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PART IV: MATHEMATICAL STRUCTURE AND ANALYSIS OF THE NONSYMMETRIC GAME

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PART IV: MATHEMATICAL STRUCTURE AND ANALYSIS OF THE NONSYMMETRIC GAME *

by

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1. Introduction

In three previous papers [1/] we have discussed and analyzed a game describing a market with an oligopolistic structure which can be used to provide the simulated environment for an oligopoly or business game to be employed for teaching and experimental research purposes.

The model programmed and analyzed was symmetric. The full meaning of symmetry will be discussed below in Section 3. The main reasons for the use of symmetry are that when using the model for gaming it is possible to start each team from the same initial position and with the same advantages; even more important is the considerable simplification in the mathematical analysis that can be obtained.

It has been suggested in Part II that three solution concepts serve as useful benchmarks in any attempt to measure competition in an oligopolistic market. They are: (1) the joint maximum solution, (2) the noncooperative equilibrium, and (3) the "beat the average" solution.

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The joint maximum solution gives us a measure of how "fat" the market is. Suppose that there were no anti-trust laws and that there would be no public outcry and political action resulting from collusive behavior; what is the monopolistic profit that could be obtained from say the automobile market if all the manufacturers colluded?

The noncooperative equilibrium presupposes that each firm concentrates on its pricing, advertising and other policies as though the actions of the others were given. Each firm is an "inner-directed" maximizer. It is neither explicitly cooperative with or hostile to the other firms. This type of solution has been at the basis of much of the writings in oligopoly theory from Cournot [2/] to Chamberlin [3/].

A solution concept which (at least in its two-person applications) belongs more to military than to economic affairs serves to give us a measure of hostility. This is known as the "beat the average" solution. Each player is expected to be more concerned with the difference between his payoff and the average payoff than he is with the actual size of his own profits.

Given the structure of the payoffs in the symmetric market used in the game it is possible to obtain explicit closed form mathematical expressions for the "solution values" of the variables for all three solutions in terms of the market parameters. This gives the user of the game considerable control inasmuch as he can immediately see the effect upon the structure of the overall market of the change in any parameter. It is possible to adjust the cutthroat and cooperative aspects of the market structure with regard to price and advertising by adjusting very few parameters. The symmetry of the model makes the market particularly easy to describe and the form of the Pareto optimal surface will be as is shown in Figure 1.

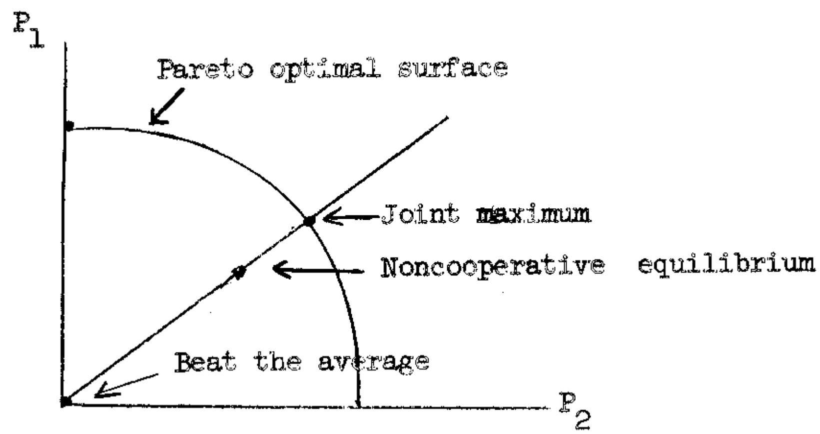

FIGURE 1

This figure is drawn for only two firms, but much the same type of surface would be relevant for any number of dimensions. The three solutions are noted on the diagram. Due to the symmetry they lie on a straight line. If the parameters of the market structure are varied the location and curvature of the optimal surface will shift as will the actual and relative positions of the three solutions; although the symmetry will still keep them on the same straight line.

More formally than we have done above, we can state the properties of the optimal surface and the three solutions. The three solutions can furthermore be regarded as special cases of a general socio-psychological view of competitiveness.

Consider N firms, each with a set of strategies S_i, where $i = 1, 2, \ldots, N$. A specific strategy belonging to the i^{th} firm is denoted by s_i. This strategy in the case of the game may be a complicated affair involving several components such as setting the price, advertising budget, level of production, dividend rate and so forth. We assume that each firm i has a payoff function which we denote by:

$$P_i(s_1, s_2, \ldots, s_N) .$$

Assuming differentiability of the payoff functions in the relevant ranges then the condition that must be satisfied by the <u>Pareto optimal surface</u> is:

$$\begin{vmatrix} \frac{\partial P_1}{\partial s_1} & \frac{\partial P_1}{\partial s_2} & \cdots & \frac{\partial P_1}{\partial s_N} \\ \vdots & & & \vdots \\ \frac{\partial P_N}{\partial s_1} & & & \frac{\partial P_N}{\partial s_N} \end{vmatrix} = 0 .$$

The condition for joint maximum is: $\max_{s_1} . \max_{s_2} . \ldots \max_{s_N} (P_1+P_2+\ldots+P_N)$, or with differentiability:

$$\frac{\partial[P_1 + P_2 + \ldots + P_N]}{\partial s_i} = 0 \quad \text{for} \quad i = 1, 2, \ldots, N .$$

The condition for the existence of a noncooperative equilibrium is that the following set of equations should be satisfied simultaneously,

$$\frac{\partial P_i}{\partial s_i} = 0 \quad \text{for} \quad i = 1, 2, \ldots, N .$$

The condition for the "beat the average" solution is that the following set of equations should be satisfied simultaneously,

$$\frac{\partial[P_i - \frac{1}{N-1} \sum_{j \neq i} P_j]}{\partial s_i} = 0 \quad \text{where } j = 1, 2, \ldots i-1, i+1, \ldots, N \text{ and } i = 1, 2, \ldots, N .$$

A different way in which the three solutions can be viewed is obtained if we introduce sociological considerations directly into the payoff functions. This can be done by considering that beyond the objective payoff functions P_i each player has a social payoff function Π_i and it is this that he is really trying to maximize. Suppose that the social payoff function can be obtained as a weighted linear combination of the objective payoffs of all players:

$$\Pi_i = \sum_{j=1}^{n} \theta_{ij} P_j \quad \text{for } i = 1, 2, \ldots, N$$

where the θ_{ij} are a set of N^2 parameters which measure the degree of concern that Player i has for the objective payoff to Player j. We consider as our metasolution a noncooperative equilibrium point in the game with the social payoff functions. This must satisfy

$$\frac{\partial \Pi_i}{\partial s_i} = \frac{\partial[\theta_{i1}P_1 + \theta_{i2}P_2 + \ldots + \theta_{iN}P_N]}{\partial s_i} = 0 \quad \text{for } i = 1, 2, \ldots, N .$$

We can note immediately that there are three settings for the parameters which are of considerable interest. When $\theta_{ij} = 1$ for all i and j this gives the <u>joint maximum.</u> When $\theta_{ij} = 1$ for $i = j$ and $\theta_{ij} = 0$ for $i \neq j$ this gives the <u>noncooperative</u> <u>equilibrium</u>, and when $\theta_{ij} = 1$ for $i = j$ and $\theta_{ij} = -\frac{1}{N-1}$ for $i \neq j$ this gives the <u>beat the average</u> solution.

For many experimental purposes and for the theoretical investigation of a considerable number of problems in competition, the symmetric game is both adequate and extremely convenient. However eventually, especially with the availability of computers it would be desirable to be able to experiment with games which can be regarded as providing a relatively good simulation of actual markets. Little work has been done on the micro-econometrics of market structure. The analysis presented here is done in an attempt to advance our knowledge of market structure. The assumption of symmetry is certainly useful as a crude first approximation which enables us to at least examine the implications of some of our theories; however if we wish to construct models of markets that are reasonably faithful in reflecting that strategic structure of the markets it becomes necessary to abandon symmetry. In doing so we complicate the analysis considerably; but hopefully, the added flexibility of the model more than compensates for the increase in complexity.

2. The basic model: the symmetric case

Although this model has been presented in Part II, we present a brief discussion of it once more in order to make this paper totally self-contained. Added background can be obtained from reading the previous papers but for the results presented it is not necessary to do so.

Suppose that we wish to represent a symmetric market with N firms in competition each selling a single product and each able to influence the demand for its product not only by price but by advertising as well. The following model is one of the simplest that can be constructed:

2.1. The Functional Form

The listing and description of variables and parameters used together with limits and bounds on their values and the functional forms of demand in this game are presented below:

A compromise has to be made between the complexity and greater richness of the functional forms chosen and the feasibility of analyzing the resulting system.

n = number of firms

p_i = price charged by Firm i

a_i = advertising expenditure by Player i

$p = (p_1, p_2, \dots, p_n)$

$a = (a_1, a_2, \dots, a_n)$

$\bar{p} = \sum_{i=1}^{n} p_i/n$

The demand for the produ c t of the i^{th} firm is given by

$$F_i(p, a) = \frac{W}{n} [\alpha - \beta(p_i + \gamma(p_i - \bar{p}))] f_i(\theta, a)(1 + \eta \sqrt{\Sigma a_i(1 + \epsilon_i))} \ , \tag{1}$$

where

$$f_i(\theta, a) = \begin{cases} \theta + (1 - \theta) \dfrac{na_i(1 + \epsilon_i)}{\Sigma a_i(1 + \epsilon_i)} & \text{if } \Sigma a_i > 0 \\ 1 & \text{if } \Sigma a_i = 0 \ , \end{cases}$$

and

$$W = r^t(1 + \lambda \sin(\omega t + \nu) + \epsilon) \ ,$$

if all the functions are nonnegative. The case where some F_i's are negative has been discussed in detail elsewhere[4] and is not relevant to the investigation at this point.

A summary of the sizes and limits on the parameters is given below.

$1 \leq n$	the number of players (integers only)
$\alpha > 0$	the "size of market" (in general, very large)
$\beta \geq 0$	price sensitivity of overall demand
$\gamma \geq 0$	inter-firm price sensitivity coefficient
η	cooperative advertising coefficient (any real number, but in general, it is likely to be ≥ 0
θ	competitive advertising coefficient
r	growth rate of the economy (any real number, but in general, ≥ 0 and near 1)
$\omega \geq 0$	angular velocity of cycle
ν	phase angle of cycle at initial point
ξ	a random variable
λ	cycle amplitude parameter
ϵ_i	$i = 1, \ldots, n$ random variables affecting advertising and are restricted to being greater than -1

2.2. Discussion of Demand Functions

For ease of discussion, the demand conditions are considered under two special cases. The first is when all firms charge the same price and have the same advertising expenditure; in other words, when the market moves are symmetric.

The functional form $F_i(p, a)$ given in equation (1), consists of four major factors. The first represents the possible effects of trend, cycle, and random element coming from the economy at large; the second deals with the effects of price and price relationships in the market; the third and fourth account for respectively the competitive and overall institutional effects of advertising on firm and industry demand.

When a functional form becomes as lengthy as this one, it is desirable to examine it by considering special cases and simple examples. Referring back to equation (1), if we ignore trend, cycle, random effects, set advertising equal to zero, and assume that prices are equal for all competitors, all that remains is a simple linear demand equation of the form:

(3) $$d_i = \frac{1}{n} [\alpha - \beta p_i] .$$

Leaving conditions as described above except for the restriction on prices, we observe that a term involving the difference between the price charged by a player and the average market price is introduced.

$$\gamma(p_i - \bar{p}) .$$

In the above expression, γ is a parameter which indicates an inherent degree of substitutability between the products.

The parameter α controls the overall market size. For example, if the structure of this game were meant to represent the automobile market, α would have a value of several million.

The parameter β controls the sensitivity of overall demand to change in average price.

The term in the function $F_i(p, a)$ which is given by

$$A_1(a) = 1 + \eta \sqrt{\Sigma\, a_i(1 + \epsilon_i)} \tag{4}$$

controls the overall or institutional effect of advertising. The parameter η controls the effectiveness of the overall impact on industry demand. If η equals zero, then advertising has no effect whatsoever on the overall demand in this industry. A square root is introduced to act on the sum of advertising expenditures to produce the effect of diminishing returns.

Figures 1 and 2 show two ways for introducing the overall industry effect of advertising. In the model constructed here, the effect as indicated in Figure 2 has been used.

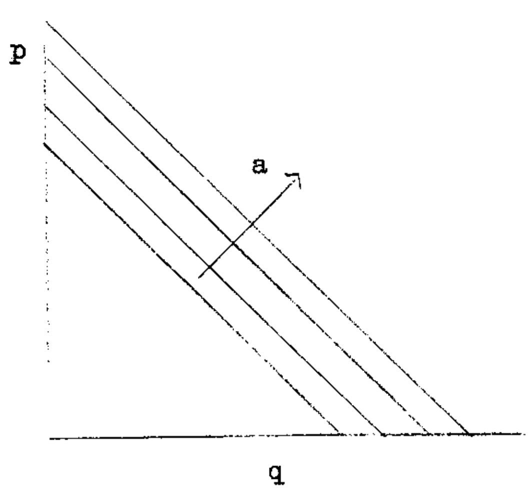

Figure 1

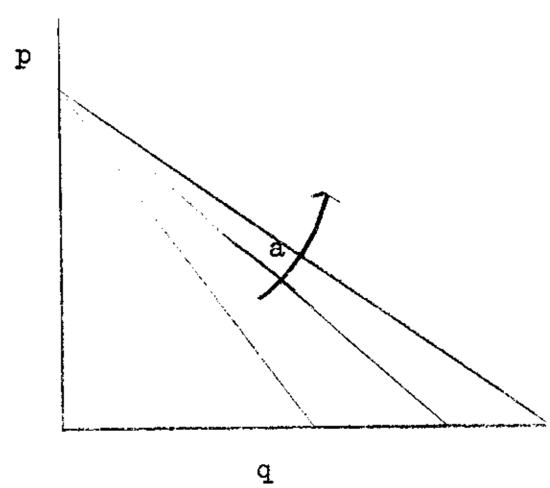

Figure 2

For the parallel shift effect, the terms involving advertising need to be applied only to the constant α in the demand function. As a case on empirical grounds can be made for either way of treating advertising, the second was chosen for ease of computation.

The term

$$A_2(a_i, a) = \theta + \frac{na_i(1 + \epsilon_i)(i - \theta)}{\Sigma\, a_i(1 + \epsilon_i)} \tag{5}$$

describes the competitive effect of advertising. The parameter θ, between zero and one, controls the competitive aspect of advertising. If it is zero, then the apportionment of the market depends completely upon advertising; if it is one, advertising has no competitive effect.

A random component $(1 + \epsilon_i)$ influences the effectiveness of each individual's advertising expenditure.

The term

$$r^t(1 + \lambda \sin(\omega t + \nu) + \epsilon) ,$$

is a simple expression for introducing the effect of a trend controlled by the parameter r and a cycle whose amplitude grows with the trend.

2.3. Costs and Revenues

2.3.1. Production Costs

The average variable costs of the firms are assumed to be constant and identical. Let the variable costs of production of q_i items be $C(q_i)$ then

(6) $$C(q_i) = cq_i$$

where c is the average cost of production.

There may be costs to changing the level of production, but these will not influence steady state solutions.

2.3.2. Advertising

Advertising costs appear merely as a single number representing the aggregated expenditures on advertising, promotion, public relations and so forth.

2.3.3. The Revenue Function

The following symbols are defined:

Π_i = before tax net revenue for the i^{th} firm,

σ_i = actual sales for the i^{th} firm,

K_i = fixed costs for the i^{th} firm,

k_i = the inventory carrying unit cost for the i^{th} firm.

The before tax profit of the firm is:

(7) $$\Pi_i = p_i\sigma_i - c_i q_i - k_i \Delta q_i - a_i - K_i ,$$

where:

Δq_i = the average inventory level during the period.

The revenue function has been written as though there were no change in production levels between periods. In a steady state we could write the revenue as:

$$\Pi_i = (p_i - c_i)q_i - \frac{k_i}{2} q_i - a_i - K_i \tag{8}$$

There are two types of symmetry to consider. They are in market interrelated conditions and in individual aspects of the firms such as costs and overheads. If the game is completely symmetric both in conditions in the market and within each firm then revenue can be written as:

$$\Pi_i = (p_i - c)q_i - \frac{k}{2} q_i - a_i - K \, . \tag{9}$$

2.4. Solutions

The steady state solutions according to the three different behavior hypotheses: joint maximization, noncooperative equilibrium* and beat-the-average, can be obtained directly by setting

$\frac{\Sigma\, \theta_{ij}\Pi_i}{\partial p_i} = \frac{\Sigma\, \theta_{ij}\Pi_i}{\partial a_i} = 0$ for $i = 1, 2, \ldots, n$ using the symmetry condition and solving the resulting equations for the appropriate values of θ_{ij}. Explicit expressions for p_i and a_i are obtained in each case as has been shown elsewhere 5/.

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* A difficulty is faced with the solution of the noncooperative price game inasmuch as it is possible that there is no pure strategy equilibrium point, but instead a range over which prices fluctuate. This is known as the Edgeworth cycle. The conditions for the existence of this phenomenon are discussed elsewhere 6/. Even if this is the case the payoffs obtained from the pure strategy "almost equilibrium point" will not be too far from the expected value of the mixed strategy solution.

These expressions are immediately useful in analyzing markets and in setting up conditions for the playing of a business game. In particular one can explore the sensitivity of the payoffs to changes in parameter values merely by differentiating the solutions with respect to the appropriate parameter. More important however, as is shown in Figure 1, it is possible to at least begin to consider structural measures of the degree of community and opposition of interests among the firms in a market.

2.5. Dynamics: A Disclaimer

In our analysis to date we have not coped with dynamic effects. Even in the relatively simple case of a constant growth in the size of market the analogues of the steady state equilibria need to be modified for the effects of costs in the change of production, interest rate and inventory charges. Furthermore it is reasonable to expect that the effect of advertising does not attenuate over one period hence as soon as any consideration is given to dynamic elements the analysis of advertising is considerably complicated. We nevertheless proceed with the steady state analysis in the belief that it serves as a reasonably good first approximation and is needed before dynamic models can be analyzed.

3. The Nonsymmetric Market Model

3.1. Types of Nonsymmetry

There are four major types of nonsymmetry which will be investigated. Each has a considerably different effect upon the steady state market model and its analysis. The direct market interaction between the firms is via price and advertising. There may be a difference in cross-elasticities of demand for the products of the firms. In terms of the model in Section 2 this means that the parameter $\beta\gamma$ must be replaced with a more general array. Similarly the effectiveness of the advertising of different firms may be different and furthermore the level of "consumer loyalty" may not be the same.

Internal to the individual firm there are two types of non-symmetry which must be considered when comparing firms. Various fixed terms and initial conditions may be different among firms. These can be overheads, capacity limitations and initial monetary and physical assets. Variable costs may also differ; thus there may be different production costs, inventory carrying costs and costs of changing production.

3.2. The Model and its Analysis

In order to help make the comparison between the symmetric and nonsymmetric models payoff functions for the i^{th} firm in each case are presented in equations (10) and (11). In each case here we have omitted the trend and cycle term W as well as the advertising term $(1 + \epsilon_i)$ and in the symmetric case we use the definition of $f_i(\theta, a)$ for the case when $\Sigma a_i > 0$. The more precise definition is given in equation (2).

$$(10) \qquad \Pi_i = (p_i - c - \frac{k}{2})(\alpha - \beta p_i + \gamma(p_i - \bar{p})(\theta + (1 - \theta)\frac{na_i}{\Sigma a_i})(1 + \eta\sqrt{\Sigma a_i}) - a_i - K.$$

$$(11) \qquad \Pi_i = (p_i - c_i - \frac{k_i}{2})[\beta w_i(V - p_i + \gamma(p_i - \bar{p}))]f_i(\theta, a_i) - a_i - K_i .$$

Comparing (10) and (11) we note that the c, k and K with subscripts stand for the unit costs of production, inventory carrying costs and the individual overheads. In the steady state, where production rates and inventory levels will be in constant proportion we can simplify both (10) and (11) by defining a new cost $c_i' = c_i + k_i/2$. In the subsequent calculations we use c_i to stand for the more complex c_i'.

In equation (11) the notation has been changed from that in (10) for convenience. Thus $V = \alpha/\beta$. This has an interpretation as the price at which demand is zero. The $\bar{p}$ is no longer the average price but $\bar{p} = \Sigma\, w_j p_j$ which is the weighted average price. The weights w_i are the means by which we introduce nonsymmetry into the demand structure as influenced by price. Nonsymmetry is introduced into the internal structure of the firms by the c_i and K_i. It is assumed that $\Sigma\, w_i = 1$; hence if all firms were to charge the same price the w_i would reflect the resultant asymmetry of market share.

Leaving aside the effect of advertising we note that the cross-derivative of the demand for firm i with respect to the price charged by firm j is given by:

$$\frac{\partial d_i}{\partial p_j} = \beta \gamma w_i w_j \ .$$

This is symmetric. No income effect is assumed.

We have used the same competitive advertising term in equation (11) as in (10). Possibly there is less justification in the nonsymmetric case than in the symmetric case for introducing the effect of advertising in a multiplicative manner. The main advantage (as has been noted in a previous paper [7]) is that the resulting equations are triangular in form. This in turn means that although the optimal advertising level depends upon price, the reverse is not true.

Before we discuss advertising further it should be stressed that one could conceivably give a different interpretation to the variable a_i. It is a variable which influences demand given price. This could be money spent on distribution, promotion or packaging or other activities which influence demand and may not be strictly described as advertising. Advertising budgets often conceal an aggregate of advertising, promotion expenditures and disguised rebates; hence the interpretation of the aggregate decision variable "advertising" may vary considerably from industry to industry.

The term $(1 + \eta \sqrt{\Sigma a_i})$ was introduced in equation (10) in order to reflect a cooperative or overall industrial effect of advertising. It seems unreasonable to attempt to introduce a lack of symmetry into this term. In our analysis of the nonsymmetric case we omit the cooperative term in this form.

The term $(\theta_i + (1-\theta) \frac{na_i}{\Sigma a_i})$ has two parts. The θ_i may be interpreted in terms of "consumer loyalty." It is that percent of customers who are not moved by advertising. The remaining term describes the type of competition for the part of the market that is influenced by advertising. The functional form was designed to conform partially to one of the folklore sayings in advertising that market share and percentage of the advertising budget should be the same.

Nonsymmetry can be introduced in several ways into this advertising term. One is by replacing θ by θ_i . This is tantamount to stating that the percentage of "loyal customers" varies among firms. A further nonsymmetry could be introduced by applying a weighting factor s_i where $\Sigma s_i = 1$ to the a_i . This would imply a different level of effectiveness for the advertising of the different firms. The justification for this appears to be hard to establish.

Given the general lack of knowledge about the nature and effectiveness of advertising it might be most reasonable to introduce it in a symmetric manner with a random term to simulate the possibility that one firm's campaign may on occasions have a higher level of effectiveness than that of another. If it were possible for all firms to measure and control the effects of advertising this would tend to make it a far less desirable weapon. It would be too powerful. This has the same paradoxical structure as does the possession of large effective nuclear weapons. If all sides have them and have a quick retaliation capability then the better they are, the less inclined all parties will be to use them. Possibly the greatest attraction that advertising and promotion have over price as a competitive weapon is that they are hardly understood, poorly controlled and the results of using them are subject to wide fluctuations, whereas price is more or less well understood, fairly well controlled and its use as a weapon by one side often brings swift retribution from another.

3.2.1. Nonsymmetry in Fixed Costs and Initial Conditions

It is evident that the steady state solutions are not dependent upon initial conditions. Although from the viewpoint of mathematical analysis this observation is both true and trivial, when we consider the model as an attempt to reflect some basic properties of an industry it is unreasonable to treat differences in initial conditions in so facile a manner. Although we do not propose to carry out a detailed analysis, at least we can indicate where the problems are and give a qualitative evaluation of the effect of different initial conditions.

(1) Capacity constraints: If the capacity constraints are sufficiently large for all firms, a lack of symmetry will have no effect. This would imply that all firms have excess capacity. If however some or all firms have relatively tight capacity constraints these will actually affect the solutions; cases of this have been studied in detail elsewhere [8]. Given tight capacity constraints one might wish to consider two possibilities in such a market. Either new firms would tend to enter or the existing firms would buy more capacity. If it were possible for new firms to enter the solutions would have to be modified to account for the increased number of competitors. If the firms were permitted to buy extra capacity then the final steady state solutions would be the same, however the transient state would be of considerable interest as it would reflect the appropriate investment policies to build up capacity.

(2) Initial Financial and Real Assets: In general markets are not static. The availability of surplus funds usually permits a firm to explore new markets and vary its product or invest in plant. In the model under consideration there are neither new markets nor products and processes. The cash position does however play three roles. When capacities are low, cash flow considerations will influence the optimal investment policy. When cash position is high and capacities are adequate, although the steady state is not influenced, the transient dividend policy is. When cash

positions are low, then possibilities of bankruptcy appear and the steady state solutions may be influenced by cutthroat competition [9].

Different levels of real assets influence the transient policies in production scheduling and may indirectly cause all of the cases noted for different levels of liquid asset holdings.

(3) General overheads: Differences in general overheads would be reflected in equation (10) by replacing K by K_i. This in general has no effect upon transient or steady state behavior except when overheads are so large that they cause cash flow, cutthroat competition and exit problems.

3.2.2. Nonsymmetry in Variable Costs

In this model the variable costs are: (1) unit cost of production; (2) the costs of change in production; (3) inventory carrying costs and (4) the rate of interest.

Production and inventory costs directly modify the period payoff functions as is shown immediately in equation (11). Differences in the costs of change in production do not change the steady state solutions. The interest rate has two indirect influences in its relation to inventory carrying costs and further in relation to depreciation costs. If the firms are permitted to buy extra capacity then the parameters K_i are not quite constants as they will depend upon capacity size and depreciation rates.

Nonsymmetry in the market both for price and advertising has already been discussed in 3.2.; we turn to the analysis of the model.

3.3. The Noncooperative Solution

Let the demand for the product of the i^{th} firm be given by:

$$d_i = w_i\beta(V - p_i - \gamma(p_i - \bar{p}))f(\theta, a_i) \tag{12}$$

The revenue of the i^{th} firm may be expressed as:

$$\Pi_i = (p_i - c_i)(w_i\beta(V - p_i - \gamma(p_i - \bar{p})))f(\theta, a_i) - a_i - K_i \tag{13}$$

For a noncooperative equilibrium we must solve the set of equations resulting from taking the derivatives of (13) with respect to p_i and setting them equal to zero.

$$\frac{\partial \Pi_i}{\partial p_i} = \beta w_i \left[- (2(1 + \gamma) - \gamma w_i)p_i + \gamma \Sigma w_j p_j + V + (1 + \gamma(1 - w_i))c_i \right] = 0 \,. \tag{14}$$

We note immediately that the advertising term factors out. All parameters and variables have been defined above in 3.2. We need to introduce extra notation in order to examine the solution of the equations (14) in matrix form.

Let $S = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$ a square matrix with 1 for every entry.

$\hat{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ a column matrix with 1 for each entry.

$W = \llcorner w_i \urcorner = \begin{bmatrix} w_1 & & & 0 \\ & w_2 & & \\ & & \ddots & \\ 0 & & & w_n \end{bmatrix}$ a diagonal matrix with entries w_i .

In equations (14) we divide by $\beta w_i \gamma$. For ease in notation set

$$\Delta = \frac{2(1+\gamma)}{\gamma}$$

The equations (14) can be written in matrix notation as:

(15) $$(\Delta I - W - SW)p = \frac{V}{\gamma}\hat{1} + (\frac{\Delta}{2} I - W)c$$

where

$$p = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}, \text{ and } c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} .$$

In order to solve (15) we wish to evaluate the inverse of $(\Delta I - W - SW)$, we write this as $[(\Delta W^{-1} - I) - S]W$. Let $\Delta W^{-1} - I = X$ and $[X - S]^{-1} = Y$. We may write δ_{ij} in terms of entries of X and Y as:

(16) $$\delta_{ij} = \sum_k (x_i \delta_{ik} - 1) y_{kj}$$

$$= x_i y_{ij} - \sum_k y_{kj}$$

call $R_j = \sum_k y_{kj}$. As Y is symmetric we can write

(17) $$y_{ij} = \frac{R_j + \delta_{ij}}{x_i} = \frac{R_i + \delta_{ji}}{x_j} .$$

If $i \neq j$ then $\frac{R_i}{x_j} = \frac{R_j}{x_i}$ hence

(18) $$R_i = \frac{R_i x_j}{x_i} .$$

But $R_j x_j$ is a constant which we may call Q and rewrite (18) as:

(19) $$R_i = \frac{Q}{x_i} \, .$$

Returning to (17) we have:

(20) $$y_{ij} = \frac{\frac{Q}{x_j}}{x_i} = \begin{cases} \dfrac{Q}{x_i x_j} & \text{for } i \neq j \\ \dfrac{Q}{x_i^2} + \dfrac{1}{x_i} & i = j \end{cases}$$

ball $Z = \begin{bmatrix} \frac{1}{x_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{x_n} \end{bmatrix}$, $z_i = 1/x_i$ then we can write:

(21) $$[X - S]^{-1} = QZSZ + Z$$

Evaluating Q :

$$\frac{Q}{x_i} = R_i = \sum_j y_{ij} = \sum_j \frac{Q}{x_i x_j} + \frac{1}{x_i}$$

hence

(22) $$Q = Q \sum_j \frac{1}{x_j} + 1 = \frac{1}{1 - \sum_j \frac{1}{x_j}} = \frac{1}{1 - \sum_j z_j} \, .$$

Now we have the inverse of $X - S$ hence the prices which can be written as:

(23) $$p = W^{-1}(QZSZ + Z) \left(\frac{V}{\gamma} \hat{1} + \left(\frac{\Delta}{2} I - W\right)C \right) \, .$$

Remembering that $z_i = \frac{w_i}{\Delta - w_i}$ and $\Sigma z_j = 1 - \frac{1}{Q}$ we may write a specific price p_i as:

$$(24) \quad p_i = \frac{z_i}{w_i} \left\{ \frac{V}{\gamma}(Q \Sigma z_j + 1) + \frac{1}{2} [(\Delta - 2w_i)c_i + Q \Sigma z_j(\Delta - 2w_j)c_j] \right\},$$

which simplifies to

$$(25) \quad p_i = \frac{1}{\Delta - w_i} \left\{ \frac{V}{\gamma} Q + \frac{1}{2} [(\Delta - 2w_i)c_i + Q \Sigma z_j(\Delta - 2w_j)c_j] \right\}.$$

There are two special cases and a check that should be made. Then when (1) all costs are the same, in other words $c_i = c$ and (2) market conditions are the same, in other words $w_i = w$. The check comes when we assume that both $c_i = c$ and $w_i = w$ and investigate the resultant simplification to see if it gives us the formula in the symmetric case.

3.3.1. A Special Case: Equal Costs

If we assume that $c_i = c$ but that the w_i are different equation (25) may be written as:

$$(26) \quad p_i = \frac{1}{\Delta - w_i} \left\{ \frac{V}{\gamma} Q + \frac{c}{2} [(\Delta - 2w_i) + Q \Sigma z_j(\Delta - 2w_j)] \right\}.$$

We wish to sum the term $\Sigma z_j(\Delta - 2w_j)$. This can be written as:

$$(27) \quad \Sigma z_j(\Delta - w_j) - \Sigma z_j w_j$$

or as

$$(28) \quad \Delta \Sigma z_j - 2 \Sigma z_j w_j .$$

However by setting $z_j = \frac{w_i}{\Delta - w_i}$ in the first part of (27) we have immediately that $\Sigma z_j(\Delta - w_j) = 1$. Furthermore as $Q = \frac{1}{1 - \Sigma z_j}$ then $\Sigma z_j = \frac{Q - 1}{Q}$ which may be substituted in the first part of (28). Letting (27) equal to (28) we have:

$$1 - \Sigma z_j w_j = \Delta \left(\frac{Q - 1}{Q}\right) - 2 \Sigma z_j w_j \qquad \text{or} \tag{29}$$

$$\Sigma z_j w_j = \Delta \left(\frac{Q - 1}{Q}\right) - 1 \qquad \text{hence} \tag{30}$$

$$\Sigma z_j(\Delta - 2w_j) = 2 - \Delta \left(\frac{Q - 1}{Q}\right) . \tag{31}$$

Substituting (31) into (26) we obtain:

$$\begin{aligned} p_i &= \frac{1}{\Delta - w_i} \left\{ \frac{VQ}{\gamma} + \frac{c}{2}(\Delta - 2w_i + 2Q - \Delta(Q - 1) \right\} \\ &= \frac{1}{\Delta - w_i} \left\{ \frac{V}{\gamma} Q + \frac{c}{2}(2 \Delta - 2w_i + Q(2 - \Delta) \right\} . \end{aligned} \tag{32}$$

This can be rewritten in a better form as:

$$\boxed{p_i = c + \frac{Q}{\Delta - w_i} \left(\frac{V - c}{\gamma}\right)} \tag{33}$$

3.3.2. <u>A Special Case: A Symmetric Market but Unequal Costs</u>

Let $w_i = w = \frac{1}{n}$ then $z_i = \frac{\frac{1}{n}}{\Delta - \frac{1}{n}}$ and $Q = \frac{n \Delta - 1}{n \Delta - 1 - n}$.

We may write (25) as:

$$\begin{aligned} p_i &= \frac{n}{n \Delta - 1} \left\{ \frac{V}{\gamma} Q + \frac{1}{2} \left[\left(\frac{n \Delta - 2}{n}\right) c_i + Q \Sigma \left(\frac{1}{n \Delta - 1}\right)\left(\frac{n \Delta - 2}{n}\right) c_j \right] \right\} \\ &= \frac{nV}{(n \Delta - 1 - n)\gamma} + \left(\frac{n}{n \Delta - 1}\right)\left(\frac{n \Delta - 2}{2n}\right) \left[c_i + \frac{1}{n \Delta - 1 - n} \Sigma c_j \right] . \end{aligned} \tag{34}$$

Substituting in $\Delta = \frac{2(1+\gamma)}{\gamma}$ we obtain:

$$(35) \qquad \boxed{p_i = \frac{V}{2 + (\frac{n-1}{n})\gamma} + \frac{1 + \gamma(\frac{n-1}{n})}{2 + (\frac{2n-1}{n})\gamma} \left[c_i + \frac{\gamma}{2 + \frac{n-1}{n}\gamma} \bar{c} \right]}$$

3.3.3. <u>The Symmetric Case: Two Checks</u>

In (33) we use the added condition of $w_i = w = \frac{1}{n}$. This gives $z_i = \frac{1}{n\Delta - 1}$ hence $Q = \frac{n\Delta - 1}{n\Delta - 1 - n}$. We obtain:

$$(36) \qquad p_i = c + \frac{n\Delta - 1}{n\Delta - 1 - n} \left(\frac{n}{n\Delta - 1}\right)\left(\frac{V - c}{\gamma}\right)$$

$$= c + \frac{1}{2\frac{(1+\gamma)}{\gamma} - \frac{1}{n} - 1} \left(\frac{V - c}{\gamma}\right).$$ This simplifies to:

$$(37) \qquad \boxed{p_i = \frac{V + c(1 + \frac{n-1}{n}\gamma)}{2 + \frac{n-1}{n}\gamma}}.$$

This is the symmetric solution as has been shown previously.[10]

In (35) we use the added condition that $c_i = c$. This immediately gives

$$(38) \qquad p_i = \frac{V}{2 + (\frac{n-1}{n})\gamma} + \frac{1 + \gamma(\frac{n-1}{n})}{2 + (\frac{2n-1}{n})\gamma} \left[\frac{c(2 + \frac{n-1}{n}\gamma + \gamma)}{2 + \frac{n-1}{n}\gamma} \right]$$

$$= \frac{V + c(1 + \frac{n-1}{n}\gamma)}{2 + \frac{n-1}{n}\gamma}.$$

3.3.4. The Noncooperative Solution: Advertising

The simplest and most parsimonious manner in which we can introduce nonsymmetric effects into competitive advertising is to use the form:

$$\theta v_i + (1 - \theta) \frac{a_i}{\Sigma a_j}, \qquad \sum_{j=1}^{n} v_j = 1 \tag{39}$$

in which the v_i serve to make the market share of "loyal customers" who are not affected by advertising different for each firm. This enables us to write the payoffs as

$$\pi_i = k_i(p_i)\ (\theta v_i + (1 - \theta)\frac{a_i}{\Sigma a_j}) - a_i \tag{40}$$

where $k_i(p_i)$ is the term of the demand involving price.

Taking derivatives and calling $k_i(p_i)\ (1 - \theta) = \varphi_i$ we obtain

$$\frac{\partial \pi_i}{\partial a} = \varphi_i \left\{ \frac{\Sigma a_j - a_i}{(\Sigma a_j)^2} \right\} - 1 \tag{41}$$

Setting these equal to zero we obtain

$$\varphi_i(\Sigma a_j - a_i) = (\Sigma a_j)^2 \tag{42}$$

In matrix notation we may write (42) as:

$$\varphi(S - I)\ a = \Sigma^2 \hat{1} \tag{43}$$

where $\varphi = \begin{bmatrix} \varphi_1 & & & \\ & \varphi_2 & & \\ & & \ddots & \\ & & & \varphi_n \end{bmatrix}, \quad a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad \Sigma = \Sigma a_j$

hence

$$(44) \qquad a = \Sigma^2 (S - I)^{-1} \varphi^{-1} \hat{1}$$

Consider $\qquad (S - I)(tS - I) = I \qquad$ or $\qquad tS^2 - tS - S + I = I$

which gives $\qquad ntS - tS - S = 0 \qquad$ or $\qquad t = \frac{1}{n-1} \qquad$ hence

$$(S - I)^{-1} = \frac{1}{n-1} S - I .$$

Rewriting (44) we have

$$(45) \qquad a = \Sigma^2 \left(\frac{1}{n-1} S - I\right)\varphi^{-1} 1$$

hence

$$(46) \qquad a_i = (\Sigma a_j)^2 \left(\frac{1}{n-1} \Sigma \frac{1}{\varphi_j} - \frac{1}{\varphi_i}\right) .$$

Summing over all i we may write

$$(47) \qquad \Sigma a_j = (\Sigma a_j)^2 \left(\frac{n}{n-1} \Sigma \frac{1}{\varphi_j} - \Sigma \frac{1}{\varphi_j}\right)$$

$$= (\Sigma a_j)^2 \frac{1}{n-1} \Sigma \frac{1}{\varphi_j}$$

which has the solutions

$$(48) \qquad \Sigma a_j = 0 \qquad \text{or} \qquad \Sigma a_j = \frac{n-1}{\Sigma \frac{1}{\varphi_j}}$$

If a firm advertises

$$(49) \qquad \boxed{a_i = \frac{n-1}{\Sigma \frac{1}{\varphi_j}} - \frac{1}{\varphi_i} \left(\frac{n-1}{\Sigma \frac{1}{\varphi_j}}\right)^2}$$

This solution might yield a negative advertising budget unless

(50) $$1 - \frac{1}{\varphi_i}\left(\frac{n-1}{\Sigma \frac{1}{\varphi_j}}\right) \geq 0$$

or $$\varphi_i \geq \frac{n-1}{\Sigma(\frac{1}{\varphi_j})} \qquad \text{or} \quad \frac{n-2}{\varphi_i} \leq \sum_{j\neq i} \frac{1}{\varphi_j}$$

or

(51) $$\boxed{\varphi_i \geq \frac{n-2}{\sum_{j\neq i} \frac{1}{\varphi_j}}}$$

For purposes of checking suppose all $\varphi_j = \varphi$. From (51) we obtain

$$\varphi \geq \frac{n-2}{n-1} \frac{1}{(\frac{1}{\varphi})}$$

which is true but indicates that there is not too much leeway in the picking of φ_i . Firms with small φ_i will not advertise.

Let $\frac{1}{\varphi_i} = \psi_i$ and consider without loss of generality that the φ_i have been ordered so that $\varphi_1 < \varphi_2 \ldots < \varphi_n$ hence $\psi_1 > \psi_2 > \ldots > \psi_n$.

We must establish that there will be a unique solution with a set of m of the "biggest firms" (i.e., the first m with the largest ψ_i) active. This would give a final general solution of:

(52) $$\boxed{\begin{aligned} a_i &= \frac{m-1}{m\,\bar{\psi}_m} - \psi_i \left(\frac{m-1}{m\,\bar{\psi}_m}\right)^2 && \text{for } i \leq m \\ &= 0 && \text{for } i > m \end{aligned}} \qquad \text{where } \bar{\psi}_m = \sum_{i-1}^{m} \frac{\psi_i}{m}$$

Lemma: There is only one set $(\psi_1, \psi_2, \ldots \psi_m)$ for which the solution (52) holds.

From (52) we may write two inequalities

$$(53) \qquad \left\{ \frac{m}{m-1} \bar{\psi} \right\} a_i = 1 - \psi_i \left(\frac{m-1}{m} \bar{\psi}_m \right) \geq 0 \quad \text{for } i \leq m$$

$$1 - \psi_i \left(\frac{m-1}{m} \bar{\psi}_m \right) < 0 \quad \text{for } i > m$$

or

$$(54) \qquad \psi_i \leq \frac{m}{m-1} \bar{\psi}_m \qquad i \leq m$$

$$> \frac{m}{m-1} \bar{\psi}_m \qquad i > m \quad .$$

Suppose that (54) was satisfied not only by m but also by $m_2 = m + \Delta$. We have

$$(55) \qquad \psi_i \leq \frac{m + \Delta}{m + \Delta - 1} \bar{\psi}_{m+\Delta} \qquad i \leq m + \Delta$$

$$> \frac{m + \Delta}{m + \Delta - 1} \bar{\psi}_{m+\Delta} \qquad i > m + \Delta \quad .$$

This is impossible as is shown immediately. Let $i = m + \Delta$ then from (55) we have:

$$(56) \qquad \psi_{m+\Delta} \leq \frac{m + \Delta}{m + \Delta - 1} \bar{\psi}_{m+\Delta} \quad .$$

From (54) we have:

$$(57) \qquad \psi_{m+1} > \frac{m}{m-1} \bar{\psi}_m \quad .$$

We may write (58) $\frac{m+\Delta}{m+\Delta-1}\,\bar{\psi}_{m+\Delta} = \frac{1}{m+\Delta-1}\sum_{i=1}^{m+\Delta}\psi_i = \frac{1}{m+\Delta-1}\left(\sum_{m+1}^{m+\Delta}\psi_i + m\bar{\psi}_m\right)$

The following inequalities must hold.

$$\underbrace{\psi_{m+1} \leq \psi_{m+\Delta} \leq}_{\text{definition (56) and (58)}} \frac{1}{m+\Delta-1}\left\{\sum_{m+1}^{m+\Delta}\psi_i + \bar{m}\psi_m\right\} < \frac{1}{m+\Delta-1}\underbrace{\left\{\Delta\psi_{m+1} + (\frac{m}{m}-1)\bar{\psi}_{m+1}\right\}}_{\text{definition (57)}}$$

$= \psi_{m+1}$ which is a contradiction.

In the symmetric case all firms will be active hence we have, from (52):

(58)
$$a_i = \frac{n-1}{n\,\psi} - \psi\left(\frac{n-1}{n\,\psi}\right)^2$$

$$= \frac{n-1}{n}\,\varphi - \frac{1}{\varphi}\left(\frac{n-1}{n}\,\varphi\right)^2$$

$$= \frac{n-1}{n^2}\,\varphi = k(p)(1-\theta)\,\frac{(n-1)}{n^2}$$

where $k(p)$ is the effect of price.

4. A Simulation of an Automobile Market

If the automobile industry were in noncooperative equilibrium we could use the noncooperative solution together with industry information to calculate parameter estimates for the automobile industry. A very crude set of parameters estimates can be obtained from the 1965 figures for the three major automobile companies. The calculations given are merely meant to be suggestive of an approach and not a careful econometric estimate of the automobile industry. The game constructed will be somewhat like the industry.

Owing to the lack of unconsolidated figures, several approximations and simplifications will be made. In particular we consider only those firms and their world wide competition. We lump all automotive units such as autos, trucks and tractors. We know that civilian nonauto products and defense accounted for $1.9 billion for General Motors or approximately 10% of sales. Rather than break out the multiproduct features explicitly we implicitly include them by inflating the price of an automotive unit so that we make the crude approximation that there is a constant ratio in multiproduct sales. Furthermore the distribution system is not accounted for explicitly. The firms obtain wholesale prices but the cars are sold at retail.

The first table gives sales (not corrected for total income which is slightly different), total assets and before tax profits.

	Sales ($x10^6$)	Assets ($x10^6$)	Profits ($x10^6$)
General Motors	20,734	11,479	2,126
Ford	11,537	7,596	710
Chrysler	5,300	2,934	233

Table 1

Variable costs are assumed to correspond to the item "costs of products sold" on the earnings statements in the annual reports. Depreciation, amortization, administrative expenses, debt servicing and various pension and retirement payments are assumed to define fixed costs. Taxes are reported separately, they include foreign taxes.

	Variable Costs ($x10^6$)	Fixed Costs ($x10^6$)	Taxes ($x10^6$)
General Motors	15,250	1,559	1,966
Ford	8,853	1,401	596
Chrysler	4,121	746	213

Table 2

A **crude indication** of the physical size of the corporations is given by the value placed on plant, equipment, property and special tools. These figures, of course, are highly influenced by accounting practices and especially when land values have increased may grossly underestimate the worth of the capital investment.

	Property and Plant $(x10^6)$	Special Machinery $(x10^6)$
General Motors	4,161	455
Ford	2,574	446
Chrysler	887	180

Table 3

World sales of automotive equipment including trucks and tractors is given in Table 4.

	Sales $(x10^3)$	%
General Motors	7,278	52.2
Ford	4,595	32.9
Chrysler	2,077	14.9
	13,950	100.

Table 4

From the above information on the basis of the assumption of linear costs we may write:

$$p_1 = 20{,}734/7.278 = 2{,}849 \qquad c_1 = 15{,}250/7.278 = 2{,}095$$

$$p_2 = 11{,}537/4.595 = 2{,}511 \qquad c_2 = 8{,}853/4.595 = 1{,}927$$

$$p_3 = 5{,}300/2.077 = 2{,}552 \qquad c_3 = 4{,}121/2.077 = 1{,}984$$

We assume that the demand for the automotive products of any firm i is given by:

(58) $$q_i = \beta w_i [V - p_i - \gamma(p_i - \Sigma\, w_j p_j)] \qquad i = 1, 2, 3$$

From (26) we have

(59) $$p_i = \frac{1}{\Delta - w_i} \left\{ \frac{V}{\gamma} Q + \frac{1}{2} [(\Delta - 2w_i)c_i + Q\, \Sigma\, z_j(\Delta - 2w_j)q] \right\} .$$

We know that $w_1 + w_2 + w_3 = 1$ or $w_3 = 1 - w_1 - w_2$ thus we have five undetermined parameters β, α ($V = \alpha/\beta$ hence we determine α rather than V), γ, w_1 and w_2.

In (59) $V = \alpha/\beta$, $\Delta = \frac{2(1+\gamma)}{\gamma}$, $z_i = \frac{1}{x_i}$,

$x_i = \Delta \frac{1}{w_i} - 1$, and $Q = \frac{1}{1 - \Sigma_j z_j}$.

From the three equations of the form

(60) $$q_i = \beta w_i (V - (1+\gamma)p_i + \gamma\, \Sigma\, p_j w_j)$$

we obtain by subtraction:

(61) $$\frac{\frac{q_1}{w_1} - \frac{q_2}{w_2}}{p_2 - p_1} = \frac{\frac{q_1}{w_1} - \frac{q_2}{w_3}}{p_3 - p_1} = (1+\gamma)\beta$$

from which we derive

(62) $$w_2 w_3 q_1 (p_3 - p_2) + w_1 w_3 q_2 (p_1 - p_3) + w_1 w_2 q_3 (p_2 - p_1) = 0$$

Let us call $q_i(p_j - p_k) = z_i$, we may rewrite (62) as

(63) $$z_1 w_2 w_3 + z_2 w_1 w_3 + z_3 w_1 w_2 = 0$$

We know that $w_1 + w_2 + w_3 = 1$ hence

(64) $$w_3 = 1 - w_1 - w_2 .$$

Substituting in (63) we obtain

(65) $$z_1 w_2(1 - w_1 - w_2) + z_2 w_2(1 - w_1 - w_2) + z_3 w_1 w_2 = 0$$

giving

(66) $$- z_2 w_1^2 - z_1 w_2^2 + (z_1(1-w_1) - z_2 w_1 + z_3 w_1)w_2 + z_2 w_1 = 0$$

Dividing (66) by $- z_1$ we obtain (67).

$$w_2^2 - \left(1 + \frac{(z_3 - z_1 - z_2)}{z_1} w_1\right) w_2 - \frac{z_2}{z_1} w_1(1 - w_1) = 0$$

set

$$\frac{z_3 - z_1 - z_2}{z_1} = r_1 , \quad \frac{z_2}{z_1} = r_2$$

We may rewrite (67) as:

(68) $$w_2^2 - (1 + r_1 w_1)w_2 - r_2 w_1(1 - w_1) = 0$$

We solve this equation to obtain w_2 as a function of w_1 . We then search through successive values of w_1 until we obtain a positive root. Returning to equation (62) we may express β as a function of w_1 and γ .

(69) $$\beta(w_1, \gamma) = \frac{\frac{q_1}{w_1} - \frac{q_2}{w_2}}{(p_2 - p_1)(1 + \gamma)}$$

Using this in equation (62) to eliminate β we may solve for V .

(70) $$V(w_1, \gamma) = \frac{q_1}{\beta w_1} + (1+\gamma)p_1 - \gamma \Sigma p_j w_j .$$

This now leaves us the problem of estimating w_1 and γ which we do by Chebychef Criterion of minimizing the maximum of the absolute value of the ratio of the deviation of predicted from observed prices $\left| \frac{p_i - \hat{p}_i}{\hat{p}} \right|$.

Using the crude aggregated data for p_i and c_i obtained from the yearly reports we have:

$p_i = 2,849$

$V = 533,678$ $\beta = 26.3$ $\gamma = 1,988$

$w_1 = .75$ $w_2 = .17$ $w_3 = .08$

maximum deviation = .0080

These estimates appear to be somewhat startling as can be seen from the β which implies that a $100 cut by all firms would result in the sale of 2,600 more automobiles! We noted previously however, that the aggregation used to obtain the observed average prices does not appear to be reasonable owing to the multiproduct nature of the firms. General Motors especially has an important part of its business (and hence costs and sales) in markets other than vehicles. We suspect that a more detailed gathering of statistics would somewhat lessen the differences in observed aggregate average prices which apparently have General Motors prices (and costs) considerably above the others. We reduce this difference (of approximately $325) by $50 and $100 in order to view the effect on our parameter estimation. Setting p_1 = $2,799 and then p_i = $2,749 we obtain:

$p_1 = 2{,}799$

$$V = 6{,}039.3 \qquad \beta = 4{,}198.5 \qquad \gamma = 9.28$$

$$w_1 = .70 \qquad w_2 = .20 \qquad w_3 = .10$$

maximum deviation = .0085

$p_i = 2{,}749$

$$V = 3{,}951 \qquad \beta = 10{,}889 \qquad \gamma = 2.2$$

$$w_1 = .65 \qquad w_2 = .24 \qquad w_3 = .11$$

maximum deviation = .0124

A quick crude check of the above shows that when $p_1 = 2{,}799$

$$e = \frac{p \, \Delta q}{q \, \Delta p} \simeq .84$$

which seems to be somewhat low. When $p_1 = 2{,}749$

$$e \simeq 2.1 ,$$

this appears to be somewhat high. It is evident that for specific econometric use of our method we need better statistics on prices and costs then the ones we have used.

It is of interest to note that for n firms we would have V, β, γ and $w_1, w_2, \ldots w_{n-1}$ to estimate or $n + 2$ parameters in toto. We have $2n$ equations which must be satisfied. In the case of the automobile market examined here we had five parameters and six equations. Our computational methods can be extended for more than three firms at the cost of some complication in the ease of manipulation.

5. Concluding Remarks

In this paper our prime concern has been the formulation and exploration of an explicit mathematical model of a nonsymmetric oligopolistic market. There are many different concepts of "solution" which may be applied to such a structure. It is our belief that (at least based upon experience with oligopoly games) the noncooperative equilibrium solution is the best short term predictor of behavior. Given that we were able to obtain various solutions for the mathematical model of the market we observed that by making both behavioral (noncooperative equilibrium) and structural (the quadratic payoff functions) assumptions we would be in a position to estimate parameters from corporate and other market information. Our first goal has been to produce a game with parameters that look "somewhat like" an actual industry and has the property that the noncooperative solution gives the profits, sales and prices of that industry at a point in time. This can be used together with a scenario stating "the game you are playing in has a simulated environment representing the automobile industry." It may then be possible to examine behavior in this somewhat specialized game.

The assumptions of a linear aggregate demand function and of constant average costs with capacity limitations lead, in the symmetric case, to quadratic payoff functions for the firms and in the nonsymmetric case to the model formulated here. Given the difficulties in microeconomic measurement and the complexity of even highly simplified n-firm market models the detailed exploration of solutions of models with more complex demand or cost functions does not appear to be merited at this time. It is not evident that any qualitatively different phenomena will be encountered by using quadratic or higher approximations for demand or cost conditions.

Apart from our gaming interests it appears that this approach may be worth following in the microeconometric investigation of market structure and in the devising of measures of oligopolistic structure and collusive behavior.*

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* The estimation program for the parameters has not been included in the paper, but a listing is available from the authors.

FOOTNOTES

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