value of the paper as contrasted with gold might initially appear slight. It obviously can have the classic values of

(1) Transportability

(2) Ease of storage

(3) Numeraire

(4) Medium of exchange.

We may add the government fiat created values of

(1) A store of value by state fiat, and

(2) A means of paying taxes.

Once a large group of more or less anonymous individuals have been "hooked into playing" a noncooperative game using dollars or blue chips they may not be able to get out of the game by individual action.

It does not require an individual belief that a trader can eat his dollars to keep him in the game and to accept the rules. Given that there are enough other players accepting the rules, considerations of custom, the value as a medium of exchange etc. may make it too expensive for the individual to get out of the game. At some point if a government or large groups are using the printing presses or playing other "money games" the individuals will abandon the rules and the paper becomes worthless. This point is usually only reached after a war, revolution or other major social upheaval. However within fairly broad bounds, once a fiat money system is in motion, a noncooperative game can provide its own internal stability.

The major original purpose of this paper was to investigate the properties of noncooperative oligopoly models in a general equilibrium system. This called for an exploration of the strategic freedom of the traders, which in turn has lead to the speculation that models of oligopoly in a closed economic system and the concept of markets and money pose closely related problems.

Strategic freedom can be introduced via a commodity money. This creates a "cash flow" problem which produces extra conditions that bound the trade possibilities in a society that does not use barter, but does not have trust or credit except in "gold" or a commodity money. The intrinsic value of the commodity provides the first reason for use. The internal circular stability of the noncooperative equilibrium and the inconvenience of other modes of trade, once the commodity money has been accepted, may enable even a symbolic commodity to support its monetary properties.

APPENDIX

The commodity money has two values imputed to it. One as a good of intrinsic worth and the other as a trading device. It can be seen from (47) and (48) below that for the noncooperative game to converge to the competitive equilibrium two different sets of conditions must be satisfied at the point. For the noncooperative equilibrium if an individual maximizes $\varphi(A_1 + x_1, A_2 + x_2, \ldots, A_{n+1} + y)$ where $p_{n+1} = 1$ the following conditions must hold:

$$(47) \qquad \sum_{j=1}^{n} \delta_j p_j x_j \leq A_{n+1} \qquad \text{where} \qquad \delta_j = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

also
$$\begin{aligned} x_1 &\geq -A_1 \\ &\vdots \\ x_n &\geq -A_n \end{aligned}$$

and

$$(48) \qquad -y = \sum_{j=1}^{n} (1 - \delta_j) p_j x_j + \sum_{j=1}^{n} \delta_j p_j x_j \, .$$

For the competitive equilibrium we have only (48). Thus we have an extra inequality for every individual in the noncooperative game. When (47) and (48) are satisfied at the same time at the competitive equilibrium it is conjectured that the noncooperative equilibrium will converge to the competitive equilibrium.* Furthermore, this appears to indicate that in such a case the "money constraint" is slack hence the value imputed to the commodity will only be its "eating value."

Given the extra conditions imposed by (47) a noncooperative game is not neutral to the selection of a commodity money. Furthermore, if no commodity exists in adequate supply no noncooperative game using a commodity as money may give the same outcome as a competitive market. Given m commodities and n traders, failure of the efficiency of a commodity money economy requires testing for each competitive equilibrium point the failure of all of m sets of n constraints, one set for each commodity in use as a commodity money.

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*If there are several competitive equilibria, it is conjectured that there will be a noncooperative equilibrium point associated with each competitive equilibrium.

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