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OPTIMAL GROWTH WITH SCALE ECONOMIES IN OVERHEAD CAPITAL

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by

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1. Summary

Following closely the approach to optimal economic growth taken in the work of Frank Ramsey [1928], a highly simplified two-sector model is presented in which the "overhead capital" sector exhibits increasing returns to scale. Basic properties of the optimal growth path are discussed and the optimal policy is explicitly demonstrated for a special case. From an economic standpoint, the model might be relevant in bearing on some issues of development programming. Mathematically, this kind of a model has an interesting structure because it is a combination of convex and concave sub-problems.

2. Introduction

In the context of development economics it is useful to distinguish two types of capital according to how round-about a role each plays in producing output. One type, denoted K_α, is the ordinary directly productive, quick-yielding capital which, when it is combined with labor,

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creates output according to classical laws of production. A second kind of capital, K_β, is the indirectly productive infrastructure which lays down the basic framework within which directly productive economic activities can function. Capital of this variety has come in for increased scrutiny by development economists. At least in part this is due to the growing suspicion that β capital, comprising those essential services without which ordinary production cannot operate, plays an especially important role in the early stages of economic growth.

For the purposes of this paper the total capital stock of the economy is thought of as being partitioned between two sectors -- K_α belonging to the α sector and K_β to the β sector. This being the case, it becomes a fair question to ask for operational criteria which can be used to distinguish α from β capital. Unfortunately it is difficult to be precise about this issue. For one thing it depends upon how aggregative a view one is prepared to take.

Considering an entire economy on the most general level, β would consist of all social overhead capital including <u>public service</u> facilities for education, scientific research, sanitation engineering, public health, and law enforcement, <u>agricultural overhead</u> such as drainage and irrigation systems, and <u>hard public utilities</u> like transportation, communications, power and water supply installations. A somewhat more satisfactory interpretation might limit β to the hard public utilities. There is even an interesting way of looking at this model which restricts the economic scenario to manufacturing and treats β as structures, α as producers' durable equipment.

For the purposes of this paper probably the most useful formulation is the middle one which treats β as overhead capital for producers' services. In any case, the basic features are taken to be the following.

(i) Capital of the β type is strongly complementary with α. Investment in α capital will be productive only if it has been preceeded by sufficient investment in β capital.

(ii) The β sector is highly capital intensive and usually consists primarily of structures and installations. It is typically characterized by a significantly higher capital-labor ratio than the α sector.

(iii) There are substantial economies of scale in creating β capacity. The main reason is that due to indivisibilities there is obvious cost lumpiness involved in creating a transportation, communications, or power and water supply system as a whole. Geometric-engineering considerations are also important in the case of many structures because the cost of an item is frequently related to its surface area while the capacity increases according to its volume.[1]

(iv) Both β and α capitals are specific to the role for which they have been created and cannot be shifted.

3. The Basic Model

The highly stylized economy under consideration is centralized and closed. A single homogeneous output, denoted Y, is produced which

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[1]In addition the usual internal economies of specialization and information handling may be present.

is perfectly general before it has been committed, and can be used for any purpose. The planners seek to maximize welfare by appropriately manipulating the available instruments -- in this case the destination of final output. For simplification the following are assumed: stationary labor force and population, constant technology, no capital deterioration.

As an abstraction of proposition (ii), it is postulated that β capital has only negligible manpower requirements. This makes the labor allocation problem trivial because all available workers will be assigned to work with α capital.

If β capital were abundant at time t, $Y(t)$ would depend only on the stock of $K_\alpha(t)$ and the labor force. Since the latter is treated as constant, the production function in this case can be written as

$$Y(t) = F(K_\alpha(t))$$

Non-increasing returns to a single factor (plus differentiability) implies $F''(K_\alpha) \leq 0$ for all $K_\alpha \geq 0$. Assume for the sake of economic interest that $F(K_\alpha(0)) > 0$ and $F'(K_\alpha(0)) > 0$.

With $K_\alpha(t)$ plentiful, the production function is simply

$$Y(t) = K_\beta(t) .$$

Note the implied asymmetry in capital measurement; K_α is gauged by the usual criterion of real production cost, whereas it will prove useful to quantify K_β in capacity units. Of course strict identification of K_β

with "capacity" would be possible only if there were a negligible elasticity of substitution between K_β and $F(K_\alpha)$, a condition which we readily assume following (i).

In the general case,

$$Y(t) = \min\{F(K_\alpha(t)),\ K_\beta(t)\}$$

At any time some output can be stored in the form of a generalized inventory, denoted X. The purpose of accumulating an inventory is the possibility of later converting it into capital stock. With α capital there is no reason for waiting to exercise this option. One unit of inventory is instantaneously convertible into a single unit of α capital. Under these terms of trade the transition from investment to α capital formation might as well be performed as soon as possible in the more usual direct form

$$\dot{K}_\alpha = I_\alpha ,$$

where I_α denotes investment in α capital.[2]

However, with β capital there is a meaningful distinction between capital accumulation and investment. Because of the presumed increasing returns to scale described in (iii) it will typically be better <u>not</u> to invest directly in β capital. Rather it will pay to first accumulate what

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[2] A dot over a variable denotes differentiation with respect to time. Variables may not be explicitly specified as functions of time if this interpretation is otherwise clear.

could be thought of as either an inventory of materials or as projects in progress. Only after a while should some generalized inventory X be transformed into new K_β available for operation.[3]

Let ΔX represent a portion of generalized inventory X earmarked for conversion into operating β capital. Naturally $- \leq \Delta X \leq X$. Suppose that ΔK_β units of K_β are created, where

$$\Delta K_\beta = G(\Delta X)$$

Reflecting economies of scale, G is taken to be a convex, monotonically increasing, continuous function of ΔX defined for all $\Delta X \geq 0$. It is assumed that $\lim_{\Delta X \to \infty} G(\Delta X) = \infty$, $G(0) = 0$, and

$$\lim_{\Delta X \to 0^+} \frac{G(\Delta X)}{\Delta X} = 0 \qquad (1)$$

Something like the latter condition is necessary to insure that economies of scale are taken advantage of and that in fact generalized inventories must be accumulated for this purpose.[4]

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[3] This reasoning is easy to spell out by a simple example. If a certain amount of material can be molded into a pipeline to carry a given volume per unit time, twice as much material would result in a pipeline able to transport four times the previous volume (assuming a given thickness of pipe). Under these circumstances it may be better to wait a while for a bigger pipeline even though larger inventory stocks would be standing idly by in the interim.

[4] Generalizing to the case where (1) does not hold is not difficult. Continuous adjustments are then possible, as well as discrete jumps.

We will also find it useful to work with the function H, defined as the inverse of G. H can be interpreted as an investment cost function relating the cost in cumulated output units of a given β capital increase according to the schedule

$$\Delta X = H(\Delta K_\beta) = G^{-1}(\Delta K_\beta)$$

Displaying decreasing unit costs, the continuous, monotonically increasing, concave cost function H is defined for all $\Delta K_\beta \geqq 0$ and possesses the properties $\lim_{\Delta K_\beta \to \infty} H(\Delta K_\beta) = \infty$, $H(0) = 0$, and $\lim_{\Delta K_\beta \to 0^+} \frac{H(\Delta K_\beta)}{\Delta K_\beta} = \infty$. An example is illustrated in Figure 1.

4. The Ramsey Model

The social utility of consuming amount $C(t)$ at time t is $U(C(t))$. The instantaneous utility function U is monotonic increasing, concave and differentiable, implying for all $C \geqq 0$ that $U'(C) \geqq 0$ and $U''(C) \leqq 0$. For simplification the condition

$$\lim_{C \to 0^+} U'(C) = \infty$$

is imposed, guaranteeing non-zero consumption for all time. Finally, it is necessary to make a boundedness qualification of the form

$$\sup_{K_\alpha \geqq 0} U(F(K_\alpha)) = B < \infty$$

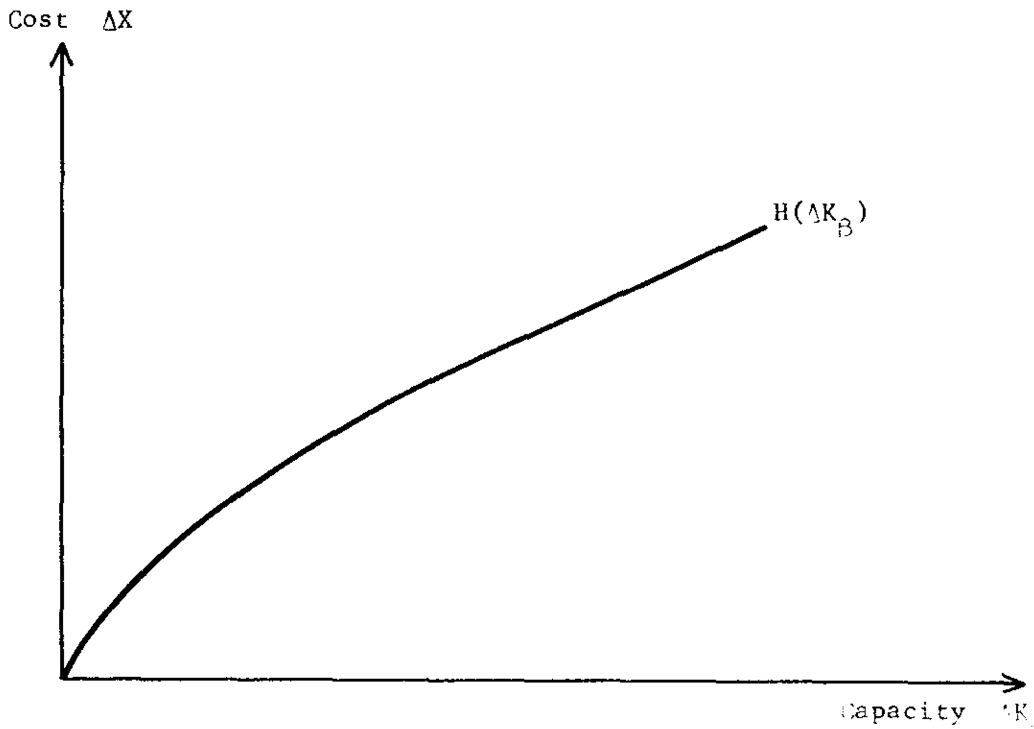

FIGURE 1

A TYPICAL INVESTMENT COST FUNCTION

Ramsey called the least upper bound B the <u>bliss</u> level. A state of bliss would be attained (in the limit) as consumers became sated with goods or as the effects of capital saturation were so pronounced as to make it impossible to increase production past a certain output no matter how much investment were undertaken.

For a given consumption path $\{C(t)\}$, Ramsey defined his social welfare criterion $V[\{C(t)\}]$ as a sum of the difference between instantaneous utility and the bliss level:

$$V[\{C(t)\}] \equiv \int_0^{\infty} [U(C(t)) - B]dt$$

There is no *a priori* reason why this evaluation integral ought to be finite for any given $\{C(t)\}$. Should V be equal to $-\infty$ for each of two consumption paths, there would be nothing to recommend one over the other; however a path yielding a finite value of V would be preferable to both.[5]

Temporarily forgetting all about β capital, Ramsey's problem is to

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[5] The *overtaking criterion* makes $\{C_1(t)\}$ preferable to $\{C_2(t)\}$ if there exists a $\bar{T}$ such that $\int_0^T U(C_1(t))dt > \int_0^T U(C_2(t))dt$ for all $T \geqq \bar{T}$. Were $\{C_1(t)\}$ preferable to $\{C_2(t)\}$ by Ramsey's evaluation integral it would also be preferable by the overtaking criterion. The converse is not necessarily true.

$$\max \int_0^\infty [U(C) - B]dt \tag{2}$$

$$\text{subject to} \quad C + I = F(K) \tag{3}$$

$$\dot{K} = I \tag{4}$$

$$C, I \geqq 0 \tag{5}$$

$$K(0) = K_\alpha(0) > 0, \quad \text{given} \tag{6}$$

Any path $\{K(t), C(t)\}$ satisfying (3)-(6) is called <u>feasible</u>.

It is presumed that U and F are such that (2)-(6) is a meaningful problem in the sense that an optimum exists.[6] The Ramsey optimal path $\{\tilde{K}(t), \tilde{C}(t)\}$ must be feasible, satisfy

$$-\infty < \tilde{V} \equiv V[\{\tilde{C}(t)\}] ,$$

and possess the property that for any feasible path $\{K(t), C(t)\}$,

$$V[\{C(t)\}] \leqq \tilde{V} .$$

Using the calculus of variations, Ramsey was able to characterize the optimal path $\{\tilde{K}(t), \tilde{C}(t)\}$ as the unique solution to the differential equation

$$U'(\tilde{C})\dot{\tilde{K}} = B - U(\tilde{C}) \tag{7}$$

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[6]It is easy to see that (2)-(6) would have to be a well-defined problem if $U(F(K_\alpha))$ reached a maximum for <u>finite</u> K_α. The various sufficiency conditions of Gale and Sutherland [1968], McFadden [1967] and von Weizsäcker [1965] are stated in terms of the overtaking criterion but many of them can be routinely modified to treat the Ramsey evaluation integral case.

$$\tilde{C} + \dot{\tilde{K}} = F(\tilde{K}) \tag{8}$$

with the given initial condition

$$\tilde{K}(0) = K_{\alpha}(0) \tag{9}$$

Equation (7) is the famous Keynes-Ramsey rule of optimal allocation. Along an optimal path both capital and consumption grow monotonically until one or the other goes to or asymptotically approaches its saturation level. The other variable goes to or asymptotically approaches a corresponding level which is determined from the production function. Thus the bliss level is reached at least asymptotically. An optimal solution is shown in Figure 2.

Define

$$\tilde{Y}(t) \equiv F(\tilde{K}(t)) \tag{10}$$

$$q(t) \equiv U'(\tilde{C}(t)) \tag{11}$$

$$\tilde{I}(t) \equiv \dot{\tilde{K}}(t) \tag{12}$$

Note that $\tilde{Y}(t)$ is a continuous monotonic function of t. The dual price $q(t)$ is interpretable as the value of an extra unit of output at time t imputed in terms of the evaluation integral. Differentiating (7) and (8) with respect to time, substituting from (11) and rearranging yields

$$\frac{\dot{q}}{q} = -F'(\tilde{K})$$

a standard relation of optimal growth theory.

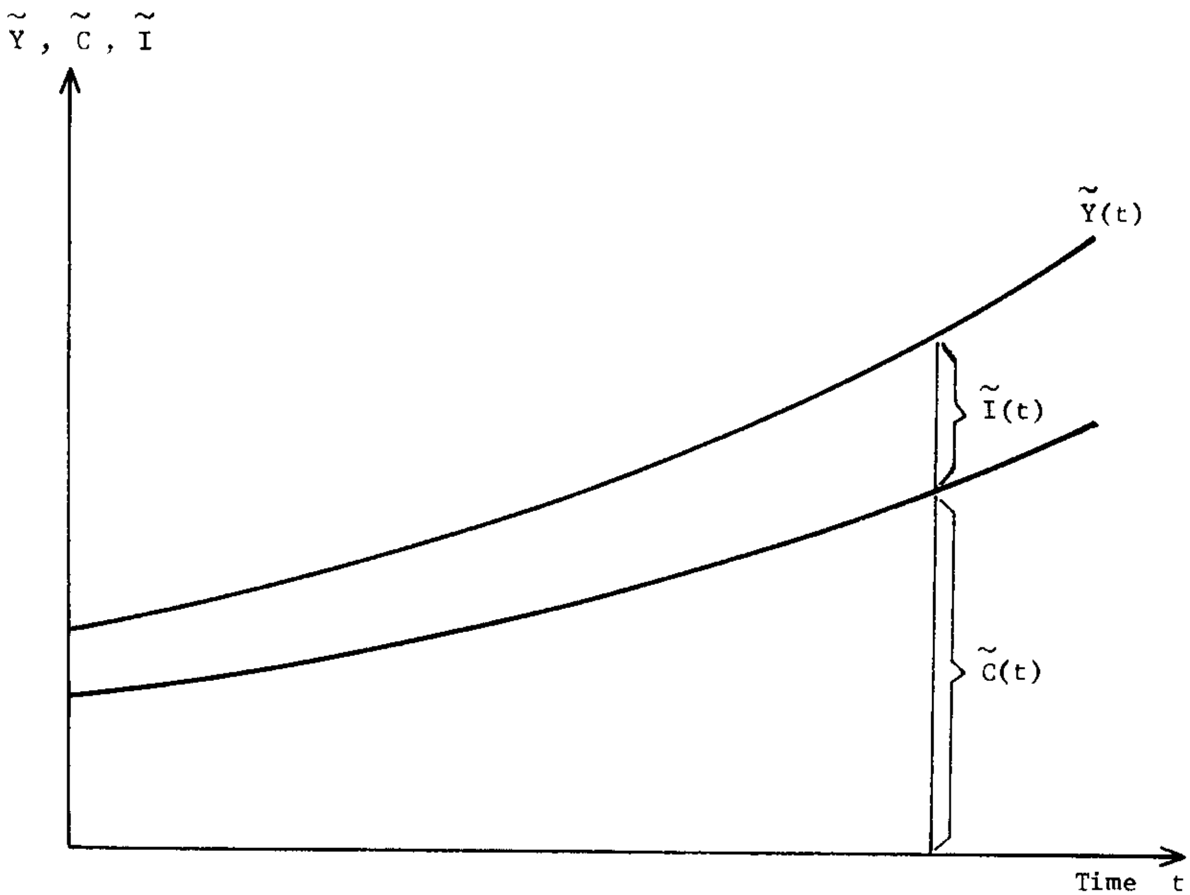

FIGURE 2

THE RAMSEY OPTIMAL GROWTH TRAJECTORY

5. Optimal Capacity Expansion

One more seeming degression is necessary before formally tackling the main problem. Now temporarily neglecting α capital entirely, treat $\{\tilde{Y}(t)\}$ and $\{q(t)\}$ as if they were prescribed data. $\{\tilde{Y}(t)\}$ is considered in the present context to be a fixed final demand schedule which must always be fulfilled. Thinking of $q(t)$ as the cost in current terms of investment funds at time t, the present discounted value of creating extra capacity ΔK at time t is $q(t)H(\Delta K)$.

The least cost capacity expansion problem is to schedule capacity $\{K(t)\}$ to meet final demands $\{\tilde{Y}(t)\}$ at minimum total present discounted cost. Mathematically, the problem is to find times $t_1, t_2, \ldots$ and capacity increments $\Delta K(t_1), \Delta K(t_2), \ldots$ which[7]

$$\min \ \phi[\{t_i\}, \{\Delta K(t_i)\}] \equiv \sum_{i=1}^{\infty} q(t_i)H(\Delta K(t_i)) \tag{13}$$

$$\text{subject to} \quad K(t) \geqq \tilde{Y}(t) \tag{14}$$

$$K(t) = K(t_{i-1}) + \Delta K(t_{i-1}) \quad \text{for} \quad t_{i-1} < t \leq t_i \tag{15}$$

$$K(0) = K_{\beta}(0) \geqq \tilde{Y}(0) \quad \text{given} \tag{16}$$

$$i = 1, 2, \ldots$$

$$t_0 \equiv 0$$

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[7] A formulation in terms of specific times when capacity is increased is a natural one because condition (1) forces a policy of positive capacity augmentation at discrete instants of time. If all the capacity increments went to a zero limit in a non-zero time interval, total investment costs would soar to ∞ because of (1). At times other than $t_1, t_2, \ldots$, $\Delta K(t)$ is thought of as being zero. Manne [1967] discusses several practical examples of optimal capacity expansion problems.

Let $\{\hat{t}_i, \Delta\hat{K}(\hat{t}_i)\}$ represent an optimal policy.[8] Define $\hat{\psi} \equiv \psi[\{\hat{t}_i\}, \{\Delta\hat{K}(\hat{t}_i)\}]$. Using (15), an optimal path can alternatively be expressed by $\{\hat{K}(t)\}$.

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[8] Although the existence of an optimal policy is not proved, we indicate why it has been assumed. Consider a feasible policy of installing δ units of extra capacity whenever $K = \tilde{Y}$. The cost of such a policy would be

$$\Omega(\delta) \equiv \frac{H(\delta)}{\delta} \sum_{j=0}^{\infty} q(\tilde{Y}^{-1}(K(0) + j\delta))\delta .$$ Passing to the limit, $$\lim_{\delta\to0^+} \sum_{j=0}^{\infty} q(\tilde{Y}^{-1}(K(0) + j\delta))\delta = \int_{K(0)}^{\tilde{Y}(\infty)} q(\tilde{Y}^{-1}(\tilde{Y}))d\tilde{Y} = \int_{t_1}^{\infty} q(t)\mathring{\tilde{Y}}dt = \int_{t_1}^{\infty} q(t)F'(\tilde{K})\mathring{\tilde{K}}dt =$$

$$\int_{t_1}^{\infty} F'(\tilde{K})[B - U(\tilde{C})]dt < \infty ,$$ where $t_1 \equiv \tilde{Y}^{-1}(K(0))$. $\frac{H(\delta)}{\delta}$ is well defined for all $\delta > 0$. It follows that for some value $\delta' > 0$ and some $M < \infty$, $\Omega(\delta') = M$. Since an upper bound on ψ exists, so does a least upper bound, denoted $\hat{\psi}$. This means that feasible policies exist with costs arbitrarily close to $\hat{\psi}$. In practice such a result is as good as an existence theorem, given the uncertainties of the data.

Strict existence of an optimal capacity schedule in the mathematical sense could be rigorously proved if we had chosen to formulate the model in terms of period analysis instead of continuous time. With M an upper bound on ψ , we can restrict $\Delta K(n)$ to values $0 \leqq \Delta K \leqq G\left(\frac{M}{q(n)}\right)$ for $n = 0, 1, 2, \ldots$ A standard application of the Tychnoff theorem (cf. Kelley [1955], p. 143) shows the Cartesian product set $Q \equiv \bigtimes_{n=0}^{\infty} [0, G\left(\frac{M}{q(n)}\right)]$ to be compact.

Define the function $\theta \equiv \min\{\psi, M\}$ for all $\bigtimes_{n=0}^{\infty} \Delta K(n) \in Q$. Being real valued and continuous under the product topology on the compact set Q , θ must attain a minimum for some value $\bigtimes_{n=0}^{\infty} \Delta\hat{K}(n) \in Q$, concluding the proof.

It is easy to see that along an optimal path, $\hat{K}(\hat{t}_i) = \tilde{Y}(\hat{t}_i)$. No extra capacity will be installed while some excess capacity already exists. With $\dot{q} \leqq 0$, it pays to take advantage of economies of scale and postpone intended construction until the day when some must be undertaken anyway because full capacity will have been reached.[9]

More specific properties of an optimal policy would have to depend upon the particular shapes of $\{\tilde{Y}(t)\}$ and $\{q(t)\}$. Later we sharply characterize an optimal capacity schedule for a certain parameterization. A typical minimum cost policy is illustrated in Figure 3.

6. Formulation and Solution of the Basic Model

Having treated separately the Ramsey and Capacity Expansion problems, we are in a position to tackle the main problem introduced in Section 3. Properties of the present model are vaguely recognizable as some sort of a rough combination of features belonging to the two simpler problems. As we shall see, the optimal solution will combine, in a well defined sense, the Ramsey and capacity expansion optimal trajectories.

We use the same social objective as Ramsey. However, in the context in which it is presently employed, $\int_0^\infty [U(C) - B]dt$ is denoted $W[\{C(t)\}]$ to avoid confusion.

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[9]This simple conclusion is an example of a "regeneration point theorem." Such a result greatly simplifies computation because the search for an optimum can be limited to full capacity regeneration points and these can typically be efficiently examined via the appropriate dynamic programming algorithm.

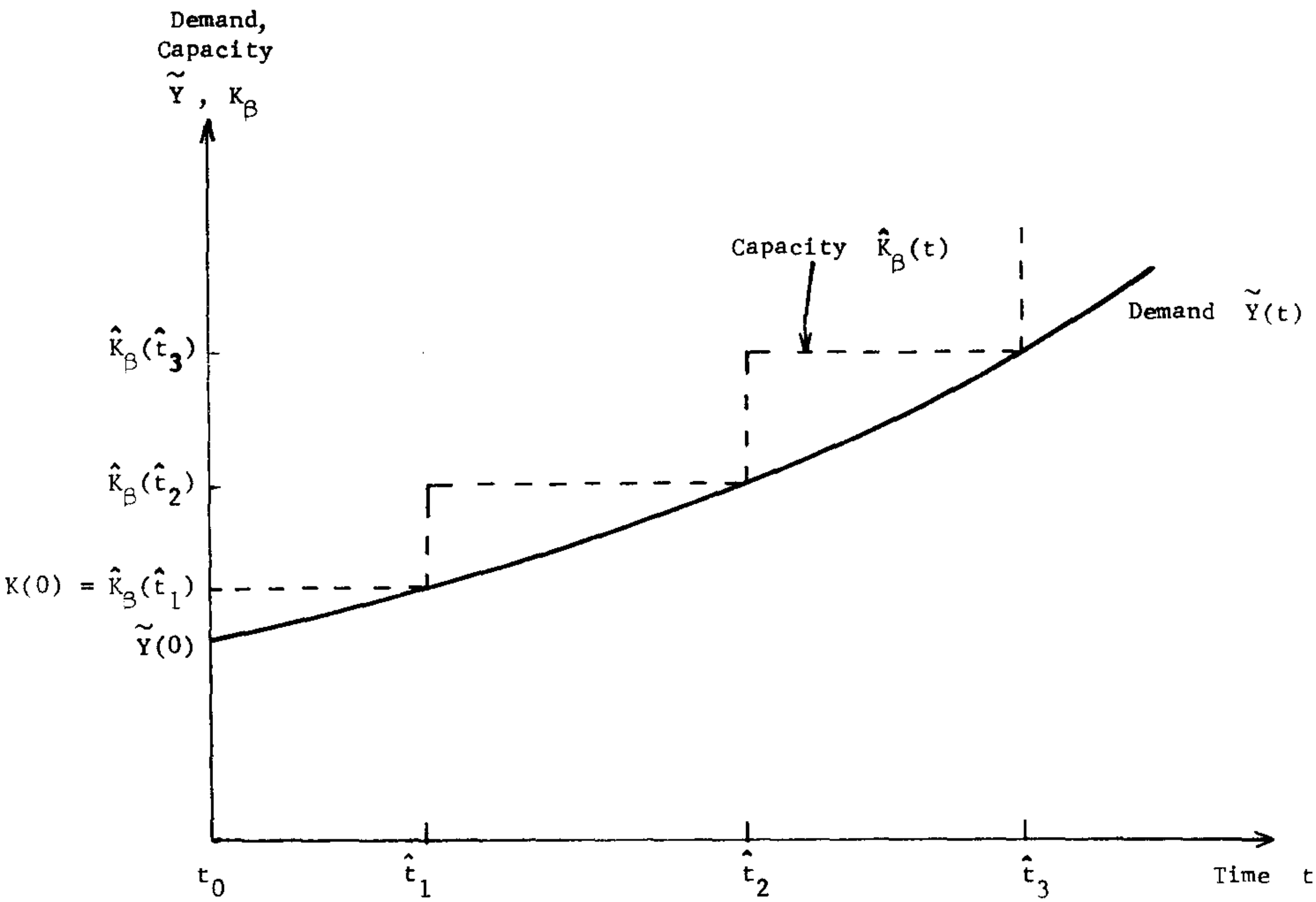

FIGURE 3

TIME PATHS OF FINAL DEMAND AND INSTALLED CAPACITY

The problem is to select times $\bar{t}_1, \bar{t}_2, \ldots$ and to choose values for the instruments $Y(t)$, $C(t)$, $I(t)$, $I_\alpha(t)$, $I_x(t)$, $\Delta X(\bar{t}_i)$, and $\Delta K_\beta(\bar{t}_i)$ to[10]

$$\max \quad W[\{C(t)\}] \equiv \int_0^\infty [U(C(t)) - B]dt \tag{17}$$

subject to

$$Y(t) \leqq F(K_\alpha(t)) \tag{18}$$

$$Y(t) \leqq K_\beta(t) \tag{19}$$

$$C(t) + I(t) = Y(t) \tag{20}$$

$$I_\alpha(t) + I_x(t) = I(t) \tag{21}$$

$$\dot{K}_\alpha(t) = I_\alpha(t) \tag{22}$$

$$\dot{X}(t) = I_x(t) \quad \text{for} \quad t \neq \bar{t}_i \tag{23}$$

$$\lim_{\epsilon \to 0^+} X(\bar{t}_i + \epsilon) = X(\bar{t}_i) - \Delta X(\bar{t}_i) \tag{24}$$

$$\Delta K_\beta(\bar{t}_i) = G(\Delta X(\bar{t}_i)) \tag{25}$$

$$K_\beta(t) = K_\beta(\bar{t}_{i-1}) + \Delta K_\beta(\bar{t}_{i-1}) \quad \text{for} \quad \bar{t}_{i-1} < t \leqq \bar{t}_i \tag{26}$$

$$Y(t),\ C(t),\ I(t),\ I_\alpha(t),\ I_x(t),\ X(t),\ \Delta X(\bar{t}_i) \geqq 0 \tag{27}$$

$$K_\alpha(0),\ K_\beta(0),\ X(0) \quad \text{given} \tag{28}$$

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[10] For economy of notation, positive values of ΔX and positive changes in K_β have been restricted *a priori* to discrete times $\{\bar{t}_i\}$. In fact this is a vacuous restriction because condition (1) implies that even with the possibility of continuous adjustments available, an optimal policy would always call for jump adjustments in K_β at certain distinct times. For times $t \neq \bar{t}_i$, it is useful to interpret $\Delta X(t)$ and $\Delta K_\beta(t)$ as being zero. Note that $\Delta X(\bar{t}_i)$ is defined as *minus* the change in X at time $\bar{t}_i$.

Where not otherwise noted, t is any non-negative time and i is any positive integer. By convention $\bar{t}_0 \equiv 0$.

With only insignificant loss of generality, we restrict attention to initial configurations of the state variables K_α, K_β, X obeying[11]

$$K_\beta(0) \geqq F(K_\alpha(0))$$

$$X(0) = 0 .$$

The following theorem is the main result.

Define $\{\underline{t}_i^*\}$ and $\{\bar{t}_i^*\}$ by the recursive equations

$$\underline{t}_i^* = \hat{t}_i + \sum_{j=1}^{i-1}(\bar{t}_j^* - \underline{t}_j^*)$$

$$\bar{t}_i^* = \underline{t}_i^* + \frac{H(\Delta\hat{K}(\hat{t}_i))}{\tilde{I}(\hat{t}_i)}$$

$$\bar{t}_0^* \equiv 0$$

$$i = 1, 2, \ldots$$

Under the assumptions of the model, an optimal solution of (17)-(28), whose variables are denoted with asterisks, can be described as follows:

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[11] Later it should become clear what to do if these initial conditions should not be met. In fact an optimal policy will always call for $K_\beta(t) \geqq F(K_\alpha(t))$ -- for all $t \geqq 0$ if $K_\beta(0) \geqq F(K_\alpha(0))$, and for all sufficiently large t if $K_\beta(0) < F(K_\alpha(0))$. The latter condition must have therefore arisen, in some sense, out of mismanagement prior to time zero. The case of $X(0)$ positive is not difficult to handle; we normalize to zero just for notational convenience.

$$\Delta K_\beta^*(t) = \Delta X^*(t) = 0 \qquad t \neq \bar{t}_i^*$$

$$\Delta K_\beta^*(\bar{t}_i^*) = \Delta\hat{K}(\hat{t}_i)$$

$$\Delta X^*(\bar{t}_i^*) = H(\Delta\hat{K}(\hat{t}_i))$$

$$\left.\begin{aligned}
K_\alpha^*(t) &= \tilde{K}(t - \sum_{j=1}^{i-1} (\bar{t}_j^* - \underline{t}_j^*)) \\
K_\beta^*(t) &= \hat{K}(\hat{t}_i) \\
X^*(t) &= 0 \\
Y^*(t) &= \tilde{Y}(t - \sum_{j=1}^{i-1} (\bar{t}_j^* - \underline{t}_j^*)) \\
C^*(t) &= \tilde{C}(t - \sum_{j=1}^{i-1} (\bar{t}_j^* - \underline{t}_j^*))
\end{aligned}\right\} \quad \bar{t}_{i-1}^* < t < \underline{t}_i^*$$

$$\left.\begin{aligned}
K_\alpha^*(t) &= \tilde{K}(\hat{t}_i) \\
K_\beta^*(t) &= \hat{K}(\hat{t}_i) \\
X^*(t) &= \tilde{I}(\hat{t}_i)(t - \underline{t}_i^*) \\
Y^*(t) &= \tilde{Y}(\hat{t}_i) \\
C^*(t) &= \tilde{C}(\hat{t}_i)
\end{aligned}\right\} \quad \underline{t}_i^* \leqq t \leqq \bar{t}_i^*$$

Fortunately a relatively simple interpretation can be placed on this formidable looking prescription.

The optimal policy is depicted in Figure 4. Suppose that all the $[\underline{t}_i^*, \bar{t}_i^*]$ sections of that diagram were compressed into a point and

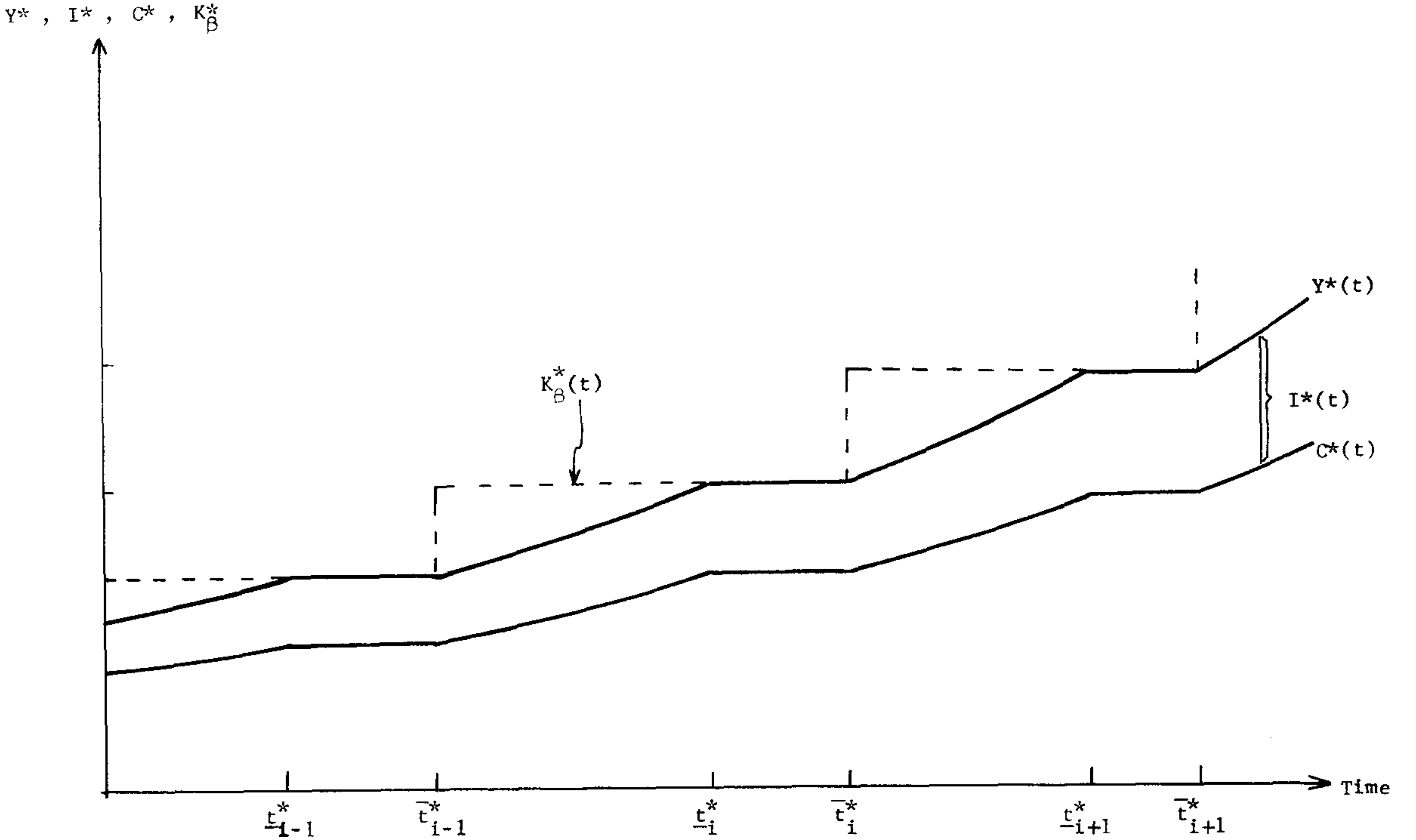

FIGURE 4

THE OPTIMAL TRAJECTORY
(chosen for the case $i = 2$)

the gaps removed by pushing together and connecting the $\{Y^*(t)\}$, $\{I^*(t)\}$ and $\{C^*(t)\}$ curves. The resulting trajectories would be exactly the $\{\tilde{Y}(t)\}$, $\{\tilde{I}(t)\}$, $\{\tilde{C}(t)\}$ graphs of Figure 2. In other words, the sole effect of introducing the $[\underline{t}_i^*, \overline{t}_i^*]$ regions into Figure 4 is to stretch out into positive intervals what would otherwise be isolated single points of the original Ramsey optimal trajectory.

An analogous interpertation is available for β capital. Except for the $[\underline{t}_i^*, \overline{t}_i^*]$ sections, the time profile of K_β in Figure 4 would look identical to the Figure 3 portrayal of an optimal capacity schedule $\{\hat{t}_i, \Delta\hat{K}(\hat{t}_i)\}$. It is easily seen that the $\{\hat{t}_i\}$ of Figure 3 are exactly those isolated points of the Ramsey optimal policy which in Figure 4 are stretched out into time segments of positive length.

While time seems to stand still during the period $[\underline{t}_i^*, \overline{t}_i^*]$, all investment is going into accumulating an inventory of $X(\overline{t}_i^*)$ starting from an initial level of zero at time $\underline{t}_i^*$. The interval $[\underline{t}_i^*, \overline{t}_i^*]$ lasts precisely long enough to build up that amount of X which will coalesce to form the $\Delta\hat{K}(\hat{t}_i)$ units of β capital dictated by the solution to the capacity expansion problem. Throughout the period $[\underline{t}_i^*, \overline{t}_i^*]$, $Y^*(t)$, $I^*(t)$, $C^*(t)$ are maintained at the constant levels $\tilde{Y}(\hat{t}_i)$, $\tilde{I}(\hat{t}_i)$, $\tilde{C}(\hat{t}_i)$. At time $\overline{t}_i^*$ the entire generalized inventory $X(\overline{t}_i^*)$ is formed into β capital and the economy picks up again at that point of the Ramsey trajectory where it left off at time $\underline{t}_i^*$ because the K_β ceiling had become binding.

In terms of economic development, $[\underline{t}_i^*, \bar{t}_i^*]$ represents a big push period.[12] During this time all investment is being funneled at a constant rate into the as yet unproductive overhead capital project. As will presently be demonstrated, big push stages are likely to be of longer duration and to occur more frequently in the earlier stages of development. From the viewpoint of social policy, the big push is probably a critical time because no real growth occurs and consumption is stagnant. It is hard to see how this kind of restraint could be extended for significant lengths of time without the imposition of economic controls.

Let $p(t)$ represent the value of an extra unit of output at time t imputed in terms of the evaluation integral. Obviously $p(t) = U'(C^*(t))$, so that

$$p(t) = q(t - \sum_{j=1}^{i-1} (\bar{t}_j^* - \underline{t}_j^*) \qquad \bar{t}_{i-1}^* < t < \underline{t}_i^*$$

$$p(t) = q(\hat{t}_i) \qquad \underline{t}_i^* \leqq t \leqq \bar{t}_i^*$$

$$i = 1, 2, \ldots$$

For t belonging to the Ramsey growth phase $(\bar{t}_{i-1}^*, \underline{t}_i^*)$, the dual price declines over time; an extra unit of output is worth more if it is received early because it could be productively invested in α capital to yield increased future returns. However, in the big push phase $[\underline{t}_i^*, \bar{t}_i^*]$, the social output price is stationary; whether received early or late in a big push stage, an extra unit of output cannot be used to increase returns but could only be invested in non-directly-productive generalized inventory.

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[12] Cf. Rosenstein-Rodan [1961], Hirschman [1958], Scitovsky [1959].

7. Proof of the Main Theorem

The strategy of proof can be easily outlined. A consumption path $\{C^e(t)\}$ is efficient in the usual sense if it is feasible and if, for any other feasible path $\{C^f(t)\}$ with the property $C^f(t) \geqq C^e(t)$ for all $t \geqq 0$, $\int_0^\infty [C^f(t) - C^e(t)]dt = 0$. We first exhibit four obvious efficiency conditions (a)-(d). Without loss of generality, any candidate for an optimal path can be restricted to satisfy these four criteria. Next, considering only comparisons among paths so restricted, we show that an optimal solution must belong to an even more exclusive family of paths with special additional features. Finally, the proposed solution is shown to be an optimal member of this family of special paths.

The four efficiency conditions are:

(a) If $F(K_\alpha(t)) < K_\beta(t)$ and $X(t) > 0$, there is no loss of generality in restricting $\Delta X(t)$ to be zero.

Condition (a) is needed only to prove (b) and will not be directly used for any other purpose. Let τ be the first time later than t when $F(K_\alpha) = K_\beta$ (if this never happens, $\tau \equiv \infty$). It certainly won't depress consumption at any future or present time to put off until τ setting $\Delta X > 0$. It may even help to wait and take advantage of increasing returns by later converting a bigger chunk of ΔX into more ΔK_β.

(b) <u>If $F(K_\alpha(t) < K_\beta(t)$, no investment in X occurs at time t.</u>

This condition specifies that all investment must go into building α capital if there is excess β capacity. As in (a), let τ be the first time later than t when $F(K_\alpha) = K_\beta$. Suppose contrary to the hypothesis that $\int_t^\tau \dot{X}(t')dt' = \gamma > 0$. From (a), we need to treat only the case $\Delta X(t') = 0$ for all $t' \in [t, \tau)$. Since nothing is going to get done with X anyway until time τ at the earliest, we can consider a different policy. Maintain at all times the same consumption levels as in the original policy. Now, however, devote all investment to first building up K_α to level $K_\alpha(\tau) = F^{-1}(K_\beta(\tau))$ and then put all investment into creating amount γ of X. Let this alternative policy take $(\tau' - t)$ time units to complete. It is easy to see that $\tau' < \tau$ because the second alternative takes advantage of the direct productivity of α capital for all times when $F(K_\alpha) < K_\beta$. Thus, the identical stocks of K_α, K_β, and X are achieved at time τ and consumption is the same for all times belonging to $[t, \tau']$, but the second alternative permits attainment of a strictly <u>higher</u> amount of consumption at level $K_\beta(\tau) = F(K_\alpha(\tau))$ for all times of $(\tau', \tau]$. It follows that the original path must have been inefficient.

(c) <u>If $F(K_\alpha(t)) \geqq K_\beta(t)$, no investment in K_α occurs at time t.</u>

This condition prohibits investment in α capital so long as full capacity already exists in that sector. Let τ be the first time <u>after</u> t when

$\Delta K_\beta > 0$. Suppose that $F(K_\alpha(t')) \geqq K_\beta(t')$ for $t' \varepsilon (t, \tau]$ and that $\int_t^{\tau} \dot{K}_\alpha(t')dt' = \gamma > 0$. It is easy to see that a superior policy is to first invest only in X and then to coalesce $H(\Delta K_\beta(\tau))$ units of X into $\Delta X_\beta(\tau)$ as soon as $X(\tau) - X(t)$ units of X have been accumulated, say at time $\tau' < \tau$. Only then should the γ units of α capital be built up. All the while the same consumption levels of the original policy are maintained. This alternative ends up with the same values of K_α, K_β, and X immediately after time τ, maintains an identical consumption level for all times belonging to $[t, \tau']$, and still allows a splash of extra consumption to occur in the interval $(\tau', \tau]$ due to taking advantage of the direct productivity of α capital.

(d) If $F(K_\alpha(t)) \geqq K_\beta(t)$, if $X(t) = 0$, and if τ is the first time after t when $\Delta X > 0$, then $\Delta X(\tau) = X(\tau)$.

This condition requires that all X built up from zero for the purpose of increasing K_β must be coalesced into ΔK_β all at once. Suppose to the contrary that $\Delta X(\tau) = \gamma < X(\tau)$. Let $\tau' < \tau$ be such that $\int_t^{\tau'} \dot{X}(t')dt' = \gamma$. If $F(K_\alpha(t)) = K_\beta(t)$, it will be better to set $\Delta K(\tau') = \gamma$ and then, duplicating the reasoning of (b), to restrict further immediate investments to α capital. The advantage of setting $\Delta K(\tau') = \gamma$ for the case $F(K_\alpha(t)) > K_\beta(t)$ is obvious. In either event, the original path is inefficient.

The following comments apply to efficient paths.

It follows from (c) and the initial condition $F(K_\alpha(0)) \leqq K_\beta(0)$ that $F(K_\alpha(t)) \leqq K_\beta(t)$ for all $t \geqq 0$. Let $\{\bar{t}_i\}$ represent the times (in order) at which $\Delta X > 0$; that is, $\Delta X(t) > 0$ if and only if $t = \bar{t}_i$ for some positive integer i. Immediately after $\bar{t}_{i-1}$, $F(K_\alpha) < K_\beta$, and all investment goes into α capital. Let $\underline{t}_i$ be the <u>first</u> time <u>after</u> $\bar{t}_{i-1}$ when $F(K_\alpha) = K_\beta$. More formally, $\underline{t}_i$ minimizes t subject to $t \geqq \bar{t}_{i-1}$ and $F(K_\alpha(t)) = K_\beta(t)$; if there is no feasible solution, $\underline{t}_i \equiv \infty$. Define $R \equiv \bigcup_{i=1}^{\infty} (\bar{t}_{i-1}, \underline{t}_i)$, with $\bar{t}_0 \equiv 0$. If $t \in R$, $F(K_\alpha(t)) < K_\beta(t)$, and, from (b), $I(t) = I_\alpha(t)$. Let $S \equiv \bigcup_{i=1}^{\infty} [\underline{t}_i, \bar{t}_i]$. For $t' \in S$, $F(K_\alpha(t')) = K_\beta(t')$, and (c) stipulates that $I(t') = I_x(t')$. Thus any efficient path can be divided into two distinct investment specialization phases. Property (d) combined with the initial condition $X(0) = 0$ implies that $\Delta X(\bar{t}_i) = X(\bar{t}_i)$, or that $X(t) = 0$ for $t \in R$.

Let ϕ_i be the value of an optimal policy starting at time $\underline{t}_i$. Note that ϕ_i is a function of $K_\alpha(\underline{t}_i)$, $K_\beta(\underline{t}_i)$ and $X(\underline{t}_i)$ but does not depend on $\underline{t}_i$ <u>explicitly</u> because the objective function and the constraints of the main problem are time autonomous. We know that all investment during the period $(\bar{t}_{i-1}, \underline{t}_i)$ is restricted to α capital alone. Consider $\bar{t}_{i-1}$, $K_\alpha(\bar{t}_{i-1})$, $K_\alpha(\underline{t}_i)$, as fixed. To be part of an optimal path, $\underline{t}_i$, treated as a variable, must be chosen to

$$\max \int_{\bar{t}_{i-1}}^{\underline{t}_i} [U(C) - B]\,dt + \phi_i \tag{29}$$

$$\text{subject to} \quad C + I_\alpha = F(K_\alpha) \tag{30}$$

$$\dot{K}_\alpha = I_\alpha \tag{31}$$

$$C,\ I_\alpha \geqq 0 \tag{32}$$

$$\bar{t}_{i-1}\ ,\quad K_\alpha(\bar{t}_{i-1})\ ,\quad K_\alpha(\underline{t}_i) \quad \text{given} \tag{33}$$

Using the calculus of variations, the solution to this free-time fixed-endpoint problem is

$$U'(C)\dot{K}_\alpha = B - U(C) \tag{34}$$

$$C + \dot{K}_\alpha = F(K_\alpha) \tag{35}$$

$$\bar{t}_{i-1}\ ,\quad K_\alpha(\bar{t}_{i-1})\ ,\quad K_\alpha(\underline{t}_i) \quad \text{given} \tag{36}$$

This, of course, is nothing but a section of the Ramsey optimal trajectory (7)-(9).

Now consider an analogous problem over the time interval $[\underline{t}_i, \bar{t}_i]$. Let ϕ_i' be the value of an optimal policy starting at time $\bar{t}_i$ with capital stocks $K_\alpha(\bar{t}_i)$, $K_\beta(\bar{t}_i)$, $X(\bar{t}_i)$. All investment during the period $[\underline{t}_i, \bar{t}_i]$ goes into building up an inventory of $X(\bar{t}_i)$ starting from zero. Also $Y(t) = Y(\underline{t}_i) = F(K_\alpha(\underline{t}_i)) = K_\beta(\underline{t}_i) = Y(\bar{t}_i)$ is stationary for $t \in [\underline{t}_i, \bar{t}_i]$.

Look upon $Y(t)$, $\underline{t}_i$, $X(\underline{t}_i)$ and $X(\overline{t}_i)$ as constants. An optimal policy must have the property that $\overline{t}_i$, treated here as a variable, is selected to

$$\max \int_{\underline{t}_i}^{\overline{t}_i} [U(C) - B]dt + \phi_i' \tag{37}$$

$$\text{subject to} \quad C + I_x = Y(\underline{t}_i) \tag{38}$$

$$\dot{X} = I_x \tag{39}$$

$$C,\ I_x \geqq 0 \tag{40}$$

$$\underline{t}_i\ ,\quad X(\underline{t}_i) = 0\ ,\quad X(\overline{t}_i)\ ,\quad Y(\underline{t}_i) \quad \text{given} \tag{41}$$

The calculus of variations solution to this free-time fixed-end-point problem is

$$U'(C)\dot{X} = B - U(C) \tag{42}$$

$$C + \dot{X} = Y(\underline{t}_i) = F(K_\alpha(\underline{t}_i)) \tag{43}$$

$$\underline{t}_i\ ,\quad X(\underline{t}_i) = 0\ ,\quad X(\overline{t}_i)\ ,\quad Y(\underline{t}_i) \quad \text{given} \tag{44}$$

Optimal values of $I_x(t)$ and $C(t)$ are constants for $t \in [\underline{t}_i, \overline{t}_i]$. They are equal to the optimal values, respectively, of $I_\alpha(t)$ and $C(t)$ in the solution of problem (29)-(33) at $t = \underline{t}_i$, since at that time equations (34)-(36) and (42)-(44) are identical. Using the same reasoning, $I_x(t)$ and $C(t)$ from (42)-(44) are also equal to the optimal values of $I_\alpha(t)$

and $C(t)$ *at time* $t = \bar{t}_i$ *for the free-time fixed-endpoint problem which* is identical to (29)-(33) except for taking place over the interval $(\bar{t}_i, \underline{t}_{i+1})$ instead of $(\bar{t}_{i-1}, \underline{t}_i)$.

These results justify the basic features of an optimal policy as they have been depicted in Figure 5. $Y^*(t)$, $I^*(t)$, and $C^*(t)$ are continuous. Each big push stage of S is just a single Ramsey optimal point stretched out into an interval of positive length. If these isolated points were reassembled back into the appropriate niches of R and the resulting conglomeration treated as if it were a set connected under continuous time, the complete Ramsey trajectory (7)-(9) would emerge. A Ramsey growth stage of R ends whenever full β capacity is encountered and it becomes necessary to devote investment to building generalized inventories. Ramsey growth continues as soon as all inventories have been converted into excess β capacity.

In our search for an optimal policy, no generality will be lost if further attention is restricted to the family of paths described above. The social objective $W[\{C(t)\}]$ can then be split up as follows

$$W[\{C(t)\}] = \int_0^\infty [U(C) - B]dt = \int_R [U(C) - B]dt + \int_S [U(C) - B]dt \ .$$

As we have seen, the value of the former integral along an optimal path must be $\tilde{V}$, the optimal Ramsey objective. As for the latter integral,

$$\int_S [U(C) - B]dt = \sum_{i=1}^{\infty} [U(C(\bar{t}_i)) - B](\bar{t}_i - \underline{t}_i)$$

$$= \sum_{i=1}^{\infty} [-U'(C(\bar{t}_i))I(\bar{t}_i)] \frac{H(\Delta K_\beta(\bar{t}_i))}{I(\bar{t}_i)}$$

$$= -\sum_{i=1}^{\infty} U'(C(\bar{t}_i)H(\Delta K_\beta(\bar{t}_i)) . \tag{44a}$$

where $\underline{t}_{i+1} = \bar{t}_i + \tilde{Y}^{-1}(Y(\bar{t}_i) + \Delta K_\beta(\bar{t}_i)) - \tilde{Y}^{-1}(Y(\bar{t}_i))$.

Think of $t_i \equiv \underline{t}_i - \sum_{j=1}^{i-1} (\bar{t}_j - \underline{t}_j)$ as the singular Ramsey optimal point which has been magnified into the interval $[\underline{t}_i, \bar{t}_i]$. In terms of the capacity expansion problem (13)-(16), (44a) is equal to

$$-\sum_{i=1}^{\infty} U'(\tilde{C}(t_i))H(\Delta K(t_i)) = -\sum_{i=1}^{\infty} q_{\not}(t_i)H(\Delta K(t_i))$$

$$= -\psi[\{t_i\}, \{\Delta K(t_i)\}] ,$$

where $t_{i+1} = \tilde{Y}^{-1}(\tilde{Y}(t_i) + \Delta K(t_i))$.

Thus $\int_R [U(C) - B]dt$ will be maximized whenever ψ is minimized subject to the constraints (14)-(16). We can translate from $\{\hat{t}_i\}$ to $\{\underline{t}_i^*\}$ and $\{\bar{t}_i^*\}$ by using the relations $\underline{t}_i^* = \hat{t}_i + \sum_{j=1}^{i-1} (\bar{t}_j^* - \underline{t}_j^*)$ and $\bar{t}_i^* - \underline{t}_i^* = \dfrac{H(\Delta\hat{K}(\hat{t}_i))}{\tilde{I}(\hat{t}_i)}$.

This concludes the proof.[13] An interesting side result is that

$$W^* = \tilde{V} - \hat{\psi}$$

The maximum social objective is less the optimal value of the Ramsey problem by the social cost of implementing the cheapest feasible capacity schedule.

8. Capacity Expansion at a Constant Geometric Rate

The features of an optimal capacity schedule can be particularized by restricting the general functions $H(\Delta K)$, $\tilde{Y}(t)$, and $q(t)$ to specific parameterizations. Henceforth we consider a constant elasticity investment cost function of the form[14]

$$H(\Delta K) = A(\Delta K)^a \tag{45}$$

where A is just a positive constant of proportionality. The exponent a measures the (constant) ratio of incremental to average costs of installed

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[13]Note the critical dependence of our results on the time autonomy of the system under consideration. In the original Ramsey problem the introduction of discounting, population growth, and time dependent depreciation or technical change does very little to change the basic properties of a solution (so long as the time dependence is exponential!). This is not the case for the present model. If time dependent features were introduced, the qualitative properties of a welfare maximizing path would be roughly similar, but there would be no possibility of cleanly decomposing the optimal trajectory into two distinct sub-problems. On the other hand, incorporating into the model such time autonomous features as Arrow's "learning by doing" [1962] would not significantly change the nature of an optimal trajectory.

[14]This parameterization is popular as a pre-design approximation to investment costs of installed capacity for such process industries as chemicals, petroleum, cement, electricity generation, and primary metals. It should be emphasized that in the cost engineering literature ΔK would refer to the extra capacity created from the establishment of a typical complete process

capacity. Consistent with the presumed existence of economies of scale,

$$0 < a < 1$$

The investment cost function (45) is depicted in Figure 1 with $a = 2/3$.

We also assume that

$$\tilde{Y}(t) = \left(\frac{q(t)}{\ell} \right)^{-s} \qquad (46)$$

for some constants ℓ and s obeying

$$0 < \ell$$

$$0 < s < \frac{1}{a} .$$

An equivalent way of writing (46) is

$$g(t) = sr(t) \qquad (47)$$

where $g(t) \equiv \dot{\tilde{Y}}/\tilde{Y}$ is the growth rate of final demand and $r(t) \equiv - \dot{q}/q$ is the discount rate, both evaluated at time t .

In the framework of the isolated capacity expansion model (13)-(16), expression (47) can be looked upon purely as a convenient parameterization of the underlying data. No formal reference need be made to any other considerations, since the capacity scheduling model is of interest in its own right. The parameterization (47) generalizes the case usually

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balanced plant and not to the creation of semi-mythical "social overhead capital" for the economy as a whole. Cost engineers usually treat the exponent a as being equal to about two-thirds. Cf. Chilton [1960], Haldi and Whitcomb [1967], and Moore [1959].

analyzed where fixed demand grows exponentially and the discount rate is constant.

However, within the context of the two-sector model we have been analyzing, relation (47) has an interesting additional interpretation. It could come about as a result of society choosing to save at the constant rate s :

$$g(t) = \frac{\dot{\tilde{Y}}}{\tilde{Y}} = \frac{F'(\tilde{K})\dot{\tilde{K}}}{\tilde{Y}} = sF'(\tilde{K}) = -s\,\frac{\dot{q}}{q} = sr(t)$$

In turn, a proportional savings function is the optimal behavior befitting any solvable Ramsey problem with a constant elasticity of marginal utility. Such a utility function is of the form

$$U(C) = B - DC^{1-\eta} .$$

where $B > 0$, $D > 0$ and $\eta = -\frac{U''}{U'}C > 1$ are given constants. The proposition about a constant savings rate is easily obtained by following the equations

$$U'(\tilde{C})\dot{\tilde{K}} = B - U(\tilde{C})$$

$$-(1-\eta)D\tilde{C}^{-\eta}\,\frac{s(t)}{1 - s(t)}\,\tilde{C} = D\tilde{C}^{1-\eta}$$

$$\frac{-(1-\eta)}{\left(\frac{1}{s(t)} - 1\right)} = 1$$

to the conclusion $s(t) = \frac{1}{\eta}$.

With the specific parameterization (45), (46) the minimum cost capacity schedule has a particularly simple characterization. When capacity must be increased (because no more slack exists), it is always incremented by a constant percentage of existing capacity.[15]

We prove this interesting result by considering a schedule $\{t_i, \Delta K(t_i)\}$ which is a candidate for minimizing current discounted cost ψ. Without loss of generality it can be presumed that $K(t_i) = \tilde{Y}(t_i)$, so that no extra capacity is installed while excess capacity is already in place.

$$\psi = \sum_{i=1}^{\infty} q(t_i) H(\Delta K(t_i))$$

$$= \sum_{i=1}^{\infty} \ell [K(0) + \sum_{j=1}^{i-1} \Delta K(t_j)]^{-\frac{1}{s}} A[\Delta K(t_i)]^a$$

$$= \ell A K(0)^{a - \frac{1}{s}} \sum_{j=1}^{\infty} \left[1 + \sum_{j=1}^{i-1} \frac{\Delta K(t_j)}{K(0)} \right]^{-\frac{1}{s}} \left[\frac{\Delta K(t_i)}{K(0)} \right]^a \qquad (48)$$

Because ψ can be written in the form (48), it is apparent that the cost minimizing values of $\left\{ \frac{\Delta K(t_i)}{K(0)} \right\}$ are independent of $K(0)$. Now

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[15]Note that if $g(t)$ (and hence $r(t)$ also) is constant, an optimal policy would call for scheduling extra capacity increments at equally spaced intervals of length $\frac{1}{g} \ln \left(1 + \frac{\Delta \hat{K}(\hat{t}_1)}{K(0)} \right)$. A constant cycle time result was proved by Srinivasin [1967] for the special case mentioned above.

consider the problem of finding a least cost capacity schedule which begins at time t_n with capacity $K(t_n)$ instead of at time 0 with capacity $K(0)$. This is a sub-problem of the original. The cost function for the new problem can be written in a form identical to (48) except for obvious index renumbering and the interchanged roles of $K(t_n)$ and $K(0)$. But the optimal incremental capacity sequence expressed in units of initial capacity is independent of the initial capacity level. Hence, for an optimal path $\{\hat{t}_i, \Delta\hat{K}(\hat{t}_i)\}$,

$$\frac{\Delta\hat{K}(t_i)}{\hat{K}(t_i)} = \frac{\Delta\hat{K}(\hat{t}_1)}{K(0)} \qquad i = 1, 2, \ldots$$

Let

$$\mu \equiv \frac{\Delta K(t_i)}{K(t_i)} \qquad i = 1, 2, \ldots$$

be the constant fraction by which capacity is always incremented. Then $\Delta K(t_i) = \mu K(0)(1+\mu)^{i-1}$ $i = 1, 2, \ldots$. Substituting into (48), we obtain

$$\psi(\mu) = \ell A K(0)^{a-\frac{1}{s}} \sum_{i=1}^{\infty} [1 + \sum_{j=1}^{i-1} \mu(1+\mu)^{j-1}]^{-\frac{1}{s}} [\mu(1+\mu)^{i-1}]^a$$

$$= \ell A K(0)^{a-\frac{1}{s}} \sum_{i=1}^{\infty} [(1+\mu)^{i-1}]^{-\frac{1}{s}} [\mu(1+\mu)^{i-1}]^a$$

$$= \ell A K(0)^{a-\frac{1}{s}} \frac{\mu^a}{1 - (1+\mu)^{a-\frac{1}{s}}} \tag{49}$$

Differentiating (49) with respect to μ and equating the derivative to zero, the following equation is obtained

$$a(1+\mu)^{\frac{1}{s} - a+1} - \frac{\mu}{s} - a = 0 \qquad (50)$$

It is not difficult to demonstrate that a unique <u>positive</u> solution to (50) exists. This value of μ, denoted $\hat{\mu}$, is the unique minimizer of $\psi(\mu)$ as can be established by noting the positive sign of the second derivative of (49) at $\hat{\mu}$ and verifying that $\lim_{\mu\to 0^+} \psi(\mu) = \lim_{\mu\to\infty} \psi(\mu) = \infty$. Table 1 gives some numerical values of $\hat{\mu}$ for given a and s.

The following results can easily be established.

1) $\frac{\partial\hat{\mu}}{\partial a} < 0$. As the degree of increasing returns diminishes there is less motivation to capture economies of scale by building significant excess capacity ahead of demand.

2) $\frac{\partial\hat{\mu}}{\partial s} > 0$. As the discount rate falls relative to the growth of demand there is greater incentive for putting off future construction costs to a later data by increasing present outlays.

A quadratic approximation to (50), valid for small values of $\hat{\mu}$ is

$$\hat{\mu} \approx \frac{2(1 - a)}{a(1 - a + \frac{1}{s})} .$$

TABLE 1

$\hat{\mu}$ AS A FUNCTION OF a AND s
(in percentage of current capacity)

a \ s	.05	.10	.15	.20	.25	.30	.35	.40	.50	.70	1.0
.50	6.4	12.9	19.7	26.6	33.7	41.0	48.5	56.2	72.0	105.8	161.8
.60	4.8	9.7	14.7	19.7	24.9	30.2	35.5	41.0	52.2	75.6	113.5
.65	4.1	8.2	12.4	16.7	21.0	25.5	30.0	34.5	43.7	63.0	93.8
.70	3.4	6.8	10.3	13.9	17.5	21.1	24.8	28.5	36.0	51.7	76.4
.75	2.8	5.6	8.4	11.2	14.1	17.0	20.0	22.9	29.0	41.3	60.7
.80	2.2	4.3	6.5	8.8	11.0	13.2	15.5	17.8	22.4	31.8	46.4
.85	1.6	3.2	4.8	6.4	8.0	9.7	11.3	13.0	16.3	23.0	33.4
.90	1.0	2.1	3.1	4.2	5.2	6.3	7.4	8.4	10.6	14.8	21.4
.95	.5	1.0	1.6	2.1	2.6	3.1	3.6	4.1	5.1	7.1	10.0

9. Concluding Remarks

In terms of our general model of economic development, capacity expansion at a constant geometric rate is optimal for a proportional global savings function, given a constant elasticity cost of extra capacity. There are some interesting implications of such a policy. The optimal big push periods $(\underline{t}_i^*, \overline{t}_i^*)$ will be of longer duration in the earlier stages of development. Later on, taking advantage of economies of scale, it will take less time to save enough at a fixed rate to increase capacity by a given fraction. Not only do the big push periods last longer in the beginning, but they occur more frequently as well. The Ramsey growth phases $[\overline{t}_{i-1}^*, \underline{t}_i^*]$ are of shorter duration early in the development process when the economy is growing faster and it takes less time for output to attain any constant multiple of its current level.

These analytic results quantify the generally accepted notion that infrastructure is somehow a much more important ingredient in the growth of an underdeveloped than of a mature economy. The longer length and increased frequency of big push stages during the early years of development means more time spent in no-growth stagnant consumption phases awaiting the completion of overhead facilities. Of course, the present model over-emphasizes certain structural rigidities, but the conclusions accord well with the customary feeling that the creation of social overhead capital is a more formidable barrier to growth in a less developed economy.

Along these lines it is interesting to ask how a national wealth statistician would record, for the present model, the relative shares of infrastructure and directly productive capital out of total capital stock. As conventionally measured in terms of real production costs or material inputs, the percentage of overhead capital in the total stock would decline over time. However in terms of capacity, the amount of indirectly productive capital would always at least match the existing quantity of directly productive capital. That the two measurement standards yield a discrepancy is just another reflection of the role of increasing returns to scale in the creation of overhead capital.

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