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SHIFTABLE vs. NON-SHIFTABLE CAPITAL: A SYNTHESIS

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by

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To an economist the study of economic development is in large part an investigation into the mechanics of capital formation. At least in theory, the output options open to a developing economy are more restricted in the case where possibilities for obtaining foreign exchange *via* trade or aid are relatively limited. Society's menu of choices is even easier to enumerate if it is further assumed that labor is surplus in the sense that labor supply is a non-binding constraint on economic development now and for some time to come. These conditions are roughly descriptive of the historical situation confronting some large underdeveloped nations wishing to industrialize rapidly; the U.S.S.R. in the thirties is a classic example.

In such situations the key to economic growth is the capacity of the domestic capital goods sector. Increasing that capacity by ploughing back a high proportion of investment goods for purposes of self-reproduction will permit high consumption levels eventually, but not just in the near future. *Vice versa* if, by bolting down a substantial percentage of invest-

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ment goods there, the consumer goods sector is presently expanded.

These thoughts underlie a very interesting model of economic development first propounded by the Soviet engineering-economist G.A. Fel'dman in 1928.[1] We are indebted to Professor Domar for first pointing out the significance of this model and for relating it to current growth theory as well as to the Soviet industrialization debate of the twenties.[2] The same model has been independently formulated by the Indian statistician P.C. Mahalanobis who places somewhat greater emphasis on making it operational enough to serve as a rough guide of sorts for Indian long term planning.[3]

In its simplest form this model splits an economy into two departments -- investment and consumption. Investment goods are general ex ante and can be used to increase the capacity of either sector. But ex post capital is specific to the department for which it has been specialized and in which it has been installed. Each department is best thought of as vertically integrated in all the intermediate materials it uses back to and including agriculture and the extraction of raw materials from the earth. Under this interpretation it is convenient to treat arable land and mineral deposits as capital. Obviously natural resources are not man made, but there is no harm in including them among the initial stocks of capital which are inherited from the past.[4]

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[1] Fel'dman [1928]; see also the interesting review of Fel'dman's life in Vainshtein and Khanin [1968].

[2] Domar [1957].

[3] Mahalanobis [1953]. It should be mentioned that Maurice Dobb has also analyzed this model, although with a less explicitely mathematical formulation; Dobb [1960] and [1967].

[4] If we wanted to we could impute constant foreign exchange earnings from trade or aid to extra investment capital. Unfortunately this only finesses the real problems of trade specialization.

Such a formulation highlights the short run limitations on output choice imposed by a capital stock structure that is specific and non-transferable. As such it is a technologically oriented model whose main purpose is to demonstrate physically attainable consumption and investment patterns, seemingly without the explicit introduction of social constraints. In practice this kind of a separation is sometimes impracticable. With too little capital budgeted to the consumption sector there may be difficulty in bartering away from an independent peasantry sufficient foodstuffs to maintain consumption levels of industrial workers.[1] This important complication aside (its effects might be negligable for less extreme savings alternatives and in any event they could be mitigated by a socialist government able to enforce the collection of grain and willing to countenance inflationary wage policies), the Fel'dman model is an important contribution. It gives revealing insight into the problems of an economic society which desires to save more but is frustrated in its attempts because capital cannot be freely shifted to assist the overburdened investment sector.

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[1]This knotty problem was first systematically studied by the Soviet economist E.A. Preobrazhinsky. Even in the early years of the relatively tranquil N.E.P. period, the far-sighted Proebrazhinsky began arguing that just as Western industrialization developed historically in the wake of ploughed-back monopoly profits (the Marxian process of "primitive capitalist accumulation"), so Soviet industrialization would have to proceed along the analogous path of "primitive socialist accumulation." Soviet development would need to be funded by the monopoly surplus value extracted from agriculture in the name of socialism; to disregard this basic trade-off would be sheer utopianism. At the time it was propounded Proebrazhinsky's thesis was strongly denounced by Stalin and Bukharin as inimical to the alliance of peasant and worker then forming the cornerstone of N.E.P. policy. But history was to show the "left deviationist" economist more prescient (or more honest?) than his critics. The "iron heel of the law of primitive socialist accumulation" was to be ground into the Soviet peasantry with more ruthless vigor by Stalin's own collectivization program than Preobrazhinsky, writing in the mid-twenties, could ever have imagined.

Despite its obvious relevance, in a broad descriptive sense, to the economic development of an economy with limited access to foreign exchange and unlimited supplies of labor, the Fel'dman model still lacks a certain elusive operational quality. How are the sectors of an economy to be divided in practice between producers' and consumers' goods? Most intermediate raw materials are used in producing both investment and consumption. The mining of coal would have to be classified in both categories, perhaps belonging "more", in some sense, to that category to which more of the (embodied) coal is ultimately destined. The same is true of many other intermediates -- chemicals, petroleum, electricity, transportation, forest products, etc. Ideally one wants to place all capacity increasing activities in one category and all those merely sustaining output at current levels in the other. But this does not tell us how to meaningfully split up the troublesome double purpose intermediate sectors.

A related problem lurks behind the whole notion of a capital stock which cannot be transferred between the consumption and investment departments of an economy. Undoubtedly wheat growing land or houses are not well suited for making tractor engines. Likewise machine tools are relatively useless for baking bread. Nevertheless, because most intermediate commodities are used, directly or indirectly, for both consumption and investment, plant and equipment engaged in producing these raw materials is transferable in the sense that a change in the intermediate product's final destination is tantamount to shifting capital from one department to the other. Railroads can transport construction materials as easily as they can sacks of flour, and it is irrelevant to the operation of a steam turbine whether its electricity goes to power factory machinery engaged in cutting metals or sewing shoes.

Clearly the Fel'dman model exaggerates the significance of capital ossification. In order to examine the consequences of greater realism it is necessary to look at an economy with more than two categories. A classification scheme for a model with three sectors is described in the following chart.[1]

symbol	sector	definition	example
I	Investment	All final and intermediate goods and services used directly or indirectly to produce <u>investment only</u>	construction, cement, machine tools, metal working
C	Consumption	All final and intermediate goods and services used directly or indirectly to produce <u>consumption only</u>	bread, flour, clothing textiles
R	Raw Materials	All intermediate goods and services used indirectly in producing <u>both investment and consumption</u>	freight transportation, fuels, chemicals, electricity

Introducing an extra sector has hardly banished the arbitrariness which must be involved in assigning certain industries to one of three categories, although the present arrangement which is along more usable functional guidelines is at least an improvement over the two sector classification. The assumption (to be made) that such gigantic sectors as these are subject to aggregate production relations (of the simple fixed-coefficients surplus-labor type, no less) should identify this model as yet another species of the non-operational "suggestive" variety.

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[1]Raj and Sen [1961] split up an economy according to similar criteria.

For evaluation purposes we take as an appropriate social objective the infinite integral of a discounted instantaneous utility function which is defined over current consumption, C . For ease of manipulation we choose the instantaneous utility function to be of the constant elasticity of marginal utility type:

$$U'(C) \equiv \frac{dU}{dC} = \frac{1}{C^{\eta}} \tag{1}$$

η = minus the (constant) elasticity of marginal utility, parameter.[1]

Likewise for convenience, the discount factor is chosen to be of the exponential form $e^{-\rho t}$

ρ = social rate of pure time discount, parameter.

From (1), the instantaneous (undiscounted) social utility of consumption at time t is given by[2]

$$U(C(t)) = \begin{cases} \frac{1}{1-\eta}\,(C(t))^{1-\eta} & \text{for } \eta \neq 1 \\ \log\,(C(t)) & \text{for } \eta = 1 \end{cases} \tag{2}$$

The three sector problem, henceforth called problem (iii), is to

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[1]Results are easily extended to the case of a general time invariant utility function along the lines of Bose [1968], appendix I. The slightly more general form $U'(C) = \frac{1}{(C-\bar{C})^{\eta}}$, $C \geqq \bar{C}$ also yields an explicit solution but the manipulations become inconveniently messy.

[2]Any time invariant utility function obtained by a linear transformation of U $(a + bU,\ b > 0)$ could serve equally well as an instantaneous utility index and would yield an identical solution for the problem under consideration.

$$\max \int_0^\infty U(C)e^{-\rho t}dt \tag{3}$$

subject to

$$\frac{I}{\gamma_I} \leqq K_I \quad : \pi_I \tag{4}$$

$$\frac{C}{\gamma_C} \leqq K_C \quad : \pi_C \tag{5}$$

$$\frac{a_I I}{\gamma_R} + \frac{a_C C}{\gamma_R} \leqq K_R \quad : \pi_R \tag{6}$$

$$\dot{K}_I = \lambda_I I - \mu K_I \quad : p_I \tag{7}$$

$$\dot{K}_C = \lambda_C I - \mu K_C \quad : p_C \tag{8}$$

$$\dot{K}_R = \lambda_R I - \mu K_R \quad : p_R \tag{9}$$

$$\lambda_I + \lambda_C + \lambda_R = 1 \tag{10}$$

$$I,\ C,\ \lambda_C,\ \lambda_I,\ \lambda_R \geqq 0 \tag{11}$$

$$[K_I(0),\ K_C(0),\ K_R(0)] = [K_I^0,\ K_C^0,\ K_R^0]\ , \quad \text{given.} \tag{12}$$

We follow the convention of writing (undiscounted) price or co-state variables to the right of a double colon following the equation to which they are dual. Variables are not explicitely specified as a function of time where this interpretation is otherwise clear.

Letting $j = I$, C , or R ,

$K_j(t)$ = capital stock in sector j at time t, state variable.

γ_j = output per unit of capital in sector j, parameter.

$\lambda_j(t)$ = fraction of investment allocated to sector j at time t, control variable.

μ = common rate of exponential deterioration of capital, parameter.

a_I = amount of input R required per unit production of I, parameter.

a_C = amount of input R required per unit production of C, parameter.

Because each sector of economy (iii) is viewed as an enclave, extensive netting out of intermediate stages internal to a sector must be assumed to have taken place. This interpretation has to be considered in translating the sectoral output-capital ratios; γ_j represents the final net-of-intermediate-stages output of sector j per unit of capital stock which is spread out in the appropriate proportions over all the stages of production internal to sector j leading up to and including the production of final product.

Let β_j represent the final output of j per unit of direct and indirect capital in all sectors. β_j differs from γ_j in that account is taken of the fact that raw materials used up in the production of I and C require the use of (direct and indirect) capital in sector R. It is easily seen that

$$\beta_I = \frac{1}{\frac{1}{\gamma_I} + \frac{a_I}{\gamma_R}} \tag{13}$$

$$\beta_C = \frac{1}{\frac{1}{\gamma_C} + \frac{a_C}{\gamma_R}} \tag{14}$$

$$\beta_R = \gamma_R \tag{15}$$

The units of I are naturally fixed in terms of K (or _vice versa_). Thus γ_I and (because a_I/γ_R is independent of the units of R) β_I are scale free. It is convenient to choose the units of C and R so that $\beta_I = \beta_C = \beta_R$; this common value is called β without ambiguity. In the context of a capital theory of value this is a natural way to define units of C and R . Note that the values of a_I , a_C , γ_C , and γ_R are automatically fixed by a choice of units for C and R .

We assume that β and η are positive and that ρ and μ are non-negative. To be able to analyze the more realistic and interesting case where genuine growth occurs, it is postulated that

$$(\beta - \mu) > \rho \ . \tag{16}$$

Otherwise the net productivity of capital is exceeded by the rate of discount being imposed on the system and it never pays to plough back any investment into the investment goods sector because society is too impatient to exploit the productivity of capital.

A final assumption is that

$$\rho > (1 - \eta)(\beta - \mu) \tag{17}$$

This requirement is necessary to insure the existence of a meaningful solution to problem (iii). Otherwise a feasible solution will exist which results in an unbounded objective.[1]

Since no one expects a condition of labor redundancy to persist forever, an infinite horizon formulation of a model incorporating a labor surplus technology might appear pointless. Such a formulation can be rationalized as an acceptable approximation so long as the day when every member of the labor force will be absorbed is in the distant future.[2]

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[1] For some M, any $\theta > 0$, $\int_0^\infty U(C^*(t))e^{-\rho t}dt \geqq \int_0^\infty U((1-\theta)Me^{(\beta-\mu)\theta t})e^{-\rho t}dt$.
If $\rho < (1-\eta)(\beta-\mu)$, the latter integral goes to infinity as $\theta \to 1$. The case $\rho = (1-\eta)(\beta-\mu)$ is essentially uninteresting because any efficient (with respect to consumption) policy is also optimal.

[2] This justification can easily be made rigorous. Let Φ_e represent the optimal value of the social objective integral (3) in problem (iii) which, it turns out, is relatively easy to solve. Now consider a harder problem (problem (h)) with the same objective function but where the potential labor force at time t is fixed, say at $\bar{L}e^{nt}$, and α_j $(j = I, C, R)$ represents the labor-output ratio for sector j under the surplus labor regime. In problem (h) the labor surplus technology described by (4), (5), (6) is appropriate only so long as $\alpha_I I + \alpha_C C + \alpha_R R < \bar{L}e^{nt}$. The moment this constraint becomes binding we must move on to other sets of techniques which economize on labor at the expense of capital. Problem (h) is thus a fully general, much more realistic three sector, non-shiftable, putty-clay model with labor a primary input and multiple production techniques available. Let the optimal value of the social objective (3) for problem (h) be denoted Φ_h.

Now consider the following feasible solution to problem (h). Follow exactly the optimal solution to problem (iii) (not yet enumerated) until time T when the equation $\alpha_I I(T) + \alpha_C C(T) + \alpha_R R(T) = \bar{L}e^{nT}$ holds for the first time. From then on follow an optimal solution with respect to the given capital

In problem (iii) utility is expressed as a function of total consumption. Only a minor adjustment is necessary to deal with utility as a function of *per-capita* consumption and exponential population growth.[1]

The behavior of model (iii) is easiest to understand in terms of simpler one and two sector models of the same family. With this in mind, it is convenient to pretend that the "real world" is being portrayed by the model (iii) and then to consider how a macro-economist would build a one and two sector model out of the same situation using the same data.

The analogous one sector model, called model (i), is to

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stocks at time T and the full production possibilities of problem (h). Let the social objective (3) for this feasible but clearly non-optimal solution to problem (h) be denoted Φ_f. Obviously $\Phi_f \leq \Phi_h \leq \Phi_e$. It is relatively straightforward to show that $\lim_{L\to\infty} \Phi_f = \Phi_e$, implying $\lim_{L\to\infty} \Phi_h = \Phi_e$. In the presence of a large initial reserve of unemployed labor the planners cannot go very far wrong in starting off by implementing the solution to problem (iii) in the early years even though they know (h) in fact to be the real situation. It is this result which can be interpreted as justifying our interest in the infinite horizon surplus labor problem.

[1]Now interpret all variables as if they are expressed *per-capita*. Interpret μ as equaling the rate of true physical depreciation plus the rate of population growth. The new value of ρ is either the same as the old if total welfare of each *generation* counts equally (except for the pure time preference factor) or is less by the rate of population growth if total welfare of each *person* counts equally (except for the pure time preference factor). In between situations are handled by in between values of ρ.

max
$$\int_0^{\infty} U(C)e^{-\rho t}dt \tag{17}$$

subject to

$$Y = \beta K \tag{18}$$

$$\dot{K} = sY - \mu K \tag{19}$$

$$C = (1 - s)Y \tag{20}$$

$$0 \leqq s \leqq 1 \tag{21}$$

$$K(0) = K^0 \text{ , given} \tag{22}$$

Using the calculus of variations, it is easy to show[1] that the complete solution calls for Y , K , C , and I to each grow at the same steady rate g , where

$$g \equiv \frac{\beta - \mu - \rho}{\eta} \tag{23}$$

This information, plus the conditions (18)-(22), specify the complete time paths of all relevant variables. In particular, it can be shown that

$$\frac{I(t)}{Y(t)} = s^*$$

for all t , where

$$s^* \equiv \frac{\beta - \mu - \rho + \eta\mu}{\eta\beta} \tag{24}$$

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[1]For details see Chakravarty [1962].

Note that $0 < s^* < 1$ by (16), (17).

The appropriate two sector non-shiftable model, called model (ii), is to[1]

$$\max \int_0^\infty U(C)e^{-\rho t}dt \tag{25}$$

subject to

$$C = \beta K_2 \tag{26}$$

$$I = \beta K_1 \tag{27}$$

$$\dot{K}_1 = \lambda I - \mu K_1 \qquad : q_1 \tag{28}$$

$$\dot{K}_2 = (1 - \lambda)I - \mu K_2 \qquad : q_2 \tag{29}$$

$$0 \leqq \lambda \leqq 1 \tag{30}$$

$$[K_1(0),\ K_2(0)] = [K_1^0,\ K_2^0]\ , \quad \text{given} \tag{31}$$

Define $s(t)$ to be the gross savings rate at time t for this economy:

$$s(t) \equiv \frac{I(t)}{I(t) + C(t)} \tag{32}$$

(We will also use this definition for model (iii).) In contrast with the one sector situation, in model (ii) the authorities are not free to choose

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[1] The production structure is that of the basic Fel'dman model. Bose [1968] has obtained a complete characterization of the optimal path using Pontryagin's principle. With slight modification we rely on his results, omitting details of the proof.

the savings rate. At any time it is fixed in terms of the prevailing capital stock structure, although it can be changed over time by manipulating λ .

Roughly speaking, the two sector economy (ii) wants to grow in the same way as the optimal solution to the one sector model (i). But only infrequently is it to be expected that $s(0) = s^*$. In all but a razor's edge case, therefore, a specialization phase comes first. All investment in this initial specialization phase is devoted to the relatively underdeveloped sector. In this way capital stock proportions are restructured to achieve the optimal gross savings rate s^* as quickly as possible. Thereafter balanced growth at rate g occurs which maintains the optimal savings rate s^* .

The solution is easily portrayed diagramatically. In Figure 1, the line N of slope s^* passing through the origin divides the half-quadrant $\{K_1, K_2: K_1 \leqq K_2 + K_1, K_1 \geqq 0, K_2 \geqq 0\}$ into two regions. Any point belonging to R_2 is one such that $s > s^*$, whereas a point is in R_1 if and only if $s < s^*$. If $[K_1(0), K_2(0)] \in R_2$ all I goes into increasing K_2 $(\lambda = 0)$ until line N is reached. In the historically more interesting case of $[K_1(0), K_2(0)] \in R_1$ all initial investment is ploughed back into sector 1 $(\lambda = 1)$ until the resulting trajectory intersects line N . The case $s(0) < s^*$ is considered to be historically more relevant in the context of an underdeveloped economy because most arable land gets counted as consumption capital. In both cases, once line N (the non-specialization phase) is reached, the optimal program remains on it forever. Thus, if $s(0) = s^*$, the solution is identical with that of the one sector model

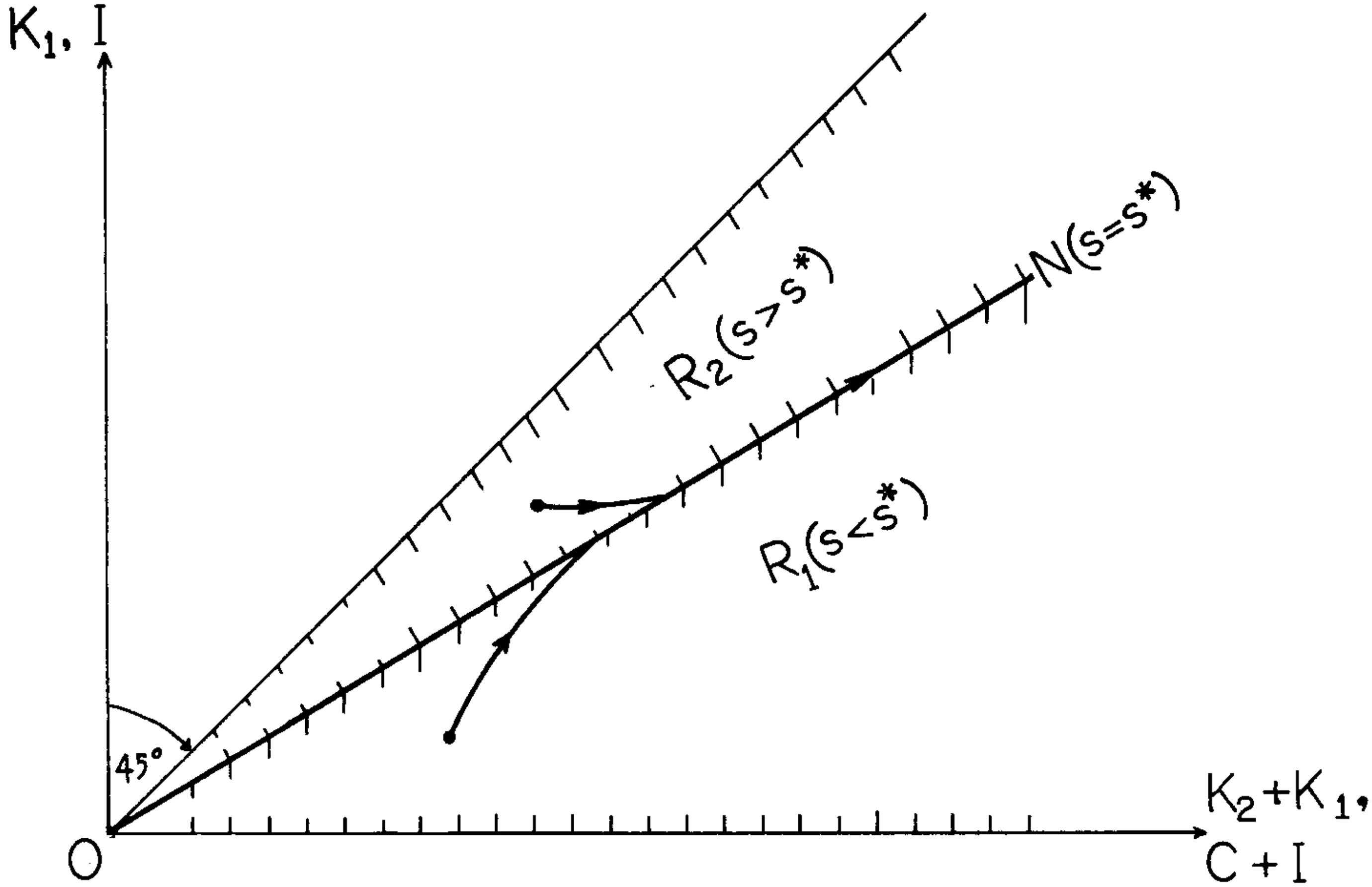

FIGURE 1.

OPTIMAL PATHS FOR DIFFERENT INITIAL CONDITIONS
IN THE TWO SECTOR MODEL.

from the very beginning.[1]

Applying the Pontryagin "maximum principle," the relevant optimality conditions[2] yield

$$\lambda(t)\begin{cases} = 1 & \text{if} \quad q_1(t) > q_2(t) \\ \in [0,\ 1] & \text{if} \quad q_1(t) = q_2(t) \\ = 0 & \text{if} \quad q_1(t) < q_2(t) \end{cases}$$

The undiscounted non-negative prices $[q_1,\ q_2]$ are continuous, satisfying

$$\dot{q}_1 = (\rho + \mu)q_1 - \beta q \tag{33}$$

$$\dot{q}_2 = (\rho + \mu)q_2 - \beta U'(C) \tag{34}$$

$$\lim_{t\to\infty} q_j(t)e^{-\rho t}K_j(t) = 0 \qquad (j = 1,\ 2)$$

where

$$q(t) \equiv \max\{q_1(t),\ q_2(t)\} .$$

Define

$$q^* \equiv q(0) .$$

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[1] With a general concave utility function the "optimal savings line" N becomes a curve. But the qualitative properties of the optimal solution remain the same -- cf. Bose [1968], appendix II.

[2] Pontryagin [1962] Theorem 1, p. 19, and the discussion of pp. 189-191; Bose [1968], p. 467.

We now derive a pair of relations between q^* and $U'(C(0))$ which will help explain the solution to model (iii). Let τ be the time spent in the initial specialization phase. It is easily seen from (33), (34) that during the non-specialization phase $(t \geqq \tau)$,

$$U'(C(t)) = q(t) = q_1(t) = q_2(t) .$$

If initial conditions start economy (ii) in region R_1 $(s(0) < s^*)$,

$$\tau = \frac{1}{\beta} \log\left(\frac{\frac{s^*}{1 - s^*}}{\frac{s(0)}{1 - s(0)}}\right), \tag{35}$$

$$q^* = q_1(0) = q(\tau)e^{(\beta-\mu-\rho)\tau} ,$$

and

$$U'(C(0)) = U'(C(\tau))e^{-\mu\eta\tau} = q(\tau)e^{-\mu\eta\tau}$$

$$= q^*e^{(-\beta+\mu+\rho-\mu\eta)\tau} .$$

Thus

$$q^* = \left(\frac{\frac{s^*}{1 - s^*}}{\frac{s(0)}{1 - s(0)}}\right)^{\eta s^*} U'(C(0)) \tag{36}$$

for $s(0) < s^*$.

With $s(0) > s^*$, $\tau = \frac{1}{\beta}\left(\frac{1 - s^*}{s^*} - \frac{1 - s(0)}{s(0)}\right)$, and

$$q^* = q_2(0) = \frac{q(\tau)}{e^{(\rho+\mu)\tau} - \int_0^\tau e^{(\rho+\mu)(\tau-v)}\beta U'(C(v))dv} \quad \text{(from (34))}$$

where

$$U'(C(v)) = \frac{1}{(\beta K_2(v))^{\eta}} = \frac{1}{(\beta e^{-\mu v}(K_2(0) + \beta K_1(0)v))^{\eta}} .$$

With $s(0) > s^*$ we also have $\frac{C(0)}{C(\tau)} = \left(\frac{\frac{I(0)e^{-\mu\tau}}{C(\tau)}}{\frac{I(0)}{C(0)}}\right)^{e^{\mu t}} = \left(\frac{\frac{I(\tau)}{C(\tau)}}{\frac{I(0)}{C(0)}}\right)^{e^{\mu\tau}} = \left(\frac{\frac{s^*}{1-s^*}}{\frac{s(0)}{1-s(0)}}\right)^{e^{\mu\tau}}$

Thus, $U'(C(0)) = U'(C(\tau))\left(\frac{\frac{s(0)}{1-s(0)}}{\frac{s^*}{1-s^*}}\right)^{\eta} e^{-\mu\eta\tau} = q(\tau)\left(\frac{\frac{s(0)}{1-s(0)}}{\frac{s^*}{1-s^*}}\right)^{\eta} e^{-\mu\eta\tau}$

When combined, these relations yield, finally,

$$U'(C(0)) = q^*[e^{(\rho+\mu)\tau} + \int_0^{\tau} e^{(\rho+\mu)(\tau-v)} U'(\beta e^{-\mu v}(K_2(0)+\beta K_1(0)v))dv]\left(\frac{\frac{s(0)}{1-s(0)}}{\frac{s^*}{1-s^*}}\right)^{\eta} e^{-\mu\eta\tau} \tag{37}$$

The expressions (36) and (37) will prove useful in the sequel.

In the full three sector model (iii), introduction of the common intermediate sector R offers the possibility of a more realistic kind of substitution between I and C . At any point of time the constraints (4), (5), (6) define a production possibilities surface (PPS) with I and C as final products. In general the PPS can have any of several shapes depending upon which configuration of effective constraints holds at the given moment. For reasons that will be made clear, two configurations are of special interest: 1) the degenerate (in the linear programming sense) case where the three constraints hold simultaneously at a single point, forming a rectangular PPS ; 2) the case where the raw materials constraint (6) cuts across the rectangle given by the two specific capacity constraints (4) and (5) so that the PPS is a pentagon. These two possibilities are represented in Figure 2. The efficient operating regions are point D in case 1) and line AB in case 2).

While there is often a temptation to analyse patterns of optimal growth parametrically, as a function of all possible initial endowments, it is unlikely that the given historical capital stocks of any real economy would take on completely arbitrary values. We assume that the economy starts off in configuration 1). The rationale behind a full capacity endowment is the notion that previous to the historical discontinuity of time zero an internally consistent ancien régime was moving along in some kind of non-specialization phase without excess capacity.[1] Après le revolution the new planning board inherits an historically determined savings rate $s(0)$ which is likely to be very different from the desired rate s^* best suited to its own newly enforceable social values.

The two sector approach suggests starting off by splitting $K_R(0)$ between departments 1 and 2. Defining $K_1(0) \equiv \left(1 + \frac{a_I \gamma_I}{\gamma_R}\right) K_I(0)$ and $K_2(0) \equiv \left(1 + \frac{a_C \gamma_C}{\gamma_R}\right) K_C(0)$, we could pretend that the resulting model

[1] In theory the general case of any initial configuration of capital endowments could be handled without difficulty by using the same methods we employ for the initial full capacity situation. The rub is that the number of possible different cases becomes unwieldy. For this reason it seems better to sacrifice full generality in favor of focusing on a particular historically interesting case.

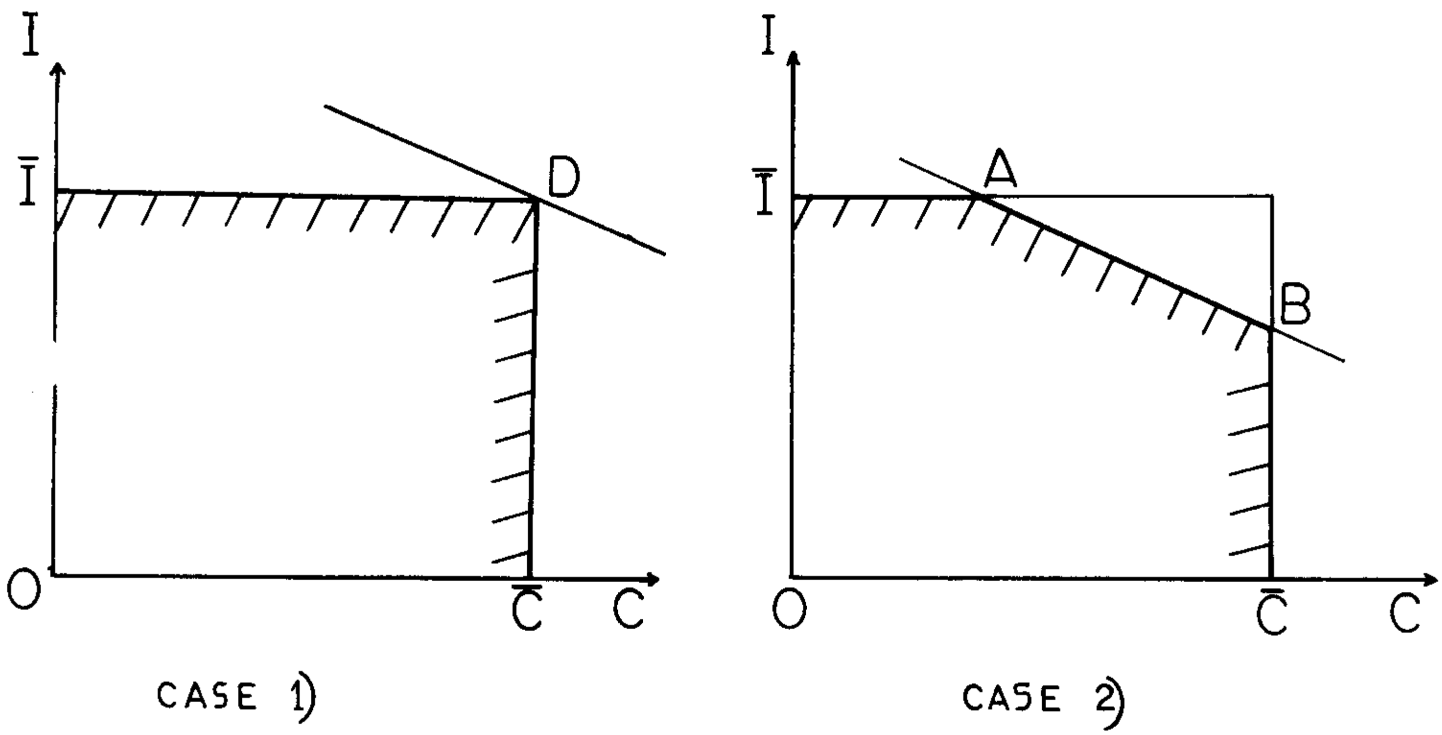

FIGURE 2.

PRODUCTION POSSIBILITIES FOR MODEL (iii).

obeyed (26)-(31) instead of (4)-(12). Translating back from model (ii) to model (iii) would be accomplished with the aid of the following relations: $K_I(t) = (1 - a_I)K_1(t)$, $K_C(t) = (1 - a_C)K_2(t)$ and $K_R(t) = a_I K_1(t) + a_C K_2(t)$. We could then proceed by optimizing model (ii) with the given initial conditions. Suppose for concreteness the historically more relevant situation $s(0) < s^*$. The solution would be to initially specialize all investment to the investment goods department (which now includes K_R capital) increasing s to s^* and thereafter maintaining it at that rate by a policy of balanced investment.

This full capacity program is certainly feasible in the context of model (iii), but is it optimal? If more investment is strongly enough desired in the beginning, it could be rapidly built up <u>via</u> a program of re-investing only in sector I and re-routing to that sector some R previously destined for sector C. This would create excess capacity in sector C, placing the economy at point A in case 2) of Figure 2.

In this situation the pseudo departments 1 and 2 would be meaningless because they would lack stability. The basic aim of this paper is to determine conditions under which stability is non-existant because capital is in effect shifted from one department to another by transferring the destination of R . The following theorem is the main result.

<u>Let q^* be the undiscounted social price of investment in model (ii) at time zero. The full capacity model (ii) solution is also optimal for model (iii) if and only if the following conditions are fulfilled</u>:

$$\text{(a)} \quad s(0) < s^* : \quad \frac{U'(C(0))}{a_C} \geqq q^* \tag{38}$$

$$\text{(b)} \quad s(0) > s^* : \quad U'(C(0)) \leqq \frac{q^*}{a_I} \tag{39}$$

$$\text{(c)} \quad s(0) = s^* : \quad \text{unconditional}$$

In (c), the solution to model (iii) is just like the solution to model (i): grow at rate g and save at rate s^*. Conditions (a) and (b) are also easy to interpret. Suppose, in model (ii) with $s(0) < s^*$, that after recalling the true three sector nature of the model it is decided to accelerate growth of the relatively underdeveloped I sector by transferring a unit of R from C to I at time zero. When all necessary sub-optimal adjustments have been made, the gain would be q^*, the value of an extra unit of investment at time zero measured in terms of the utility index. Consumption would be diminished by $\frac{1}{a_C}$ units, resulting in a utility decrease of $\frac{U'(C(0))}{a_C}$. The relevant question is whether the loss of utility is matched by a sufficiently high value of investment to justify the transfer of R. A similar interpretation can be made for case (b).

Conditions (38) and (39) can be translated into a form where they are expressed only in terms of the original parameters and initial capital stocks of problem (iii). If $s(0) < s^*$ it follows from combining (36) with (38) that the full capacity model (ii) solution is optimal for model (iii) if and only if

$$\left(\frac{\frac{s(0)}{1 - s(0)}}{\frac{s^*}{1 - s^*}}\right)^{\eta s^*} \gtreqqless a_C \tag{40}$$

Other things being equal, condition (40) would be more likely to hold the lower is a_C . If little R is used per unit of C , it would be foolish to sacrifice a great deal of consumption to free a small amount of intermediate commodities for investment purposes.

Condition (40) is also more likely to prevail, ceterus paribus, the closer is $s(0)$ to s^* . An economy starting off with a savings rate near that which it desires to attain is less likely to be in such a hurry to speed up the growth of investment as a tolerate excess capacity in the consumption goods sector.

The most meaningful consumption goods in the context of an economically underdeveloped country are primarily agricultural products, soft goods, and housing services. It seems reasonable to suppose that the transportation, fuel, electricity, and selected industrial materials necessary to maintain consumption levels at existing capacity are probably negligable compared with the loss of consumption entailed by transferring these intermediate materials to the investment sector.[1] Even though investment goods were top priority in post-1928 Soviet development strategy, it would have

[1]Casual playing with numbers supports the feeling that (40) holds. The biggest unknown is η . Suppose a logarithmic utility function $(\eta = 1)$. Even with $s(0)$ as low as .05, s^* as high as .3 , and a_C as high as .5 , condition (40) would be fulfilled.

been foolish to have /conveyed capital goods in vehicles which formerly carried consumption goods in order to avoid having to invest in the transportation system.

These kinds of arguments suggest that a_C may be low enough so that in practice capital might not be shifted even though in theory it could be. The Fel'dman story about a two-department specific capital economy may not be literally true, but it probably makes a good parable. The effectiveness of the parable depends upon a certain stickiness in model (iii) around the initial capital stocks. Although it costs a rouble to increase consumption by a rouble, salvaging a rouble's worth of intermediate commodities requires sacrificing <u>more</u> than a rouble's worth of consumption. The higher this asymmetric adjustment cost, the less profitable it becomes to shift capital.[1]

The case $s(0) > s^*$ can be treated analogously. Substituting (37) into (39), one can transform (39) into a condition that depends only on parameters and initial values. This condition can be given a similar interpretation to that which was placed on (40).

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[1]The behavior of model (iii) <u>vis a vis</u> model (ii) should be contrasted with the behavior of (ii) <u>vis a vis</u> (i). In all but a razor's edge case the two sector model yields an optimal growth path different from that which would prevail in the one sector case. As we have seen, however, the behavior of (iii) may well duplicate that of (ii). The mathematical reason is that although in (iii) an extra sector has been added, an extra initial condition has also been snuck in with the stipulation that economy (iii) starts off without any excess capacity.

<u>Proof of the theorem</u>: We prove only part (a). The proof of part (b) is completely analogous and part (c) is trivial.

Let τ be specified by (35) and q^* by (36). Define g and s^* by (23) and (24).

For $t < \tau$, set

$$p_I(t) = p_R(t) = q^* e^{-(\beta-\mu-\rho)t} \tag{41}$$

$$p_C(t) = \frac{q^* e^{-(\beta-\mu-\rho)\tau}}{(1-\eta s^*)(1-a_C)}[e^{\mu\eta(t-\tau)} - \eta s^* e^{(\rho+\mu)(t-\tau)} - a_C(1-\eta s^*)e^{-(\beta-\mu-\rho)(t-\tau)}] \tag{42}$$

$$\lambda_I(t) = 1 - a_I \tag{43}$$

$$\lambda_C(t) = 0 \tag{44}$$

$$\lambda_R(t) = a_I \tag{45}$$

$$I(t) = I(0)e^{(\beta-\mu)t} \tag{46}$$

$$C(t) = C(0)e^{-\mu t} \tag{47}$$

$$R(t) = a_I I(t) + a_C C(t) \tag{48}$$

$$K_I(t) = K_I(0)e^{(\beta-\mu)t} \tag{49}$$

$$K_C(t) = K_C(0)e^{-\mu t} \tag{50}$$

$$K_R(t) = \frac{a_I}{1 - a_I} K_I(t) + \frac{a_C}{1 - a_C} K_C(t) \tag{51}$$

For $t \geqq \tau$, set

$$p_I(t) = p_C(t) = p_R(t) = q^*e^{-(\beta-\mu-\rho)t} \tag{52}$$

$$\lambda_I(t) = s^*(1 - a_I) \tag{53}$$

$$\lambda_C(t) = (1 - s^*)(1 - a_C) \tag{54}$$

$$\lambda_R(t) = s^*a_I + (1 - s^*)a_C \tag{55}$$

$$I(t) = I(\tau)e^{g(t-\tau)} \tag{56}$$

$$C(t) = C(\tau)e^{g(t-\tau)} \tag{57}$$

$$R(t) = R(\tau)e^{g(t-\tau)} \tag{58}$$

$$K_I(t) = K_I(\tau)e^{g(t-\tau)} \tag{59}$$

$$K_C(t) = K_C(\tau)e^{g(t-\tau)} \tag{60}$$

$$K_R(t) = K_R(\tau)e^{g(t-\tau)} \tag{61}$$

In (56)-(61) $I(\tau)$, $C(\tau)$, $R(\tau)$, $K_I(\tau)$, $K_C(\tau)$, $K_R(\tau)$ are fixed by (46)-(51), whereas $I(0)$ and $C(0)$ in (46), (47) are simply full-capacity output values at time zero.

We must show that in the model (iii) solution with (38) holding no shifting or excess capacity occurs. The optimal policy is to devote all investment initially to building up the I and R sectors together until time τ when $s = s^*$. Thereafter all sectors grow at rate g. The method of proof for the if part of the theorem is to verify that the system (41)-(61) satisfies all the relevant Pontryagin conditions.[1]

Using the dual variables to equations (4)-(6), the following undiscounted Hamiltonian form is introduced:

$$He^{\rho t} \equiv U(C) + p_I(\lambda_I I - \mu K_I) + p_C(\lambda_C I - \mu K_C) + p_R(\lambda_R I - \mu K_R) \qquad (62)$$

The proposed solution (41)-(61) is feasible because it satisfies (4)-(12). In addition, the following conditions (A) and (B) must be fulfilled.

(A) At each t, control variables $(I, C, \lambda_I, \lambda_C, \lambda_R)$ are set at feasible values which maximize $He^{\rho t}$.

Let

$$p(t) \equiv \max\{p_I(t),\ p_C(t),\ p_R(t)\} .$$

Maximizing (62) with respect to non-negative $(\lambda_I, \lambda_C, \lambda_R)$ subject to (10) yields

$$He^{\rho t} = U(C) + pI - \mu(p_I K_I + p_C K_C + p_R K_R) \qquad (63)$$

For $t < \tau$, $p_C(t) < \min\{p_I(t), p_R(t)\}$ implies $\lambda_C(t) = 0$; no new information of this sort is revealed for $t \geqq \tau$ since $p_I(t) = p_C(t) = p_R(t)$ during that period.

Maximizing (63) over non-negative I and C satisfying (4)-(6)

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[1]See Pontryagin [1962], Theorem 1, p. 19 and the discussion of pp. 189-191. Due to convexity in production and strict concavity of utility, we are assured that the Pontryagin necessary conditions are also sufficient for the proposed solution to be optimal.

and (49)-(51), (59)-(61) will always call forth the full capacity solutions (46)-(48), (56)-(58). In this sub-problem, π_I, π_C, and π_R are dual, respectively, to equations (4), (5), and (6). From duality theory, we have the following equations[1] for all $t \geqq 0$:

$$p(t) = \frac{\pi_I(t)}{\gamma_I} + \frac{a_I \pi_R(t)}{\gamma_R} \tag{64}$$

$$U'(C(t)) = \frac{\pi_C(t)}{\gamma_C} + \frac{a_C \pi_R(t)}{\gamma_R} \tag{65}$$

$$\pi_I(t),\ \pi_C(t),\ \pi_R(t) \geqq 0 \tag{66}$$

(B) <u>$p_I(t)$, $p_C(t)$, and $p_R(t)$ are non-negative, continuous, and must satisfy</u>

$$\dot{p}_I = (\rho + \mu)p_I - \pi_I \tag{67}$$

$$\dot{p}_C = (\rho + \mu)p_C - \pi_C \tag{68}$$

$$\dot{p}_R = (\rho + \mu)p_R - \pi_R \tag{69}$$

$$\lim_{t\to\infty} p_j(t)e^{-\rho t}K_j(t) = 0 \qquad (j = I,\ C,\ R) \tag{70}$$

The proposed set of prices (p_I, p_C, p_R) defined by (41), (42), (52) are clearly continuous and non-negative. Using the conditions (16), (17) and the equations (59), (60), (61), it is easy to verify that the transversality conditions (70) hold.

In order for (67)-(69) to be consistent with (52) during $t \geqq \tau$ we have

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[1] (64) and (65) must hold with full equality because in our proposed solution C and I are strictly positive.

$$\pi_I(t) = \pi_C(t) = \pi_R(t) = \beta p(t) \ . \tag{71}$$

It is easily seen that conditions (64)-(66) are fulfilled for values (71) so long as

$$U'(C(t)) = p(t) \ ,$$

which holds for $t \geqq \tau$ by (1), (52), (57).

As for $t \leqq \tau$, (67)-(69) will be consistent with (41) only if

$$\pi_I(t) = \pi_R(t) = \beta p(t) \ . \tag{72}$$

Obviously (64) is satisfied by values (72), and substitution of (72) into (65) yields

$$\pi_C(t) = \gamma_C(U'(C(t)) - a_C p(t))$$

It follows that for $t < \tau$, $\pi_C(t) \geqq 0$ if and only if $U'(C(t)) \geqq a_C p(t)$. Substituting from (41) and (47), $\pi_C(t) \geqq 0$ if and only if

$$U'(C(0)) \geqq a_C q^* e^{-(\beta-\mu-\rho+\mu\eta)t}$$

which will hold for <u>all</u> $t < \tau$ if and only if

$$\frac{U'(C(0))}{a_C} \geqq q^* \ .$$

It remains only to verify that the price solutions (41), (42), (52) satisfy the differential equations (67)-(69). This is straightforward, if tedious, and finishes the necessity part of the proof.

For the <u>only if</u> part of the theorem, it is not difficult to show that so long as $s(0) < s^*$, and whether or not (38) holds, the initial phase must consist of $\lambda_C = 0$. In order for no excess capacity to be created in the first phase, we must have λ_I and λ_R both positive. This will be possible only if $p_I(t) = p_R(t) > p_C(t)$ during the initial phase which in turn is not allowable (as we have seen) if $U'(C(0))/a_C \leqq p(0)$. If capital is chosen to be shifted, the initial value of an extra unit of investment must be higher than if no shifting is allowed, implying $p(0) > q^*$. Thus, if (38) does <u>not</u> hold, non-shiftability must be violated, demonstrating the sufficiency part of the theorem and finishing the proof.

For completeness we conclude with a brief description of optimal three sector growth in the case where capital is shifted and excess capacity is created. The proof is omitted; it is mostly a tedious verification of optimality conditions, somewhat in the spirit of the proof of the previous theorem.

Suppose the historically more interesting case $s(0) < s^*$ (the case $s(0) > s^*$ is analogous). If capital is to be shifted, we must be given that

$$\frac{U'(C(0))}{a_C} < q^* .$$

In the first phase all investment goes into I $(\lambda_C = \lambda_R = 0)$ and R is transferred from C to I, creating excess capacity in the C sector $(\pi_C = 0)$. This places output at point A of Figure 2, case 2). The purpose of phase 1 is to rapidly build up I, which grows at the fast rate $(\gamma_I - \mu)$. In this stage $p = p_I > p_R > p_C$, $U'(C) < a_C p$, and $U'(C)/p$ increases over time.

Phase 2 begins as soon as $U'(C) = a_C p$. This is a stage when the full capacity C output is recouped by investing in R at a faster rate than in I . Throughout this phase $U'(C) = a_C p$, $\lambda_C = 0$, $\pi_C = 0$, and $p = p_I = p_R > p_C$.

Phase 3 begins when no excess capacity exists anywhere in the economy, as in the beginning of phase 1. But this time I and R are built up in balanced fashion just like in the opening phase of the optimal path described in the previous theorem. Throughout this stage $U'(C)/a_C p$ increases from unity, $\lambda_C = 0$, $p = p_I = p_R > p_C$ and no excess capacity exists in the economy $(\pi_I, \pi_C, \pi_R > 0)$.

Phase 4 is the balanced growth phase which begins when $s = s^*$. All stocks and flows grow at rate g , and $p = p_I = p_C = p_R$.

The solution in terms of C is depicted in Figure 3. For ease of understanding, the case $\mu = 0$ is depicted.

If $s(0) < s^*$ and $U'(C(0))/a_C \geqq q^*$, the optimal program starts off in phase 3 of the present situation and only the last two growth phases are relevant. It is only in this kind of circumstance that the story about non-shiftable capital in a two-department economy can be defended as a meaningful parable.[1]

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[1] This is aa good a place as any to say a few words about what happens if the number of sectors is made greater than three.

Increasing the number of consumption goods changes very little. Non-shiftability now holds if and only if conditions (38) or (39) are true for <u>each</u>

consumption good. If $s(0) < s^*$, this puts the greatest pressure for shifting, other things being equal, on those consumer goods having the highest content of salvageable raw materials.

An enlarged number of common intermediate raw materials sectors can also be treated by an easy modification. Now interpreting a_C and a_I as the total common raw materials per unit output of, respectively, C and I, the necessary and sufficient conditions for non-shiftability in the present case are still (38) or (39). However, the description of exactly what happens if capital is shifted can become very complicated.

As usual, working with more than one investment goods sector opens a Pandora's box of practical difficulties. There does not seem to be an easy way of cataloging results in this case.

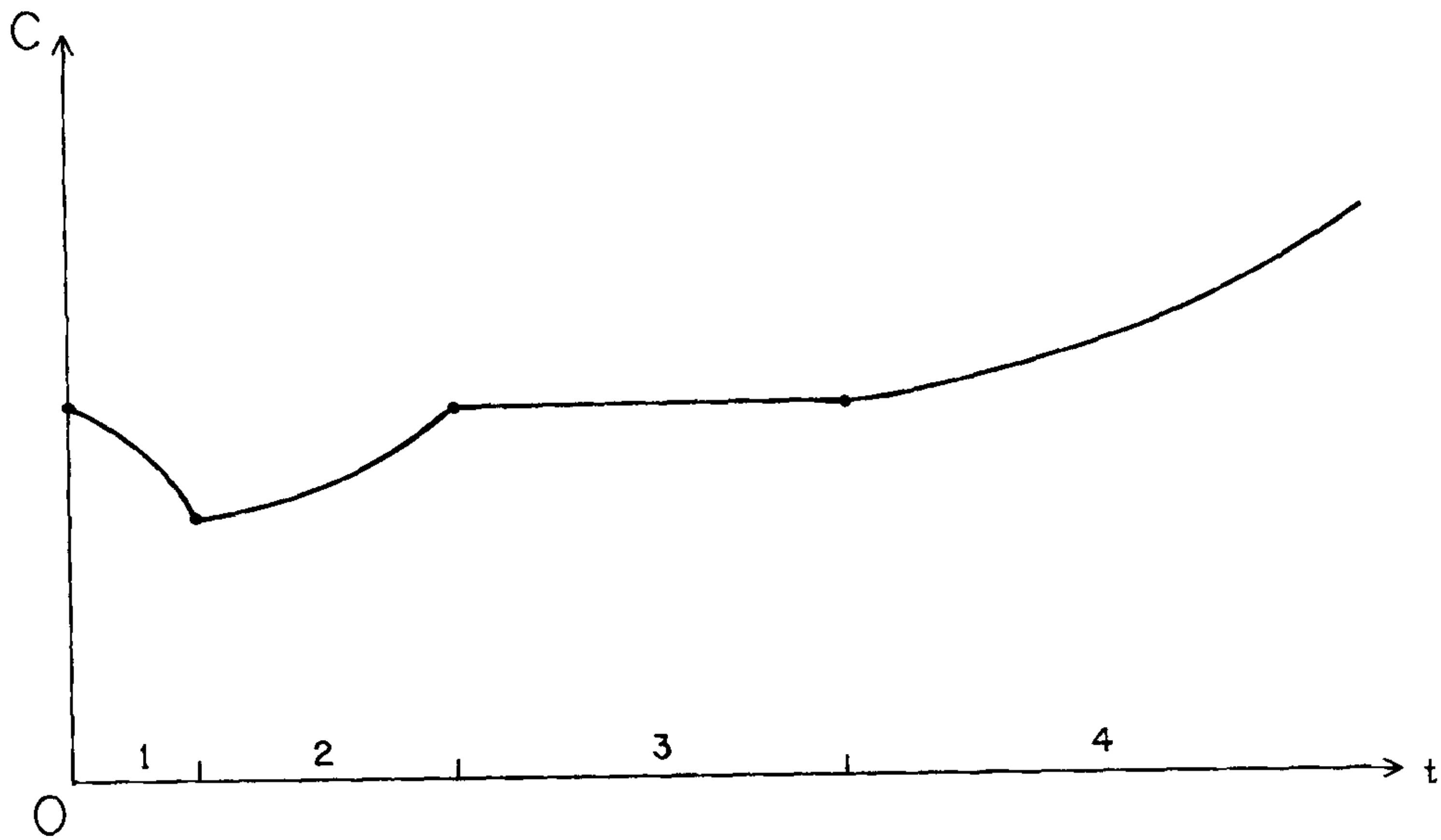

FIGURE 3.

OPTIMAL GROWTH PROFILE OF C WHEN CAPITAL IS SHIFTED FROM C TO I .

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