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THEORY OF LIQUIDITY PREFERENCE AND THE TERM STRUCTURE OF INTEREST RATES

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THEORY OF LIQUIDITY PREFERENCE AND THE TERM STRUCTURE OF INTEREST RATES*

by

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It is reasonable to assume that individuals do not desire wealth for its own sake, but for the consumption that it provides. A long term (say n period) bond is a perfectly safe asset in terms of consumption in the n^{th} period, but a risky asset in terms of consumption in preceding and following periods. A one period bond is safe in terms of consumption next period, but risky in terms of consumption in all following periods. A consol provides a perfectly safe income stream although its capital value is uncertain. Traditional theory, unfortunately, has focused on one period capital valuations.[1] If our hypothesis that individuals desire wealth for the consumption it provides is accepted, then it is not correct,

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*This paper is part of a more general study of portfolio theory being conducted with David Cass. The author is indebted to M. Miller and C. von Weizsacker for extremely helpful comments and suggestions. Both have been studying similar questions in the context of different models; Miller uses the quadratic utility function general equilibrium model, developed by Lintner [8], Sharpe [12], and Mossin [10]; von Weizsacker uses a dynamic programming approach, assuming constant elasticity utility functions and particular parameterizations of the distribution functions. It is reassuring that the preliminary results of both authors seem in accord with the findings here. I would also like to acknowledge the many useful discussions with my colleagues at the Cowles Foundation. The research was supported by a grant from the National Science Foundation.

[1] A notable exception is Tobin's recent work [15].

in spite of common usage in economics dating at least back to Keynes, to consider long term bonds as riskier than short term bonds, and a theory of the demand for money based on those considerations (such as that of Tobin [14]) may be misleading. The purpose of this paper is to provide an alternative, consumption-oriented theory of liquidity preference.

In Part I, we present the basic model, and show how it provides an answer to the first fundamental question of the theory of liquidity preference: why do some individuals hold liquid (short term) assets when they can obtain a higher return (on average) from holding long term bonds. In Part II, we analyze how changes in interest rates, uncertainty, wealth, and the degree of risk aversion affect the demand for short term bonds. In Part III, we indicate how the model may be extended.

Part I

1. The Basic Model

We begin our investigation by considering an individual who has a given amount of wealth to invest, W_0 . He can buy one period bonds, which yield a (certain) return of $(1+r_1)$ or he can buy two period bonds which, at the end of two periods, will yield a return of $(1+R)$.[1,2] At the end of the first period, his net wealth is given by

$$(1) \qquad W_1 = W_0[a(1+r_1) + (1-a)\frac{(1+R)}{(1+r_2)}] ,$$

where a is the percentage of the portfolio invested in short term bonds and where $(1+r_2)$ is the return on a one period bond purchased at the beginning of the second period. What r_2 will be is unknown to the individual at the time he has to make his original allocation. He then must allocate W_1 between consumption and investment in bonds. Our problem is to calculate the optimal allocation a, i.e. that allocation which maximizes expected utility of consumption over the two periods, $EU(C_1, C_2)$. U will be assumed to be concave, so

$$(2) \qquad U_{11} < 0 , \quad U_{22} < 0 , \quad U_{11}U_{22} - U_{12}^2 > 0$$

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[1] In order to make our analysis as close as possible to that of traditional monetary theory, we will follow the conventional practices of ignoring all non-monetary assets and other sources of income. (As Tobin has expressed it, "Liquidity preference theory takes as given the choices determining how much wealth is to be invested in monetary assets and concerns itself with the allocation of this amount among cash and alternative monetary assets." [14])

[2] We will not in this paper be concerned with why individuals hold money rather than short term bonds (demand deposits rather than savings deposits). The explanation of this must lie outside the purely portfolio analysis of this paper. We will assume that the one period safe asset has a positive rate of return, since such an asset is always available to the individual; the question we are interested in is what determines individuals' demands for liquid (short term) assets.

and such that both C_1 and C_2 are superior goods, i.e., as wealth increases, at any given interest rate, consumption in both periods increases; this implies that[1]

(3) $$U_{11} - (1+r)U_{21} < 0 \quad \text{and} \quad U_{22}(1+r) - U_{21} < 0 \ .$$

One interpretation of the concavity restriction which will be useful in Section 2 is the following. Assume an individual is given an uncertain wealth, with mean $\bar{W}$ and small variance σ^2, with which he will buy consumption goods in the two periods. What is the certainty equivalent of the uncertain wealth, i.e., if $V(W, 1/1+r)$ is the maximum value of U attainable with wealth W and interest rate r, for what value of x is

$$EV(W, 1/1+r) = V(x\bar{W}, 1/1+r) \ .$$

Taking a Taylor Series expansion, around $\bar{W}$ we find

(4) $$\frac{1}{2}\,\frac{V_{11}\sigma^2}{V_1\bar{W}} = \frac{1}{2}\,\frac{V_{11}\bar{W}}{V_1}\left(\frac{\sigma}{\bar{W}}\right)^2 = x-1 \ .$$

But $V_1 = U_1$, so $V_{11} = (U_{11}U_{22} - U_{21}^2)/\left\{\frac{-U_{11}}{(1+r)^2} + \frac{2U_{21}}{(1+r)} - U_{22}\right\}$.

The denominator is unambiguously positive, so the sign of $x-1$ (i.e. whether the individual requires positive or negative compensation for the removal of the uncertainty) is the same as the sign of $(U_{11}U_{22} - U_{21}^2)$. $\hat{\rho} \equiv V_{11}\bar{W}/V_1$ we call the measure of relative risk aversion for a two period model.

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[1]Throughout the paper, where there is no ambiguity, we shall drop the subscript on r_2, i.e., r (with no subscript) stands for r_2 .

For any given value of W_1 and $(1+r_2)$, the individual allocates his income between the two periods in the usual way, i.e., he maximizes

(5) $$U(C_1, C_2)$$

subject to the budget constraint

(6) $$C_1 + C_2/(1+r_2) = W_1 .$$

The necessary condition for an optimum is, of course, that

(7) $$U_1 = (1+r_2)U_2$$

and provided the indifference map is quasi-concave, this is sufficient.

From (1), W_1 is just a function of $(1+r_2)$, for given values of a, $(1+r_1)$ and $(1+R)$. Thus the solution will be simply a function of $(1+r_2)$:

(8) $$C_1 = \psi[(1+r_2);\ a,\ (1+r_1),\ (1+R)]$$

$$C_2 = (W_1 - C_1)(1+r_2)$$

Hence, the problem of finding the optimal a is to find that a which maximizes

(9) $$EU(C_1, (W_1 - C_1)(1+r_2))$$

where C_1 and W_1 are functions of the random variable r_2 and the control variable a For an interior maximum, we require

$$E\{(U_1 - (1+r_2)U_2)dC_1/da + U_2\{(1+r_1)(1+r_2) - (1+R)\}W_0\} = 0$$

But from (7), this simply requires

$$(10) \qquad EU_2[(1+r_1)(1+r_2) - (1+R)] = 0$$

From this, we immediately derive the result that if the individual is risk neutral, an interior solution requires the long rate to be the product of the expected short rates:

$$(11) \qquad (1+r_1)E(1+r_2) = (1+R)$$

If at $a = 0$

$$EU_2[(1+r_1)(1+r_2) - (1+R)] < 0$$

an individual specializes in long term bonds, while if at $a = 1$,

$$EU_2[(1+r_1)(1+r_2) - (1+R)] > 0$$

he specializes in short term bonds.[1] For a risk neutral person, this

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[1]It is easy to see that there is a unique interior maximum:

$$\left.\frac{\partial^2 EU}{\partial a^2}\right|_{\frac{\partial EU}{\partial a} = 0} = E\left\{\left\{U_{21}\frac{dC_1}{dW_1} + U_{22}\frac{dC_2}{dW_1}\right\}[(1+r_1)(1+r_2) - (1+R)\}\frac{dW_1}{da}\right\}$$

$$= E\left\{\frac{(U_{12}^2 - U_{11}U_{22})}{\left(\frac{-U_{11}}{(1+r_2)^2} + \frac{2U_{21}}{(1+r_2)} - U_{22}\right)}\frac{[(1+r_1)(1+r_2) - (1+R)]^2}{(1+r_2)}\right\} < 0$$

(from 3).

means that if the long rate is greater than the product of the expected short rates, he purchases only longs, and conversely.

2. Patterns of specialization for Risk Averse Individuals

Whether an individual is at an interior maximum will depend both on his subjective probability distribution for interest rates next period and on his utility function. We will employ two assumptions about his expectations of next period's interest rates. The first corresponds to the assumption of "market risk neutrality" and has been extensively used in the literature on the term structure of interest rates.[1] If the absence of uncertainty, market equilibrium would require

$$(1+r_1)(1+r_2) = (1+R)$$

By market risk neutrality we mean that

Hypothesis A. $$(1+r_1)E(1+r_2) = (1+R)$$

The second assumption which we shall investigate is that the expected return from holding the two period bond for one period is equal to the safe return on the short term bond. We shall call this

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[1]For extensive discussions of this, see [8a] and [9].

Hypothesis B. $E(1+R)/(1+r_2) = (1+r_1)$

(a) Hypothesis A has been the subject of much debate in the literature. Hicks, for instance, argued that in order to induce individuals to hold long term bonds, there would have to be a risk premimum, i.e.

$$(1+r_1)E(1+r_2) < (1+R).$$

As he put it,

> "If no extra return is offered for long lending, most individuals (and institutions) would prefer to lend short, at least in the sense that they would prefer to hold their money on deposit in some way or other." [6]

As we shall now see, this presumption is not in general correct.

First, a risk averse individual will always buy some long bonds. For, letting $a = 1$, straightforward calculations show that[1]

$$(12)\quad \frac{dU_2}{d(1+r_2)} \sim \left[(U_{11}U_{22} - U_{12}^2)C_2/(1+r_2)^2 - U_2(U_{22} - (1+r_2)U_{12})\right]$$

which, by (2) and (3), is unambiguously negative. Hence, using (11)

$$(1+R)EU_2 = (1+r_1)E(1+r_2)EU_2 > (1+r_1)EU_2(1+r_2)$$

and the result is immediate.

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[1]The symbol $\sim$ means "is of the same sign as."

This provides us with our first contradiction to classical liquidity preference theory: Long term bonds will always be held by a risk averse individual in his portfolio when there is no "risk premium" on them. (By continuity, they will also be held even when there is a small negative risk premium.)

More striking is the following result: With no risk premium a risk averse individual may even specialize in long term bonds.

To see this, we consider first the case of an additive utility function (i.e., $U = U(C_1) + (1-\delta)U(C_2)$).[1] Assume $a = 0$. Then, straightforward calculations show that

$$\frac{dC_2}{d(1+r_2)} \sim \left\{ \frac{-U_{11}(C_2 - (1+R)W_0)}{(1+r_2)^2} + U_2 \right\} .$$

Using (1), (6), and rearranging terms, we obtain

$$(13) \qquad \frac{dC_2}{d(1+r_2)} \sim \left\{ 1 - \frac{-U_{11}C_1}{U_1} \right\}$$

The expression $\frac{-U_{11}C_1}{U_1}$ is the familiar expression for relative risk aversion introduced by Pratt [11] and Arrow [1]. Hence, if relative risk aversion is between 0 and 1, $dC_2/d(1+r_2)$ is positive,[2] while if the

[1] Where δ is the pure rate of time preference.

[2] More accurately, $dC_2/d(1+r_2)$ is non-negative, since if relative risk aversion is below one, the indifference curves hit the axes, and the individual may not change his consumption C_2 for some changes in r_2 .

individual is very risk averse, i.e., has a relative risk aversion greater than unity, $dC_2/d(1+r_2)$ is negative. If the utility function is logarithmic in consumption (the Bernoulli utility function) then $dC_2/d(1+r_2)$ is identically zero. Using this with (10), one immediately observes that if there is no risk premium for long term bonds, i.e., if hypothesis A is true, and if relative risk aversion is less than or equal to one, the individual specializes in long term bonds, while if relative risk aversion is greater than one, the individual always buys both kinds of bonds.

An extension to non-additive utility functions is possible. We use the concept of relative risk aversion introduced earlier (p. 4) for the non-additive utility function. Straightforward calculations show that

$$(14) \qquad \frac{dU_2}{d(1+r_2)} \sim (U_{22}U_{11} - U_{12}^2)\frac{C_1}{(1+r_2)} - U_2(U_{22} - \frac{U_{21}}{(1+r_2)})$$

$$\sim \frac{\hat{\rho}}{\frac{d \ln C_1}{d \ln W_1}} - 1$$

where $\hat{\rho}$ is the measure of relative risk aversion for a two period utility function. Hence, if the indifference map is homothetic, $dU_2/d(1+r_2) \gtrless 0$ as $\hat{\rho} \gtrless 1$. Otherwise, $dU_2/d(1+r_2) \gtrless 0$ as $\hat{\rho} \gtrless$ wealth elasticity of C_1.

(b) Hypothesis B. Under Hypothesis B, on the other hand, all individuals buy some short term bonds, and some may even specialize in them. Rewriting (10) as

$$EU_2(1+r_2)\left[(1+r_1) - \frac{1+R}{1+r_2}\right]$$

it is clear that if $U_2(1+r_2)$ is an increasing (decreasing) function of r_2 at a given a, then that a is too large (small). But, for the additive utility function, letting $U_2 \equiv U_2'$, $U_{22} \equiv U_2''$, etc.,

$$\left.\frac{dU_2'(1+r_2)}{d(1+r_2)}\right|_{a=0} = \frac{U_2'U_2''(1+r_2)^2\rho + U_2'U_1''}{U_1'' + U_2''(1+r_2)^2} \geq 0 \tag{15}$$

$$\left.\frac{dU_2'(1+r_2)}{d(1+r_2)}\right|_{a=1} = \frac{U_2'U_1''(1-\rho)}{U_1'' + U_2''(1+r_2)^2} \gtrless 0 \quad \text{as} \quad \rho \lessgtr 1 \tag{16}$$

so if Hypothesis B is true, all individuals hold some short term bonds and if $\rho \leq 1$, only short term bonds.

3. Interpretation

The results presented in the above discussion run counter to most of the literature on the term structure of interest rates and liquidity preference, which say the short term rates ought to be less than or equal to long term rates for an interior solution. There are at least three reasons for the discrepancy between our view and the traditional one:

a) Capital gains. Keynes rightly emphasized the importance of the capital gains or losses in long term bonds from a change in the (short term) interest rate. When the short term interest rate falls, the price of a long term bond increases: the holder of the bond experiences a capital gain. In our example, at the end of one period, the value of the two period bond is

$$\frac{(1+R)}{(1+r_2)}$$

so the expected return from holding a two period bond for one period is

$$E\,\frac{(1+R)}{(1+r_2)} > \frac{(1+R)}{E(1+r_2)}$$

by Jensen's inequality. Hence even if $(1+R) = (1+r_1)E(1+r_2)$, the expected return on a two period bond held for one period would exceed that for a one period bond, so a risk averse individual would still buy both long and short bonds, and a person who is only slightly risk averse would specialize in long term bonds.[1]

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[1] von Weizsacker has emphasized this point in his analysis of the problem.

b) Negative covariance with interest rate. We have already noted that the value of long bonds varies inversely with the (short term) rate of interest. But it is easy to see that, in a portfolio consisting only of short term bonds, the level of utility increases with the rate of interest. If the individual had bought only short term bonds, his wealth at the end of the initial period would be $W_0(1+r_1)$. His budget constraint would then rotate as indicated in Figure 3.1, as the interest rate increased. On the other hand, note that when all of the assets are in long term bonds, the individual is better off with low interest rates. The reason is obvious: the lower the interest rate, the greater the capital gain. This can be seen diagramatically in Figure 3.2. In this case, the first effect (the capital gains effect) and the positive covariance with utility effect offset each other. The individual may want to buy more or less long term bonds. In the case of additive utility functions, we have shown that which effect is stronger will depend on the degree of relative risk aversion.

In practice, the covariance effect is probably even stronger than suggested by our model which is limited to two types of bonds, since the typical portfolio consists also of stocks, the return to which is positively correlated with the rate of interest and therefore negatively correlated with the return on long bonds.[1]

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[1] Miller in his work has emphasized this aspect of the problem.

c) Consumption patterns. The typical analysis of demand for bonds follows the Keynesian tradition of separating out the savings decision from the portfolio allocation problem. This is, as we have observed, not really legitimate. The two period bond is safe in terms of consumption one period hence. In these terms, there is no answer to the question, which is the safe asset. Thus the variance in the one period return from a long term asset does not mean that it is not a relatively safe asset from the point of view of the individual's consumption pattern.

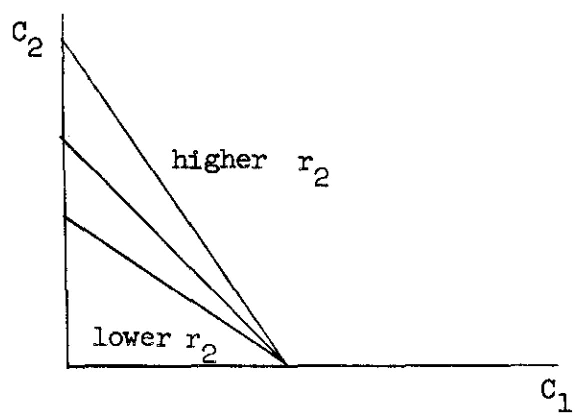

FIGURE 3.1

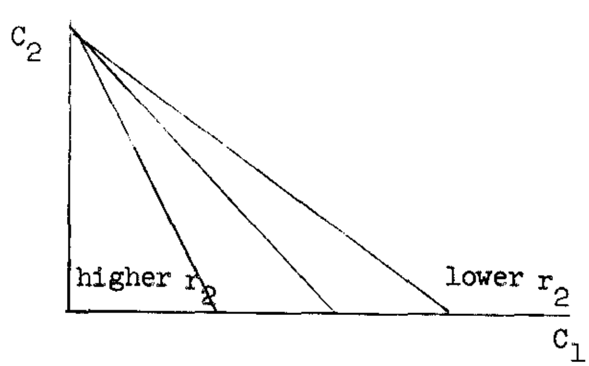

FIGURE 3.2

Part II

The theory of liquidity preference has traditionally focused on two questions:

(1) Why do individuals hold short term bonds (liquid assets) when they can get (on average) a higher rate of return on long term bonds? In the previous section we have suggested that the answer that short term bonds are safer is not really an adequate explanation. Indeed, we have shown that individuals will hold long term bonds even when they can get (on average) a higher rate of return from short term bonds. And the common explanation for both phenomenon is that individuals wish to avoid uncertainty in their consumption stream.

(2) Why is the liquidity preference schedule downward sloping? Tobin [14] has shown that although in general it may not be, for low interest rates it always will be downward sloping. We will show that, _if_ the term structure of interest rates remains unchanged, the presumption for a downward sloping schedule is even weaker. We also will show that whether expectations are elastic or inelastic makes little difference for these results.

But before we can establish this, we need some more detailed information on how the individual allocates his portfolio. To obtain this, we first examine how (a) uncertainty, (b) the level of wealth, and (c) the degree of relative risk aversion affect the demand for liquid assets. (Section 4, 5, and 6, respectively.)

4. Effects of Increasing Variance

(a) A single period analysis. Before analyzing how increasing uncertainty affects the individual's allocation between short term and long term bonds, it will be useful to consider how, in a one period problem, the allocation between safe and risky assets is affected by an increase in uncertainty. There is, not surprisingly, no unique way of characterizing an increase in uncertainty. We shall employ several approaches.

The first is to consider a symmetric "spread" in the distribution. Then, if $e(\lambda)$ is the (random) return on a risky asset, with mean $\bar{e}$,

$$(17) \qquad e(\lambda) = \lambda e(1) + (1-\lambda)\bar{e}$$

$$V(e(\lambda)) = E(e(\lambda) - \bar{e})^2 = \lambda^2 E(e(1) - \bar{e})^2$$

As λ increases, the variance increases.

Consider an individual who wishes to maximize his expected utility of wealth, $EU(W)$, at the end of the period. If W_0 is his initial wealth, $r(> 0)$ the return on a safe asset, and $\underline{a}$ the portion of his portfolio in the safe asset, then expected utility maximization requires

$$(18) \qquad EU'(W_0(ar + (1-a)e(\lambda))(r - e(\lambda)) = 0$$

It is clear that

$$\frac{da}{d\lambda} \sim EU''(1-a)(e-\bar{e})(r - e(\lambda))W_0 - U'(e-\bar{e})$$

$$= E\left\{\frac{U''W}{U'\lambda} - \frac{U''}{U'}\,(\bar{e}(1-a) + ra)W_0\right\}U'(r - e(\lambda)) - U'(e-\bar{e})$$

Thus, $da/d\lambda$ is positive if there is increasing (or constant) relative risk aversion and decreasing (or constant) absolute risk aversion. (Note that constant relative risk aversion implies decreasing absolute risk aversion and constant absolute risk aversion implies increasing relative risk aversion.)

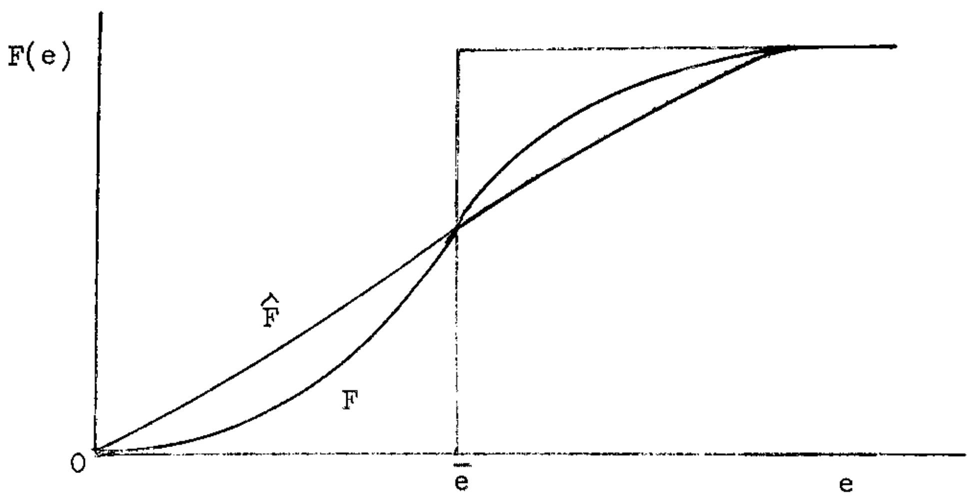

FIGURE 4.1

The second approach is essentially a generalization of the first. In Figure 4.1 we have drawn the distribution function for e . We have also drawn the corresponding functions for a perfectly safe asset. Any asset whose distribution function is such that it lies between the distribution function of e and that for the safe asset will be said to be less risky than e . $\left(\int e\,dF = \int e\,d\hat{F}\right)$

If a is the allocation between safe and risky assets when the distribution of e is F, and $\hat{a}$ is the allocation when the distribution of e is $\hat{F}$, where $\hat{F}$ is riskier than F, then, if $\int U'(r-e)dF = 0$,

$$\hat{a} \gtrless a \quad \text{as} \quad \int U'(r-e)d\hat{F} = \int U'(r-e)\{d\hat{F} - dF\} \gtrless 0$$

But, integrating by parts,

$$(19) \quad \int U'(r-e)\{d\hat{F} - dF\} = \int \left\{ \frac{U''W}{U'} - \frac{U''rW_0}{U'} + 1 \right\} U'\{\hat{F} - F\}de$$

$$= \int \left\{ \frac{d\ -\frac{U''W}{U'}}{dW} - rW_0 \frac{d\ -\frac{U''}{U'}}{dW} \right\} (1-a)W_0\gamma(e)de$$

$$+ \gamma(e) \left\{ \frac{U''W}{U'} - \frac{U''rW_0}{U'} + 1 \right\}_{e=\max\{e\}}$$

where

$$\gamma(e) \equiv \int_0^e U'\{\hat{F} - F\}de \geq 0 \qquad \gamma(0) = 0$$

so that if relative risk aversion is less than or equal to unity (in the relevant range) and there is increasing relative and decreasing absolute risk aversion, then $a < \hat{a}$.

Stronger results may be obtained if we consider situations where the variance of the risky asset is small. Then, we can approximate the first order condition for utility maximization by the first two terms of a Taylor series expansion:[1]

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[1]We assume throughout that the risky asset (in subsequent sections, r_2) is bounded with probability one.

$$(20) \qquad 0 = EU'(r-e) \approx U'_{e=\bar{e}}(r-\bar{e}) + \left.\frac{d^2U'(r-e)}{de^2}\right|_{e=\bar{e}} \frac{E(e-\bar{e})^2}{2}$$

But for an interior solution, $\bar{e} > r$, so the coefficient of $E(e-\bar{e})^2$ must be positive; thus an increase in variance must lead to an increase in the demand for the safe asset.

(b) The two period problem.[1] The problem of defining an increase in "uncertainty" for our two period model is even more difficult than for our one period model. As we increase the variance of r_2, do we keep the mean of r_2 or the mean of $1/1+r_2$ constant? The two are clearly not equivalent; consider the case where r_2 has small variance:

$$E\,\frac{1}{1+r_2} \approx \frac{1}{1+\bar{r}_2} + \frac{E(r_2 - \bar{r}_2)^2}{(1 + \bar{r}_2)^3}$$

As the variance of r_2 increases, keeping the mean of r_2 constant, the mean of $1/1+r_2$ increases. Since the one period expected return on the two period bond is $1+R/E(1/1+r_2)$, it seems more reasonable to keep the mean of $1/1+r_2$ constant. (This will mean, of course, that if hypothesis A was satisfied before the increase in uncertainty it will not be satisfied after the increase.)

Taking a Taylor series expansion of (10) around $E\frac{1+R}{1+r_2} = \bar{e}$, and letting $\frac{1+R}{1+r_2} = e$

$$(21) \quad 0 = EU_1'(1+r_1 - e) \approx U_1'[(1+r_1) - \bar{e}] + \left.\frac{d^2U_1'((1+r_1 - e)}{de^2}\right|_{e=\bar{e}} E(e-\bar{e})^2$$

[1] In the remainder of the paper, we shall limit ourselves to additive utility functions.

If, as we would expect, $\bar{e} > 1+r_1$, then for an interior solution the second term of the left hand side of (21) must be positive, so that an increase in the variance leads to an increase in the demand for short term bonds.[1]

(c) Uncertainty and utility. Does uncertainty make an individual worse off? The answer usually provided is yes; the extent to which he is worse off is measured by the concept of certainty equivalence: the amount of wealth an individual would be willing to forego to avoid the uncertainty. In this model, however, if we compare the level of utility attained when $1+r_2$ is known for certain, and so must equal $1+R/1+r_1$, and when $(1+r_2)$ is random, but with mean $(1+R)/(1+r_1)$, he is better off in the latter situation than in the former. In the former situation, the individual chooses consumption bundle $(\hat{C}_1, \hat{C}_2)$, where $(\hat{C}_1, \hat{C}_2)$ is the solution to the problem

$$\text{(22)} \qquad \begin{aligned} &\text{Maximize} \quad U(C_1, C_2) \\ &\text{subject to} \quad C_1 + \frac{(1+r_1)}{(1+R)} C_2 = W_0 \end{aligned}$$

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[1]On the other hand, an increase in the variance of r_2, keeping the mean of r_2 fixed, leads to an actual increase in the demand for long term bonds, provided $(1+r_1)E(1+r_2) > 1+R$. Taking a Taylor series expansion of (10), we have

$$0 = EU_2'[(1+r_1)(1+r_2) - (1+R)] \approx U_2'[(1+r_1)(1+\bar{r}_2) - (1+R)] + \left.\frac{d^2U_2'[(1+r_1)(1+r_2) - (1+R)]}{d(1+r_2)^2}\right|_{r_2=\bar{r}_2} E(r_2 - \bar{r}_2)^2$$

since the first term is positive, the second must be negative.

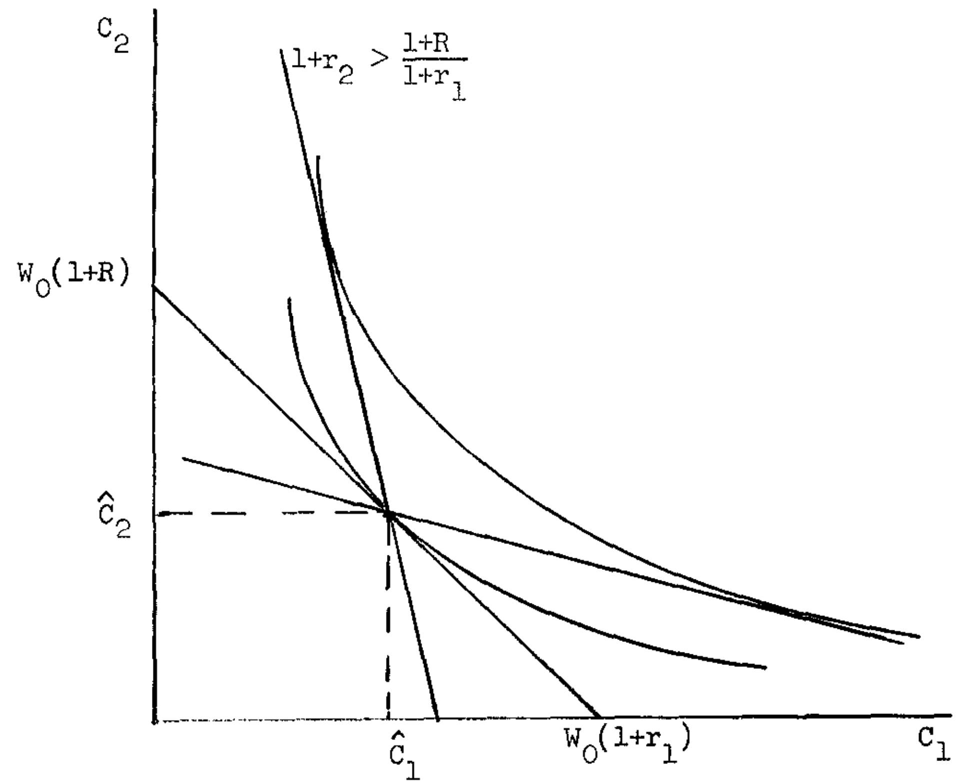

FIGURE 5.1

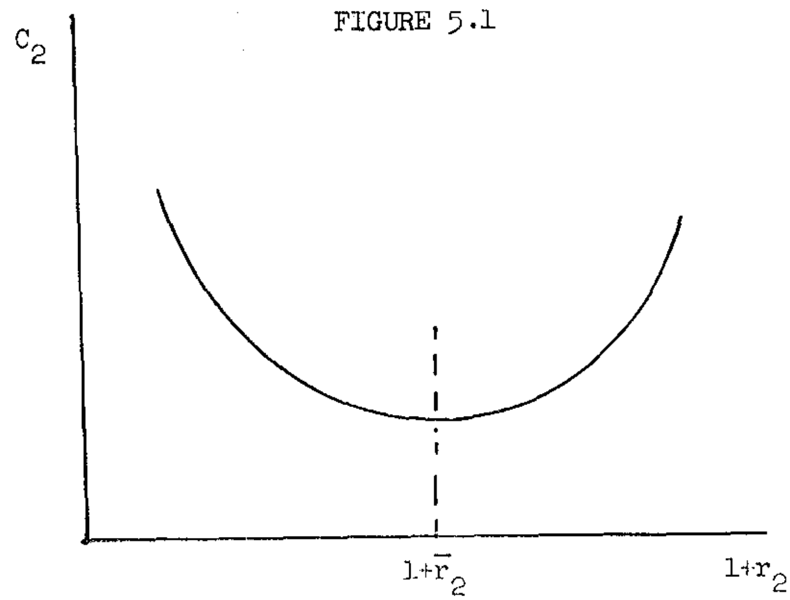

FIGURE 5.2

Now, under the uncertain situation, the individual could have set $\hat{a} = \hat{C}_1/1+r_1W_0$. Then, when $(1+r_2) \neq (1+r_1)/(1+R)$ he could consume some bundle other than $(\hat{C}_1, \hat{C}_2)$ and increase his level of utility. Finally, he can always increase his expected utility still further by choosing a optimally. Thus, if a^* is the optimal allocation,

$$EU(C_1(r_2, a^*), C_2(r_2, a^*)) \geq EU(C_1(r_2, \hat{a}), C_2(r_2, \hat{a})) \geq U(\hat{C}_1, \hat{C}_2)$$

with strict inequalities holding in all cases with finite relative risk aversion.

5. Wealth Effects

Traditional theory of liquidity preference has little to say on how the composition of the portfolio ought to change as wealth increases. Before we can procede, it will be necessary to establish some further properties of the individual's optimal allocation. We consider first the situation under Hypothesis A. Assume there were no uncertainty, and the individual allocated his portfolio so his income from one period bonds equalled his first period consumption, i.e., $a(1+r_1)W_0 \equiv \hat{C}_1$, and similarly $(1-a)\dfrac{(1+R)W_0}{(1+r_2)} \equiv \hat{C}_2$. Then, from Figure 5.1, it is clear that for $r_2 > \bar{r}_2$, $C_2 > \hat{C}_2$, and for $r_2 < \bar{r}_2$, $C_2 < \hat{C}_2$. Hence $EU_2'\{(1+r_1)(1+r_2) - (1+R)\} < 0$. Thus $a < \hat{C}_1/(1+r_1)W_0$. Indeed, at the optimal allocation $U_2'(C_2)$ cannot be a monotonic function of r_2 .

We shall now assume that the variance in r is small. Then the relation between C_2 and $(1+r_2)$ can be depicted as in Figure 5.2. Thus, if the variance in r is small, $C_2(\bar{r})$ must be in the region of minimum C_2 for the optimal a. But since

$$U_2'(C_2) = \frac{U_1'}{1+r_2} , \quad \text{and} \quad \frac{dU_1'/1+r_2}{d(1+r_2)} = \frac{U_1'' dC_1/d(1+r_2)}{1+r_2} - \frac{U_1'}{(1+r_2)^2}$$

this means $\left.\dfrac{dC_1}{d(1+r_2)}\right|_{r_2=\bar{r}_2} < 0$.

Similarly, under hypothesis B, it can be shown that $(1-a) < \hat{C}_2/(1+R)W_0$ and $dC_2/d(1+r_2)\Big|_{r_2=\bar{r}_2} > 0$.

It is now easy to determine the effect of a change in W_0 on a ; letting $e = \dfrac{1+R}{1+r_2}$

$$(23) \quad \frac{da}{dW_0} \sim E - \rho(C_2) \frac{d \ln C_2}{d \ln W_1} U_2'(r_2-\bar{r}_2) \sim -E\rho(C_1) \frac{d \ln C_1}{d \ln W_1} U_1'(1+r_1 - e)$$

under hypothesis A, and under hypothesis B,

$$(24) \qquad \frac{da}{dW_0} \sim E - \rho(C_2) \frac{d \ln C_2}{d \ln W_1} U_2'(1+r_2 - \alpha)$$

where $\alpha = 1/E(1/1+r_2)$. If the individual has constant relative risk aversion, then his indifference map is homothetic, so $d \ln C_2/d \ln W_1 = 1$, and $da/dW_0 = 0$. Because C_2 is not monotonic in r_2 , it is not possible to say in general what happens if relative risk aversion is not constant. But if there is small variance in r_2 , and the indifference curves are homothetic, then we can take a Taylor Series expansion of (23) to obtain

$$\frac{da}{dW_0} \sim -\rho'(C_1(\bar{e})) \left.\frac{dC_1}{de}\right|_{e=\bar{e}} EU_1'(1+r_1 - \bar{e})^2 \tag{25}$$

so

$$\frac{da}{dW_0} \gtreqless 0 \text{ as relative risk aversion is } \begin{cases} \text{increasing} \\ \text{constant} \\ \text{decreasing} \end{cases} \tag{26}$$

Thus, if, as Arrow has argued [1], there is increasing relative risk aversion, then an increase in wealth leads to a smaller portion of one's wealth being held in long term assets, which are usually considered the risker assets; recall that in a one period model, the portion of one's wealth in the risky asset decreases (increases) if one has increasing (decreasing) relative risk aversion [1, 13].

Similar results obtain under Hypothesis B, if the Taylor Series expansion is taken around the value of r for which $1/1+r = E(1/1+r)$.

6. Comparisons among Individuals

(a) Additive constant elasticity utility functions. We have already observed that the more relatively risk averse of two individuals may buy more or less long term bonds, depending on the expected returns.

If the individual has an additive constant elasticity utility function, we can show under hypothesis A, for instance, that the more risk averse individual buys more short term bonds (at least for small variance); since under hypothesis A, if relative risk aversion is less than unity, the individual specializes in long term bonds, we consider only the case when relative risk aversion is greater than unity; the utility function can be written

$$U = - C_1^n - C_2^n(1 - \delta), n < 0 \ .$$

Since $-U_i''C_i/U_i' = n-1$, $n-1$ is the measure of relative risk aversion. Hence, letting

$$x = \frac{1}{1 + [(1+r_2)(1-\delta)]^{n/n-1}}$$

we obtain

$$\frac{da}{dn} \sim EC_2^{n-1} \left\{ \ln C_2 + (n-1)\frac{d \ln C_2}{dn} \right\} (r_2 - \bar{r}_2)$$

$$\approx EC_2^{n-1} \frac{(r_2 - \bar{r}_2)^2}{1+r_2} \left\{ a - \frac{x}{1+x} + \frac{x}{(1+x)^2} \frac{n}{(n-1)^2} \ln (1+r_2) \right\}_{r_2=\bar{r}_2}$$

But we have already shown that (p. 22)

$$a < \frac{x}{1+x}$$

so if $n < 0$

$$\frac{da}{dn} < 0 .$$

On the other hand, it is easy to show, for instance, that if hypothesis B is satisfied then $\frac{da}{dn} > 0$ for small variances in r_2 . Thus again, which of the bonds behaves like a risky asset is indeterminate.

The demand curves of **a** as a function of $\frac{1+R}{1+r_1}$ for individuals with different values of n may be depicted as in Figure 6.1 (as usual, we assume the range of r_2 is finite).

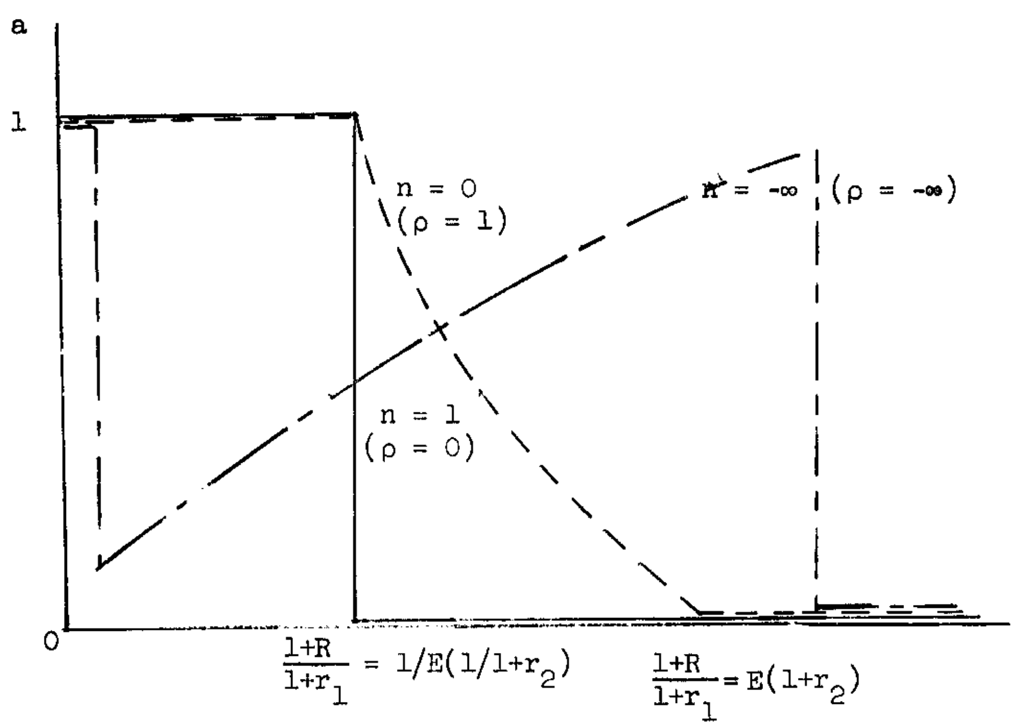

FIGURE 6.1

(b) Homogeneous utility functions. It is not clear, however, that $(n-1)$ is a desirable measure of risk aversion in this context. As n changes, both the shape and the "degree of homogeneity" of the utility function change. Each indifference curve of the family $|U| = C_1^n + C_2^n(1-\delta)$ is the same shape as a constant elasticity of substitution isoquant of elasticity $1/1-n$. To separate out the effects of shape from those of homogeneity, we introduce the utility function

$$U = (C_1^n + C_2^n)^{Z/n} \quad (n \neq 0) \quad \text{and} \quad U = [\ln C_1 + \ln C_2]^Z$$

Then the first order condition may be written

$$E\hat{W}^Z(1+x)^{Z(1-n/n)}[(1+r_1)(1+r_2) - (1+R)] = 0$$

where $\hat{W} = a(1+r_1)(1+r_2) + (1-a)(1+R)$; so, letting $n/n-1 = \xi$

$$\frac{da}{d\xi} \sim E[\ln(1+x) - \frac{x}{1+x} \ln x]\hat{W}^Z(1+x)^{-Z/\xi}[(1+r_1)(1+r_2) - (1+R)] < 0$$

since

$$\frac{1+r_2 d(\ln(1+x) - \frac{x}{1+x} \ln x)}{d(1+r_2)} = -\frac{x \ln x}{(1+x)^2}\,\frac{n}{n-1} < 0$$

This means that as the elasticity of substitution of the indifference curves gets smaller, the demand for the short term bond decreases.

The effects of a change in Z may be similarly analyzed; under hypothesis A, for small variance,

$$\frac{da}{dZ} \sim E(\ln \hat{W} - \frac{1}{\xi}\ln(1+x))\hat{W}^Z(1+x)^{-Z/\xi}(r_2 - \bar{r}_2)$$

$$\approx E(a - \frac{x}{1+x}\Big|_{r_2=\bar{r}_2})(r_2 - \bar{r}_2)^2\hat{W}^Z(1+x)^{-Z/\xi} < 0$$

Similarly if hypothesis B is true, (for small variance), taking a Taylor series expansion around the value of r_2 for which $\frac{1}{1+r_2} = 1/\alpha$,

$$\frac{da}{dZ} \sim E(a - \frac{x}{1+x})(1 + r_2 - \alpha)^2\hat{W}^Z(1+x)^{-Z/\xi} > 0 \ .$$

7. Effects of Changes in Interest Rates

One of the difficulties in analyzing the effects of an increase in the short term rate of interest is the determination of what happens to (a) the price of long term bonds and (b) expectations about future short term rates of interest as a result. Much of the familiar liquidity preference analysis has made confusing, if not contradictory, assumptions. For instance, one argument for the negative slope of the liquidity preference schedule may be sketched as follows: as the interest rate falls, the price of long term bonds, say consols, rises -- if the long term bond is a consol, the price is equal to $1/r_1$. But since expectations about future short rates are unaffected by what happens today, the expected capital loss from holding the consol increase, so the demand for them decreases, and the demand for short term bonds increases. But why should the price of a consol rise in proportion to the fall in r_1 ? Only if future expected short rates also fall as r_1 falls. But the second part of the argument requires that expectations about future expected short rates be unaffected by the change in short term interest rates.

We shall now show (under hypothesis A) that if future expected short rates are in fact unaffected, and if the price of the long term bond changes so that hypothesis A continues to be satisfied, i.e., $1+R = (1+r_1)E(1+r_2)$, the demand for short term bonds may be completely unaffected by a change in r_1 , for constant W_0 . For

$$\frac{da}{dr_1} \sim E\, \frac{U''}{U'}\, C_2\, \frac{d \ln C_2}{d \ln W_1}\, \frac{1}{1+r_1}\, U'(r - \bar{r}) \tag{27}$$

It is clear that under constant relative risk aversion, a does not change at all. Moreover, observing that (23) and (27) are identical, it is clear that an increase in the short run rate of interest, keeping W_0 constant, with appropriate adjustments in the price of the long term bond increases, leaves unchanged, or decreases the proportion of one's assets in short term bonds as the proportion of one's assets in short term bonds increases, is unchanged, or decreases as wealth increases (e.g., if r has small variance, then as relative risk aversion is increasing, constant or decreasing).[1] Note that identical results can be obtained for hypothesis B.

So far, we have assumed that changes in the interest rate today have no effect on expectations of short rates tomorrow. This is an extreme form of inelasticity of expectations. Keynesian theory suggests that this inelasticity of expectations is an important aspect of the speculative demand for money: if expectations about future interest rates fall as the interest rate today falls, a capital loss from holding a long term bond would be no more likely at a low interest rate than at a high one. We shall show that in fact if there is a positive elasticity of expectations, and prices of bonds adjust so hypothesis A continues to be satisfied, the elasticity of the demand schedule may actually increase.

Changes in r_1 can effect expectations about r_2 in either an additive or multiplicative way. In the latter, assume the initial value of $1+r_1 = 1+\bar{r}_1$; now let $1+r_1 = h(1+\bar{r}_1)$. Then, considering r_2 as a function of h

$$1+r_2(h) = (1 - \mu + \mu h)(1+r_2(1))$$

$$E(1+r_2(h)) = (1 - \mu + \mu h)E(1+r_2(1))$$

where μ is the expectations elasticity coefficient (equals zero in the previous analysis).

---

[1]All of this ignores the fact that as r_1 rises, if individuals have some of their wealth in long term assets, the price of those long term assets will fall, and hence W_0 falls. See [14].

The return on the two period bond held for one period is (under hypothesis A)

$$\frac{(1+r_1)E(1+r_2(h))}{1+r_2(h)} = \frac{(1+r_1)(1+r_2(1))}{E(1+r_2(1))}$$

independent of h or μ . Hence, the elasticity of expectations have no affect on the slope of the liquidity function: (27) still obtains.

In the additive formulation, we assume

$$1+r_1 = 1+\bar{r}_1 + \mu h$$

$$1+r_2(h) = 1+r_2(0) + \mu h$$

$$E(1+r_2(h)) = E(1+r_2(0)) + \mu h$$

The return on a two period bond held for one period is now

$$\frac{(1+r_1)E(1+r_2 + \mu h)}{1+r_2 + \mu h}$$

so the sign of

$$\frac{da}{dh} - \frac{da}{dh}_{\mu=0} \sim \mu E U_2'' \frac{dC_2}{dr_2}(r-\bar{r}) - \mu(1-a)(1+r_1)W_0 E\rho(C_2)\frac{d \ln C_2}{d \ln W_1}\frac{U_2'(r-\bar{r})}{W_1(1+r_2)}$$

If the variance of r_2 is small and the indifference map is homothetic the first term will always be negative and the second positive, and without further restrictions it cannot be ascertained which is larger in absolute value.

If, as the above analysis suggests, changing the short term interest rate with a corresponding change in the price of long term bonds has little effect (or even the wrong effect) on the demand for money, how deos monetary policy work? One way, to which we have already drawn attention, is that as r_1 changes, the value of wealth, and hence demand for both short and long term bonds changes; another way seems to be through a change in the term structure of interest rates.

Consider the effect of a simple change in R keeping r_1 and expectations on r_2 unchanged.

$$(28) \qquad \frac{da}{d(1+R)} \sim -E\left\{ U_2' + \rho \frac{d \ln C_2}{d \ln W_1} \frac{(1-a)W_0 U_2'[(1+r_1)(1+r_2) - (1+R)]}{W_1(1+r_2)} \right\}$$

Since $dW_1(1+r_2)/dr_2 > 0$, it is clear that the second term may be (e.g., if ρ is constant) negative, while the first term is positive, so the net result is ambiguous. However, if the variance of r_2 is sufficiently small, the sign of the above derivative is negative if before the change in R hypothesis A was satisfied (again assuming homotheticity and finite ρ)

$$\frac{da}{d(1+R)} \approx -EU_2' + \frac{(1-a)(1+r_1)}{(1+R)^2} (U_2'\rho)_{r=\bar{r}} E(r - \bar{r})^2 \approx EU_2' \quad .$$

In particular, if the utility function has constant elasticity and expectations are inelastic, to change a , the proportion of assets held in short term bonds, the term structure <u>must</u> change.

Part III

So far, we have considered a two period model, in which at the end of the first period, the individual could only buy a safe one period bond. Now, we assume that each period the individual can choose between a one and two period bond. The analysis of the individual's behavior under these circumstances requires first an analysis of consumer savings behavior under uncertainty.

8. Savings under Uncertainty: A Digression

We consider an individual with given wealth, W_1. He wishes to maximize his expected utility:

$$EU(C_1, C_2) = U((1-s)W_1, sW_1)(a(1+\tilde{r}) + (1-a)(1+r))$$

where $\tilde{r}$ is the (random) rate of return on the (from the single period point of view) risky asset, and r is the return on the safe asset. Both a and r must be chosen optimally. The first order conditions are

$$EU_1 = EU_2\{a(1+\tilde{r}) + (1-a)(1+r)\}$$

$$EU_2((1+\tilde{r}) - (1+r)) = 0$$

(a) Comparison of savings rate under certainty and uncertainty.

There are at least two stories of how uncertainty affects savings: (a) a risk averse individual, in order to ensure his "minimum standard of living" saves more in the face of uncertainty; (b) a risk averse individual is discouraged from saving for the future by the uncertainty of the return. The former story seems to be perferred by those who have studied consumption behavior, to help them explain the seemingly higher rate of savings of groups facing greater uncertainty [3 , 4]. But as the following analysis suggests, it is by no means clear that this will in general be the case. It is not obvious what is the most meaningful way to compare certain and uncertain situations (see [2]). For our purposes, we compare the savings rate with (for simplicity) a simple risky asset (with mean $\bar{r}$) with one where there is only a safe asset, with return equal to $\bar{r}$. Then it should be clear that the savings rate under uncertainty, s , will be greater or less than that under certainty, s^* as (assuming the utility function is additive),

$$U_2'(s^*W_1(1+\bar{r}))(1+\bar{r}) \lesseqgtr EU_2'(s^*W_1(1+\tilde{r}))(1+\tilde{r})$$

i.e., as $U_2(1+r)$ is a convex (concave) function of r . But

$$\text{(29)} \qquad \frac{dU_2'(1+r)}{d(1+r)} = U_2(1+ \frac{U_2''}{U_2'} C_2) = U_2(1 - \rho(C_2))$$

so

$$\text{(30)} \qquad \frac{d^2U_2'(1+r)}{d(1+r)^2} = (U_2'' C_2/1+r)(1 - \rho(C_2)) - U_2'\rho'(C_2) .$$

Hence, $s > s^*$, if relative risk aversion is greater than one and there is decreasing or constant relative risk aversion; while $s < s^*$ if relative risk aversion is less than one and there is increasing or constant relative risk aversion. Note if ρ is constant at unity, savings is unaffected by uncertainty. (See [5])

(b) The effect of an increase in "uncertainty" may be similarly analyzed. It is easy to show that if $U_2(1+r)$ is a concave (convex) function of r, and if we increase the variance by a spread of the distribution (see above pp. 16-19), then savings are reduced (increased).[1]

(c) Effects of increased wealth on average propensity to save. First, we must consider how s changes with W in this model in the absence of uncertainty. The first order condition is then simply $U_1' - U_2'(1+r)(1-\delta) = 0$. This defines an implicit relation between s and W, which yields

$$\frac{ds}{dW} = -\rho(C_1) + \rho(C_2)$$

But since $C_1 \gtrless C_2$ as $(1+r)(1-\delta) \gtrless 1$, we obtain the result that

$$\frac{ds}{dW} \sim \rho'[(1+r)(1-\delta) - 1)] \tag{31}$$

If there is constant relative risk aversion or $(1+r)(1-\delta) = 1$,

---

[1] As M. Rothschild has argued, an increase in uncertainty (with fixed mean) of a distribution means a lowering of expected utility for any risk averse individual, i.e., the expectation of any concave moment of r is reduced.

then the average propensity to save does not change with wealth; if, however, there is increasing relative risk aversion and $C_2 > C_1$, or decreasing relative risk aversion and $C_2 < C_1$, then the average propensity to save will be rising.

In the case of uncertainty,

$$\frac{ds}{dW} \sim -U_1' \rho(C_1) + E\rho(C_2)[U_2'(1+r)(1-\delta)]$$

Provided $(1+r)(1-\delta) \neq 1$, for sufficiently small variance this can be approximated by the first terms in a Taylor series expansion, so that (31) still is true.

9. Towards A More General Model

We now assume that at the end of the first period the individual can buy either a one period bond, which yields a return r_2 , or a two period bond, at a price of $1/1+R_2$, which yields, at the end of the second period a return of $(1+R_2)/(1+r_3) - 1$, where r_3 is the one period bond rate for the third period. In order to make his initial allocation decision, he now must form expectations not only on r_2 , but also on R_2 , and on r_3 (conditional on r_2).
This is, clearly, a considerably more complicated problem.

The individual wishes to maximize

$$(32) \qquad E\left\{E\left\{U\left[(1-s)W_1,\ sW_1\left(\hat{a}(1+r_2) + (1-\hat{a})\frac{1+R_2}{1+r_3}\right)\right]\right\}\right\}$$

where the inside expectation is over all possible values of r_3 given (r_2, R_2) and the outside expectation is over all sets of (r_2, R_2) ; where

$$W_1 = W_0(a(1+r_1) + (1-a)(1+R_1)/1+r_2) ;$$

and where $\hat{a}$ is the allocation in the second period, a in the first, between one and two period bonds. Straightforward differentiation with respect to a , using the first order conditions for $\hat{a}$ and s , yields

$$\text{(33)} \qquad E[\{EU_2\}\{(1+r_2)(1+r_1) - (1+R_1)\}] = 0$$

which is identical to the corresponding earlier result.

First we analyze the patterns of specialization in the second period of his life. We show that in the second period of his life, under hypothesis A, he never specializes in short term bonds. (Again, we assume additive utility functions.)

Letting $\hat{a} = 1$, we observe that

$$EU_2'(sW_1(1+r_2))(1+r_2 - (1+R_2/1+r_3)) = U_2'E(1+r_2 - (1+R_2/1+r_3)) < 0$$

by Jensen's inequality, if hypothesis A is satisfied. Note that if hypothesis B is satisfied, $\hat{a}$ is always equal to one.[1] Now, assume

---

[1]Since $\hat{a} = 1$, under hypothesis B, the analysis with two assets available the second period is identical to that of Parts I and II, and nothing more need be said about it here.

$\hat{a} = 0$. Then

$$EU_2'(sW_1(1+R_2/1+r_3))(1+r_2 - (1+R_2/1+r_3)) = E\,\frac{U_2'}{1+r_3}\,\{(1+r_2)(1+r_3) - (1+R_2)\}$$

But $\dfrac{dU_2'/(1+r_3)}{d(1+r_3)} = \dfrac{U_2'}{(1+r_3)^2}\{\rho - 1\}$ so if $\rho > 1$, he purchases both kinds of bonds, but if $\rho \leq 1$, he specializes in long bonds.

In order to procede with the analysis for the pattern of specialization with respect to a , further assumptions about the nature of the stochastic process describing short term rates of interest must be made. We will employ two assumptions. The first is that the distribution of r_t is independent of time. The second is that $E(1+r_t) = 1+r_{t-1}$ (i.e., the Martingale assumption). More generally, we shall assume, if r^* is the normal short term rate of return, and if $\tilde{\gamma}$ is a random variable with mean one, then

$$1+\tilde{r}_{t+1} = (\mu\gamma_t + (1-\mu))\tilde{\gamma}(1+r^*)$$

$$E(1+\tilde{r}_{t+1}) = (\mu\gamma_t + 1-\mu)(1+r^*) = \mu(1+r_t) + (1-\mu)(1+r^*)$$

The first case is the one for which $\mu = 0$, the second for which $\mu = 1$. The Keynesian hypothesis that when interest rates are low, they are more likely to increase (return to the normal level) is represented by a low value of μ .

Under hypothesis A, then, we have

$$1+R_t = (1+r_t)E(1+r_{t+1}) = [\mu(1+r_t) + (1-\mu)(1+r^*)](1+r_t)$$

and

$$\frac{1+R_t}{1+\tilde{r}_{t+1}} = \frac{1+r_t}{\tilde{\gamma}}$$

which does not depend on μ .

We now observe that if $a = 0$

(34)
$$C_2 = sW_0\ (1+\bar{r}_2)(1+r_1)(\hat{a} + \frac{E(1+r_3)}{1+r_3}(1-\hat{a}))$$
$$C_1 = (1-s)W_0\ \frac{E(1+r_2)}{1+r_2}(1+r_1)$$

Substituting this into the first order conditions for expected utility maximization, we obtain the result that

(35)
$$\frac{dEU_2'}{d(1+r_2)} \sim -(1-\rho)\left\{E\ \frac{U_2''C_2}{s}\ E\left[U_2''\ (1+\bar{r}_2)(1+r_1)\left(1-\frac{E(1+r_3)}{1+r_3}\right)^2\right]\right\}$$
$$-\frac{EU_2''C_2}{s}\left(1-\frac{E(1+r_3)}{1+r_3}\right)E\left[U_2''\ (1+\bar{r}_2)(1+r_1)\left(1-\frac{E(1+r_3)}{1+r_3}\right)\right]$$

If there is constant relative or absolute risk aversion, (35) reduces simply to

$$\frac{dEU_2'}{d(1+r_2)} \sim -(1-\rho)$$

so that the results obtained earlier still obtain. Indeed, by using (34), it is easy to show that the term in the bracket in (35) is always positive.

Similarly, if we set $a = 1$, we observe that

$$\frac{dEU_2'}{d(1+r_2)} \sim -\left(EU_2' + \frac{EU_2''C_2}{1+r_2}\right)\left\{\frac{EU_2''C_2}{s}(1+r_2)EU_2''sW_0(1+r_2)(1+r_1)\left(1 - \frac{E(1+r_3)}{1+r_3}\right)^2\right.$$
$$\left. - \frac{EU_2''C_2}{s}\left(1 - \frac{E(1+r_3)}{1+r_3}\right)EU_2''sW_0(1+r_2)(1+r_1)\left(1 - \frac{E(1+r_3)}{1+r_3}\right)\right\} < 0$$

Hence, as before,

$$E\left\{EU_2'[(1+r_2) - E(1+r_2)]\right\} < 0$$

so the individual does not specialize in short term bonds.

The comparative statics analysis procedes in much the same manner. We observe that if there is constant relative risk aversion, then, under our stochastic assumptions,

$$\frac{da}{dr_1} = \frac{da}{d\mu} = \frac{da}{dW_0} = 0$$

so that neither the degree of inelasticity of expectations, the level of short run interest rate, nor the level of wealth affect the allocation between the two kinds of assets.

It should also be noted that this model has provided us with a simple example showing the inapplicability of the separation theorem to dynamic problems. Even if the individual had a constant elasticity, additive utility function (the separation theorem holds then in a single period problem) individuals in different years of their lives would hold different portfolios. Moreover, the portfolio allocation cannot be done myopically [15]: expectations about r_3 and R_2 affect the allocation of the first period.

10. Concluding Comments

This paper has attempted to present a theory of liquidity preference based on consumption valuations rather than capital valuations. The limitations of the analysis are clear. We have already drawn attention to a number of special assumptions. Perhaps most striking is the absence of a supply side to the analysis: what determines the supply of bonds offered by, for instance, corporations. This will need to be the subject of a separate paper, and until such a study is completed, only limited statements about the equilibrium in the monetary markets can be made. Subject to these qualifications, the major conclusions of this study may be summarized as follows:

(1) This model provides an explanation of why individuals hold short term assets even when they could get, on average, a higher return from holding long term bonds. But it also provides an explanation of why individuals hold long term bonds even when they could get,

on average, a higher return from holding short term bonds.

(2) It is not correct to treat the long term bond as a risky asset and the short term bond as a safe asset. Since, from different horizons each are safe and from different horizons each are risky, at times each may act "more" like a safe asset than the other. For instance, under some expectations hypotheses, individuals which are not very risk averse may specialize (speculate) in long term bonds; under others they may specialize in short term bonds; while in both situations very risk averse individuals will buy some of both.

(3) If it is observed, as seems likely, that few individuals specialize in long term bonds, then $1+R < (1+r_1)E(1+r_2)$: instead of normal backwardation of interest rates, long term rates are smaller than the product of the expected short term rates.

(4) If, as also seems likely, there are relatively few individuals specializing in short term bonds,[1] then $(1+R) > (1+r_1)\cancel{E}(E'/1+r_2)$: long term bonds have a somewhat higher one period return than short term bonds.

(5) In depression situations because of increasing uncertainty the demand curve for short term assets shifts to the right, so that it may appear as if there is a liquidity trap, even if there is none.

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[1] One ought probably to treat permanent life insurance policies, annuities, etc., at least partially as ownership of long bonds. It is hard to obtain a clear test of the segmentation hypothesis for individuals; transaction costs, special taxation provisions, etc. obfuscate the matter further. See [8a].

(6) Inelasticity of expectations does not seem to make any important qualitative difference to the behavior of the individual, at least for the simple parameterizations investigated. Moreover, when short run interest rates change _and_ prices of long term bonds adjust so that, for instance, the long rate is still equal to the product of the expected short rates, the proportion of assets in short term bonds may either increase or decrease; in the case of constant relative risk aversion, nothing changes.

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