COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
AT YALE UNIVERSITY

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New Haven, Connecticut

COWLES FOUNDATION DISCUSSION PAPER NO. 191

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August 30, 1965

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This paper explores the implications for the Golden Rule of Accumulation, and hence for efficient economic growth, of the existence of a delay between the construction of new capital goods and their use in production.[1] The latter phenomenon has been discussed widely, especially in the context of less developed countries, under the heading of "absorptive capacity".

There appear to be two distinct uses of the term "absorptive capacity". One concept involves fixed proportions models. In these models, absorptive capacity is said to be reached when the stock of capital -- both in existence and in use -- is sufficiently large to employ the entire labor force. If the labor force or the "effective" or "augmented" labor force is growing, continuously positive net investment will be possible without exceeding absorptive capacity. It will be wasteful to invest permanently in excess of this rate, and a waste of foreign aid to finance investment permanently in excess of this rate.

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1. This paper was stimulated by unpublished work of Branko Horvat in which he contended that the growth path which is designated the Golden Rule path in conventional models is inefficient in models containing absorption difficulties. The formulation here is quite different from Horvat's.

This counsel not to exceed absorptive capacity can be viewed as advice not to maintain the capital stock permanently in excess of the Golden Rule value, for in fixed-proportions models of the conventional sort the path on which there is continuously full employment of both capital and labor is the Golden Rule path.

The second type of absorption problem is dynamic in nature. Because it takes time for newly produced capital goods to be installed and possibly for firms to acquire the labor and knowledge to operate them, there may be a gap between the stock of finished capital goods in existence and the stock of capital goods in use or in full use. Absorptive capacity, *in strictu sensu*, is reached when (at full employment) the stock of capital in existence is so much larger than the capital stock in use that an increase in the former will not produce an increase in the latter for some finite interval of time. *Absorption difficulties*, a weaker concept, will be said to prevail when (at full employment) the capital stock in existence exceeds the capital stock in use.

The analysis below focuses on the latter, dynamic type of absorption problem. I shall confine the analysis to the variable-proportions production function. The model is aggregative and excludes capital-embodied (and labor-embodied) technical progress. It is thus a somewhat unrealistic vehicle for study of the problem but I believe that it captures enough of the absorption phenomenon to be of interest. I begin with the simpler of two models.

I. Absorptive Difficulties

The model begins in a conventional way. The labor force, L, grows exponentially at some non-negative rate γ:

(1) $$L(t) = L_o e^{\gamma t}, \quad \gamma \geq 0$$

Output, Q, is a function of capital in use, K, and augmented labor. The rate of labor augmentation is a constant, λ.

(2) $$Q(t) = F[K(t), e^{\lambda t} L(t)], \quad \lambda \geq 0, \quad \gamma + \lambda > 0$$

The production function exhibits constant returns to scale. Marginal products are everywhere positive, diminishing and continuous. Labor is required for positive output. Hence, if $k(t)$ denotes the ratio of capital in use tc augmented labor at time t,

(3) $$k(t) = \frac{K(t)}{e^{(\lambda + \gamma)t} L_o},$$

then the production function may be written

(4) $$Q(t) = e^{(\lambda + \gamma)t} L_o f(k(t)),$$

$$f(0) = 0, \; f'(k) > 0, \; f''(k) < 0, \; f'(\infty) = 0$$

Let X denote capital in existence so that $\dot{X} = dX/dt$ is the rate of investment. Then if C denotes consumption,

$$C(t) + \dot{X}(t) = Q(t) \tag{5}$$

Now let $x(t)$ denote the ratio of capital in existence to augmented labor at time t :

$$x(t) = \frac{X(t)}{e^{(\lambda + \gamma)t} L_o} \tag{6}$$

Differentiating (6) we obtain

$$\frac{\dot{X}(t)}{e^{(\lambda + \gamma)t} L_o} = (\lambda + \gamma)\, x(t) + \dot{x}(t) \tag{7}$$

Hence, dividing both sides of (5) by $e^{(\lambda + \gamma)t} L_o$ and using (4) and (6) we have

$$\frac{C(t)}{e^{(\lambda + \gamma)t} L_o} + \left[(\lambda + \gamma)\, x(t) + \dot{x}(t) \right] = f(k(t)) \tag{7}$$

Suppose now that the investment-output ratio is a constant, s . Then

(8) $$\frac{C(t)}{e^{(\lambda + \gamma)t} L_o} = \frac{Q(t) - \dot{X}(t)}{e^{(\lambda + \gamma)t} L_o} = (1 - s)\, f(k(t)), \quad 0 < s \leq 1 .$$

From (7) and (8) we therefore obtain the following differential equation:

(9) $$\dot{x}(t) = s\, f(k(t)) - (\lambda + \gamma)\, x(t)$$

If $x(t) = k(t)$, as in conventional models, then (9) reduces to the standard differential equation for analyzing Solow-type growth models. But I shall suppose that there may be absorption difficulties causing $x(t) > k(t)$.

I postulate the following linear absorption mechanism:

(10) $$\dot{K}(t) = \alpha \cdot [X(t) - K(t)]$$

or equivalently

(10') $$\frac{\dot{K}(t)}{e^{(\lambda + \gamma)t} L_o} = \alpha \cdot [x(t) - k(t)] , \quad \alpha > 0, \quad x(t) \geq k(t) .$$

According to (10), the rate at which capital in existence is absorbed into use is proportional to the gap between the two, the absorption coefficient, α , being a positive constant. The second model will introduce a non-linear mechanism. The greatest doubt surrounding (10) is perhaps the assumption that the growth of the labor force or perhaps the growth of the "efficiency" of the labor force does not serve to increase α over time.

Differentiating (3) we obtain

$$(11) \qquad \frac{\dot{K}(t)}{e^{(\lambda + \gamma)t} L_o} = (\lambda + \gamma)\, k(t) + \dot{k}(t)$$

Equations (10') and (11) yield

$$(12) \qquad \dot{k}(t) = \alpha\, x(t) - (\alpha + \lambda + \gamma)\, k(t)$$

Equation (12) together with (9) comprise a system of two differential equations in the variables $x(t)$ and $k(t)$. This system is in equilibrium -- a golden-age equilibrium -- when $\dot{x}(t) = \dot{k}(t) = 0$ for all t. In equilibrium, therefore, we have, letting g denote $\lambda + \gamma$,

$$(13) \qquad 0 = s\, f(k) - gx$$

$$(14) \qquad 0 = \alpha x - (\alpha + g)k\ ,$$

or equivalently

$$(13') \qquad x = \frac{s\, f(k)}{g}$$

(14') $$x = \left(\frac{\alpha + g}{\alpha}\right) k$$

These two equations are diagrammed in Figure 1. The diagram shows that a unique equilibrium with $k > 0, x > 0$ exists if and only if the curve representing (13') is steeper at the origin then the line representing (14'). Hence such an equilibrium exists if and only if

(15) $$s\ f'(0) > g \left(\frac{\alpha + g}{\alpha}\right)$$

As $\alpha \to \infty$ we obtain the analogous condition in the conventional model without absorption difficulties, viz., $s\ f'(0) > g$.

Such an equilibrium is globally stable for all $k(0) > 0$, $x(0) \geq k(0)$, as shown by the arrows in Figure 1 whose directions are determined by (9) and (12).

From Figure 1 it can be seen that an increase of the saving ratio leads asymptotically to higher levels of $k(t)$ and $x(t)$, and hence to a higher equilibrium path of output. But after a point, say $\hat{k}$, the Golden Rule level of $\underline{k}$, an increase of $\underline{k}$ due to an increase of $\underline{s}$ actually reduces consumption, as will now be shown.

From (7) we have

(16) $$C(t) = \left\{f(k(t)) - (\lambda + \gamma)\ x(t) - \dot{x}(t)\right\} e^{(\lambda + \gamma)t}\ L_o$$

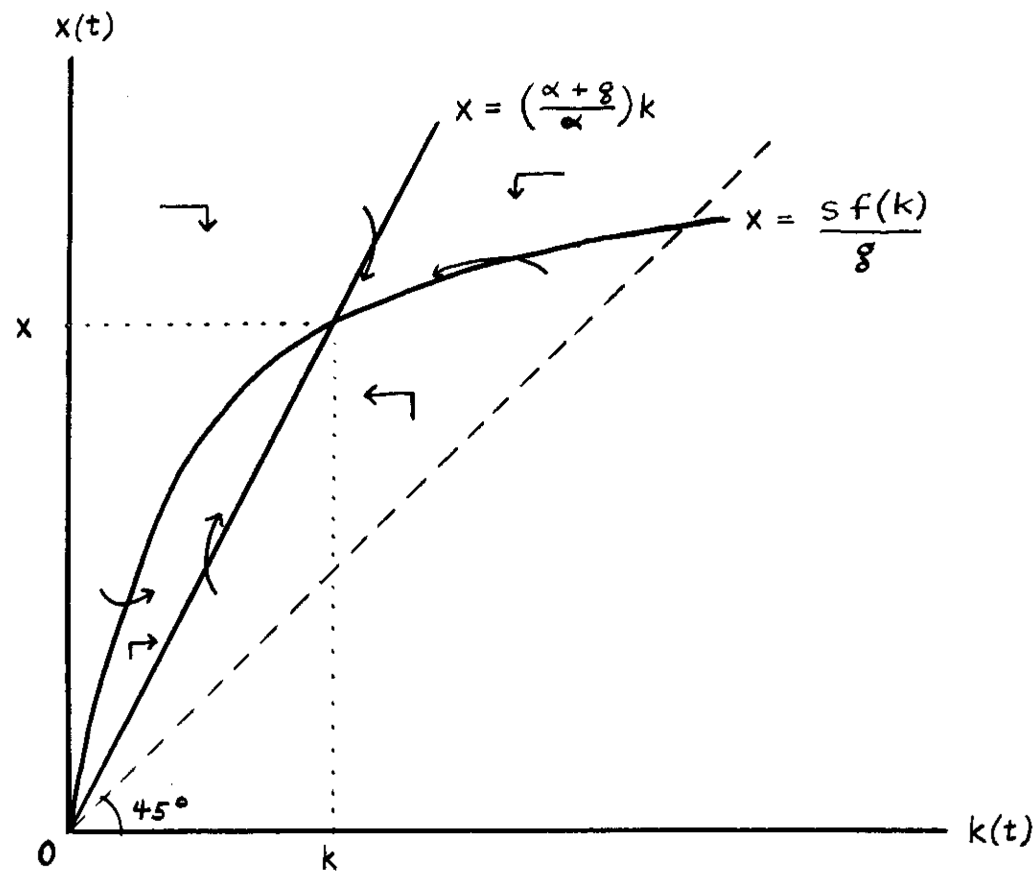

Figure 1

Hence, in golden-age equilibrium, in which $\dot{k}(t) = \dot{x}(t) = 0$, we have

$$(17) \qquad C(t) = \left\{ f(k) - (\lambda + \gamma)\, x \right\} e^{(\lambda + \gamma)t}\, L_o$$

Equating to zero the derivative of $C(t)$ with respect to <u>k</u> , so as to find the consumption-maximizing golden age path, we find

$$(18) \qquad f'(k) - (\lambda + \gamma) \frac{dx}{dk} = 0$$

From (13') we have

$$(19) \qquad \frac{dx}{dk} = \frac{\alpha + g}{\alpha} > 1 , \qquad g = \lambda + \gamma .$$

Hence, by (18) and (19), the consumption-maximizing or Golden Rule value of <u>k</u> , say $\hat{k}$, is defined by

$$(20) \qquad f'(\hat{k}) = g \left(\frac{\alpha + g}{\alpha} \right) > g$$

In the present model, therefore, the marginal product of <u>capital in use</u> exceeds the natural growth rate on the Golden Rule path. (There is equality of marginal product and growth rate in the limit as α approaches infinity.) The path on which the marginal product of capital in use equals the golden-age growth rate is dynamically inefficient in this model; such a path entails excessive capital intensity from the standpoint of consumption. The reason that the Golden Rule marginal product exceeds the growth rate is that here a one-unit increase of capital in use requires more than a one-unit increase of capital in existence in golden age equilibrium, so that high intensity of capital in use is more expensive in terms of investment requirements than in the conventional model without absorption difficulties.

Nevertheless one can, somewhat artificially, rescue the generality of the usual formula for the Golden Rule path -- equality of capital's marginal product and the golden-age growth rate -- by stating that on the Golden Rule path, in either the present or the conventional model, the "marginal product of capital in existence" is equal to the golden-age growth rate. For (18) may be written

$$f'(k)\frac{dk}{dx} - (\lambda + \gamma) = 0 \tag{21}$$

and we can interpret the first term as the equilibrium marginal product of capital in existence.

II. Absorptive Capacity

In the previous model, absolute absorptive capacity could never be reached: $X(t)$ could never be so large in relation to $K(t)$ that an increase of the former would not immediately increase $\dot{K}(t)$ and hence increase the rate of growth of output. Let us consider a model now in which absorptive capacity could be reached.

In place of (10) I postulate the following nonlinear absorption mechanism:

$$(22) \qquad \frac{\dot{K}(t)}{e^{(\lambda + \gamma)t} L_o} = \Phi\,[x(t) - k(t)], \quad \Phi(0) = 0 ,$$

$$\Phi' > 0,\ \Phi'' < 0 \quad \text{for} \quad x(t) - k(t) < m$$

$$\Phi' = 0 \quad \text{for} \quad x(t) - k(t) \geq m ,$$

$$m > 0 .$$

This implies that $\dot{K}$ is a linear homogeneous function of X, K and $e^{gt} L_o$. The rate of absorption per unit effective labor is at a maximum when the gap between $x(t)$ and $k(t)$ equals or exceeds some positive number $\underline{m}$. Using (11) we may write (22) in the form

$$(23) \qquad \dot{k}(t) = \Phi\,[x(t) - k(t)] - (\lambda + \gamma)\, k(t)$$

Our two differential equations are now (9) and (23). An equilibrium is defined as a path on which $\dot{k}(t) = \dot{x}(t) = 0$ for all $\underline{t}$. In equilibrium, therefore,

$$(24) \qquad x = \frac{s\, f(k)}{g}$$

$$(25) \qquad \Phi(x - k) = gk$$

Once again a unique equilibrium with $\underline{k} > 0$ will exist if and only if the curve representing (24) is steeper at the origin than the curve representing (25), i.e., if and only if

$$(26) \qquad s\, f'(0) > g \left[\frac{\Phi'(0) + g}{\Phi'(0)} \right]$$

This equilibrium can be shown to be globally stable for all $k(0) > 0$.

An increase of the saving ratio once again increases equilibrium $\underline{x}$ but here it is possible that $\underline{k}$ will attain an upper bound, given by the vertical postion of the curve depicting (25), so that an increase of $\underline{s}$ beyond a certain point will not increase equilibrium $\underline{k}$, and hence not increase golden-age output. In these equilibria the economy is operating at absorptive capacity. (Of course, $s = 1$ may occur before absorptive capacity is reached.)

Turning to the Golden Rule path, we again use (17), the equation for golden-age consumption, which may be written.

$$(27) \qquad C(t) = \left\{ f(k) - gx \right\} e^{gt} L_o$$

On the Golden Rule path, as before,

$$(28) \qquad f'(k) - g \frac{dx}{dk} = 0$$

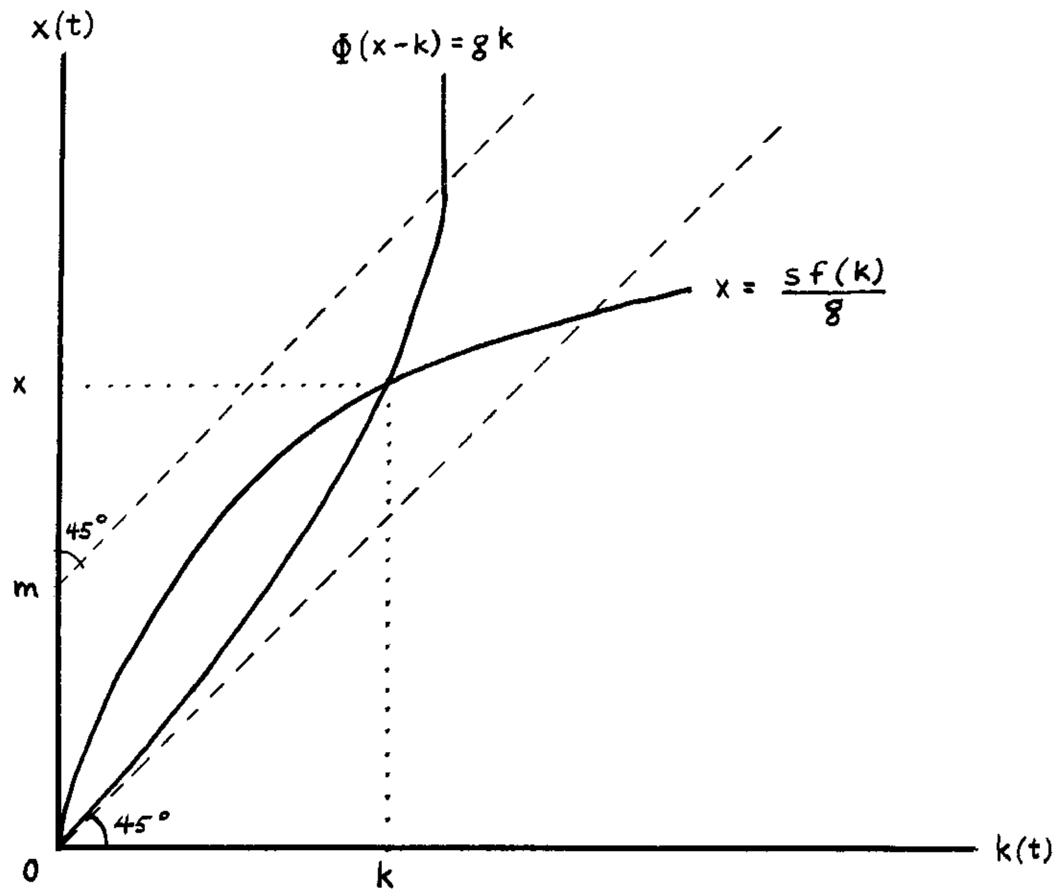

Figure 2

Differentiation of (25) yields

$$(29) \qquad \frac{dx}{dk} = \frac{\Phi'(x - k) + g}{\Phi'(x - k)} > 1$$

Hence, on the Golden Rule path

$$(30) \qquad f'(\hat{k}) = g \left[\frac{\Phi'(\hat{x} - \hat{k}) + g}{\Phi'(\hat{x} - \hat{k})} \right] > g$$

This condition together with (25), which links <u>x</u> to <u>k</u>, determine the Golden Rule path.

The relation of the Golden Rule path to absorptive capacity may be of some interest. If $\Phi'(x - k)$ is continuous, so that $\Phi'(x - k) \rightarrow 0$ as $x - k$ approaches <u>m</u> from below, then the Golden Rule gap, $\hat{x} - \hat{k}$, will be smaller than <u>m</u>; absorptive capacity will not be reached. If $\Phi'(x - k)$ is allowed to be discontinuous, one can conceive of Φ functions which make the Golden Rule $\hat{x}$ the smallest consistent with absorptive capacity.

III. Concluding Remarks

The two absorption mechanisms postulated here have the property that an excess of capital in existence over capital in use persists even into long-run, golden-age equilibrium. I briefly consider two absorption mechanisms which do not have this property.

One might postulate that

$$\dot{k}(t) = \Phi[x(t) - k(t)] \tag{31}$$

or equivalently

$$\frac{\dot{K}(t)}{e^{(\lambda + \gamma)t} L_o} = \Phi \ [x(t) - k(t)] + (\lambda + \gamma) k(t) , \tag{31'}$$

where Φ has the same properties as in (22).

This mechanism implies that in golden-age equilibrium, where $\dot{k}(t) = 0$, $x(t) = k(t) =$ constant. Hence there is no persistent gap between capital in existence and capital in use. In this model, the Golden Rule path will be characterized by equality between the marginal product of <u>capital in use</u> and the golden-age growth rate.

But (31') is not entirely reasonable. Is absorption really faster the greater is the <u>rate of increase</u> of the labor force or of the "effective" labor force? It is conceivable that as firms become accustomed to a high γ they learn to absorb new capital faster but I do not find this very plausible. The role of λ in (31') is even more mysterious.

Another absorption mechanism is

$$(32)\qquad \frac{\dot{K}(t)}{e^{(\lambda+\gamma)t} L_o} = \psi \left[x(t) - k(t),\ L_o\, e^{(\lambda+\gamma)t} \right] \quad , \quad \psi_1, \psi_2 > 0$$

with ψ increasing without bound as the effective labor force increases without bound. This mechanism implies that absorption per unit effective labor is faster the larger the effective labor force, given the gap per unit effective labor. The mechanism suggests that as the effective labor force increases without bound, all capital in existence will tend to be absorbed.

This returns-to-scale mechanism is questionable if population density per land area is constant; for it may be that absorption is actually slower in large countries. Further, (32), like (22) and (31), uses <u>effective</u> labor; this may be acceptable in production functions but it is less plausible perhaps in absorption equations.

A variety of dynamic absorption mechanisms have been considered. One cannot choose decisively among them without more information about absorption problems than is yet available. This paper has shown that, for some mechanisms, a Golden Rule path may exist and that, expressed in terms of the marginal product of capital in use, the formula for the Golden Rule path differs from that in models without absorption difficulties.