COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut

COWLES FOUNDATION DISCUSSION PAPER NO. 164

A TWO-COUNTRY, THREE COMMODITY, DYNAMIC MODEL OF INTERNATIONAL TRADE

T. N. Srinivasan

December 9, 1963

A TWO-COUNTRY, THREE COMMODITY, DYNAMIC MODEL OF INTERNATIONAL TRADE *

T. N. Srinivasan

1. Introduction

The literature on dynamic models of international trade and growth is somewhat scanty as compared to the vast literature on the static and comparative static aspects of trade theory (See Bhagwati [1] for a survey of the existing literature). Among the few discussions of the effects of accumulation and/or population growth on international trade are Bensusan-Butt [2], Brems [3], Johnson [4] and Verdoorn [5].

Bensusan-Butt studies the effect of accumulation in a highly suggestive two-country multi-commodity model. There are two processes for producing a commodity in each country, a mechanized and a non-mechanized process. Each country starts with zero past accumulation using only the non-mechanized process for production of each commodity. An accident starts off the accumulation process in one country. Bensusan-Butt then traces the course of progressive mechanization, emergence of comparative advantage and export industries in the accumulating country. Capital movement from the accumulating country to the other comes at a certain phase in the growth

---

* This research was supported by the Economic Growth Center at Yale University, New Haven, U.S.A., and the Institute of Economic Growth at Delhi, India. I wish to thank Professors Jagdish Bhagwati, Bela Balassa and Harry Johnson for their comments on an earlier draft. The revision of and extension of the results of the earlier draft was carried out under a grant from the National Science Foundation.

sequence. The impact of size difference between the two countries on this sequence is also discussed. While his discussion is rich in its insights regarding the possible evolution over time of his international economy, the failure of Bensusan-Butt to state and analyze his model in explicit mathematical terms leaves a student of theory wondering as to which elements of his growth sequence will stand in a rigorous analysis.

Brems [3] and Johnson [4] apply Harrod-Domar type growth theory to the international economy. Brems proposes an elaborate two sector (firms and households) model involving 52 equations. In its technological aspects, Brems' model is similar to a dynamic Leontief model and for that reason can result in negative values for capital stock and gross outputs. The mechanism by which current trade deficits or surplus get adjusted is not explicitly mentioned. It is assumed implicitly that relative prices within a country and terms of trade of traded items remain constant over time. Once again the mechanism by which this is brought about is not mentioned.

Johnson's model is essentially a single sector, single factor one. Constant savings and import propensities are assumed. In Part I of the article Johnson postulates fixed exchange rates with current deficits or surpluses erased by capital movements. After analyzing growth in a single country with exponentially growing exports, Johnson introduces a two-country international economy with each country having its own constant propensities to save and import. Time paths of equilibrium output in the two countries and their rates of growth are analyzed. In Part II of the article exchange rate movements keep trade in balance with no capital movements. Johnson's model also yields time paths involving negative outputs in certain situations. Johnson, however, confines his attention to those intervals of time for which output is non-negative.1/

1/ This procedure is somewhat unsatisfactory. An alternative and perhaps better procedure will be to modify some of the behavior equations in such a situation.

Verdoorn [5] considers an open economy producing a single commodity using 3 factors of production viz. capital, labor and imports. The factors are assumed to be complementary in production with total output determining uniquely the demand for each. On the supply side, a Keynesian savings function, exponential growth of labor force and an export demand function are postulated. Export prices as a ratio of import prices are assumed to grow exponentially with the rate of growth being a policy parameter. Part of exports, having an autonomous character, grow exponentially. Besides the rate of growth of export prices, the propensity to save and rate of growth of labor force (net of emigration) are treated as instrument variables. International trade balance is postulated. The problem of deriving an optimum constellation of instruments for achieving (say) maximum national income in a particular year is solved. Thus Verdoorn studies the case of growth of a single country in an international economy.

In Section 2 we present a two-factor, three commodity, dynamic general equilibrium model of international trade. In considering two primary factors of production,[1] labor and capital, and neutral technical change in production our model is more general than those of Johnson and Brems while it differs from Verdoorn's in the sense that unitary elasticity of substitution between factors is assumed. Our model is similar to Johnson's and Brems' in that it takes into account the interaction on each other of the growth of the two trading countries. However, our model is less general in that it leaves out monetary problems altogether. It is concerned with tracing out the time paths of such variables as income (defined in terms of a numéraire), capital, barter terms of trade, etc. Two alternative models of capital flow between the trading countries are considered: (a) One country aids the other by agreeing to meet a fixed proportion of the other country's import bill, and (b) The capital flow between the countries depends on the difference in the rates of return to capital in the two countries.

1/ An important consequence of this is that given the production functions of our model, savings ratio has no influence on the asymptotic growth rates of income.

2. The Model With Foreign Aid

There are two countries in our international economy. Country 1 produces a consumer good (type 1) and a capital good. Country 2 produces only a consumer good (type 2). Country 2 exports part of its output of its consumer good and imports capital goods produced in country 1. Apart from these two goods traded internationally, no other goods or factors of production move between the two countries. Production of each good takes place under constant returns to scale and neutral technical change in both countries. There are two factors of production, labor and capital, which is identical with the capital good produced in country 1. Capital once created lasts forever. Labor force is assumed to grow exponentially in both countries. Production functions in the two countries are of the Cobb-Douglas type. A unit of consumer good produced by country 1 is taken as the numéraire.

Country 1 devotes a constant proportion $(1-s_1)$ of its income for consumption expenditure. Out of this consumption expenditure a constant proportion ϵ is spent on domestically produced consumer goods and the remaining proportion $(1-\epsilon)$ is spent on consumer goods imported from country 2. Country 2 exports a constant proportion s_2 of its output of consumer goods, and imports capital goods from country 1. The relative price of a unit of capital goods exported by country 1 in terms of a unit of consumer goods imported by it is determined by a balance of payments constraint which says that aproportion λ of the import bill of country 2 is financed through its exports, $\lambda < 1$ being a case of capital transfer from country 1 to country 2, $\lambda = 1$ one of zero capital flow and $\lambda > 1$ one of capital transfer from country 2 to country 1. This flow is assumed to be an outright grant of capital by one country to the other with no necessity of repayment in any form. Transportation costs will be assumed to be zero. Perfect competition within each country and free trade between countries will be assumed. Time will be treated as continuous. The problem is to trace the time paths of income, outputs of various commodities, factor prices, terms of trade, etc. The effect of changing some of the parameters of the model on the time paths will also be evaluated. The following notation will be used:

$Y_i(t) \equiv$ Income at t of country i in terms of the numeraire.

$C_i(t)$ = Consumption expenditure of country i at time t

$p_i(t)$ = Price per unit of consumer goods produced in country i at time t ($p_1(t) \equiv 1$ by choice of numéraire)

$q_1(t)$ = Price per unit of capital goods at t

$r_i(t)$ = Rental per unit of time per unit of capital stock in country i at time t

$w_i(t)$ = Wage rate per unit of labor at time t in country i

$Q_{ic}(t)$ = Output of consumer goods in country i at time t

$Q_{1k}(t)$ = Output of capital goods in country 1 at time t

$\dot{K}_i(t)$ = Stock of capital in country i at time t

$\dot{K}_i(t)$ = Rate of change of $K_i(t)$ with respect to t

$L_i(t)$ = Labor force in country i at time t

$X_i(t)$ = Exports in physical units of country i at time t

$K_{1j}(t)$ = Stock of capital devoted to the production of goods j $(j = c, k)$ in country 1

$L_{1j}(t)$ = Labor services devoted to the production of goods j $(j = c, k)$ in country 1

θ_i = Proportionate rate of growth of labor force in country i

σ_{ij} = Proportionate rate of growth of neutral technical change in country i $(i = 1, 2)$ in the production of goods j $(j = c, k)$. σ_{2k} is not defined as country 2 does not produce any capital goods

α_1 = Elasticity of the output of capital goods with respect to capital input in country 1

β_i = Elasticity of the output of consumer goods in country i with respect to capital input

The following equations summarize the model in algebraic terms:

<u>Country 1</u>

(1) $$Y_1(t) \equiv C_1(t) + q_1(t)\,\dot{K}_1(t) + q_1(t)\,\overset{X_1}{K_1}(t) - p_2(t)\,X_2(t)$$

(2) $$C_1(t) = (1-s_1)\,Y_1(t)\,, \qquad 0 < s_1 < 1$$

(3) $$C_1(t) \equiv Q_{1c}(t) + p_2(t)\,X_2(t)$$

(4) $$Q_{1c}(t) = \epsilon C_1(t)\,, \qquad 0 < \epsilon < 1$$

(5) $$\dot{K}_1(t) + X_1(t) = Q_{1k}(t)$$

(6) $$Q_{1c}(t) = e^{\sigma_{1c}(t)}\,K_{1c}^{\beta_1}(t)\,L_{1c}^{1-\beta_1}(t)\,, \qquad 0 < \beta_1 < 1$$

$$(7) \qquad Q_{1k}(t) = e^{\sigma_{1k}(t)} K_{1k}^{\alpha_1}(t)\, L_{1k}^{1-\alpha_1}(t) \,, \qquad 0 < \alpha_1 < 1$$

$$(8a) \qquad K_{1k}(t) + K_{1c}(t) = K_1(t)$$

$$(8b) \qquad \dot{K}_1(t) = \frac{dK_1(t)}{dt}$$

$$(9) \qquad L_{1k}(t) + L_{1c}(t) = L_1(t)$$

$$(10) \qquad L_1(t) = e^{\theta_1 t}$$

$$(11) \qquad \beta_1\, e^{\overset{\sigma_{1c}}{\cancel{\alpha_k}} t} \left[\frac{K_{1c}(t)}{L_{1c}(t)}\right]^{-(1-\beta_1)} = q_1(t)\alpha_1\, e^{\sigma_{1k} t} \left[\frac{K_{1k}(t)}{L_{1k}(t)}\right]^{-1-\alpha_1} = r_1(t)$$

$$(12) \qquad (1-\beta_1)\, e^{\sigma_{1c} t} \left[\frac{K_{1c}(t)}{L_{1c}(t)}\right]^{\beta_1} = q_1(t)\,(1-\alpha_1)\, e^{\sigma_{1k} t} \left[\frac{K_{1k}(t)}{L_{1k}(t)}\right]^{\alpha_1} = w_1(t)$$

<u>Country 2</u>

$$(13) \qquad Y_2(t) \equiv C_2(t) + q_1(t)\, \dot{K}_2(t) + p_2(t)\, X_2(t) - q_1(t)\, X_1(t)$$

$$(14) \qquad C_2(t) = (1-s_2)\, Y_2(t) \,, \qquad 0 < s_2 < 1$$

$$(15) \qquad p_2(t)\, X_2(t) + C_2(t) = p_2(t)\, Q_{2c}(t)$$

$$(16) \qquad \dot{K}_2(t) = \frac{dK_2(t)}{dt} = X_1(t)$$

(17) $$Q_{2c}(t) = e^{\sigma_{2c}t} K_{2c}^{\beta_2}(t) L_{2c}^{1-\beta_2}(t) , \qquad 0 < \beta_2 < 1$$

(18) $$L_{2c}(t) = e^{\theta_2 t}$$

(19) $$r_2(t) = p_2(t) \beta_2 K_{2c}^{\beta_2 - 1}(t) L_{2c}^{1-\beta_2}(t)$$

(20) $$w_2(t) = p_2(t)(1-\beta_2) K_{2c}^{\beta_2}(t) L_{2c}^{-\beta_2}(t)$$

Balance of Payments Constraint

(21) $$\lambda q_1(t) X_1(t) = p_2(t) X_2(t) , \qquad \lambda > 0$$

The above equations are explained as follows:
Equations (1) and (13) define national income of a country as the sum of expenditures on consumption, investment and exports less imports (the sign $\equiv$ is used to indicate equality by definition). Equations (2) and (19) express the relation that consumption expenditures form a constant proportion of income. Equation (3) defines total consumption expenditure in Country 1 as the sum of expenditures on the two consumer goods, viz., domestically produced and imported ones. Equation (4) expresses the relation that the expenditure on domestically produced consumer goods forms a constant proportion of total consumption expenditure. Equation (5) states that the output of capital goods equals the sum of the addition to capital stock at home and exports to country 2. Equations (6), (7) and (17) are

production function relations. Equations (8), (9), (10) and (18) are self-explanatory. Equations (11) and (12) are the factor allocation equations which express the requirement that marginal value products of capital and labor in the consumer goods industry equal their prices and their corresponding values in the capital goods industry. Equation (15) states that the value of output in country 2 is the sum of sales to domestic and foreign consumers. Equation (16) states that the addition to stock of capital in country 2 equals its import of capital goods. Equations (19) and (20) state that the marginal value products of capital and labor in country 2 equal their prices. Equation (21) states that the value of exports of country 2 forms a proportion λ of the value of its imports.

In solving the model it is convenient to introduce the following factor ratio variables:

$$\delta_1(t) = \frac{K_1(t)}{L(t)}\ , \ \delta_{1c}(t) = \frac{K_{1c}(t)}{L_{1c}(t)}\ , \ \delta_{1k}(t) = \frac{K_{1k}(t)}{L_{1k}(t)}\ , \ \delta_2(t) = \frac{K_2(t)}{L_2(t)}$$

It is shown in Appendix A that the solution to this model is given by the following:

Let

$$\eta = \frac{\beta_1(1-\alpha_1)}{\alpha_1(1-\beta_1)}\ , \qquad \mu = \frac{\epsilon(1-\beta_1)(1-s_1)}{(1-\alpha_1)[s_1+(1-\epsilon)(1-s_1)]}\ ,$$

$$\Phi(\lambda) = \left[1 - \frac{(1-\epsilon)(1-\alpha_1)\mu}{\lambda\,\epsilon(1-\beta_1)}\right], \qquad g = \frac{\sigma_{1k}}{1-\alpha_1} + \theta_1$$

$$\bar{\delta}_1(0) = \left[\frac{\Phi(\lambda)}{g(1+\mu)^{1-\alpha_1}(1+\eta\mu)^{\alpha_1}}\right]^{\frac{1}{1-\alpha_1}}$$

. Let $\delta_1(0)$ be the initial ratio of total capital stock to labor force in country 1. Then

Country 1

(22) $$\delta_1(t) = \left[\left\{\delta_1(0)^{1-\alpha_1} - \bar{\delta}_1(0)^{1-\alpha_1}\right\} + \bar{\delta}_1(0)^{1-\alpha_1} e^{g(1-\alpha_1)t}\right]^{\frac{1}{1-\alpha_1}} e^{-\theta_1 t}$$

(23) $$K_1(t) = e^{\theta_1 t}\,\delta_1(t)$$

(24) $$\delta_{1k}(t) = \left(\frac{1+\mu}{1+\eta\mu}\right)\delta_1(t), \qquad \delta_{1c}(t) = \frac{\eta(1+\mu)}{(1+\eta\mu)}\,\delta_1(t)$$

(25) $$Q_{1c}(t) = \left[\frac{\mu\eta^{\beta_1}}{(1+\mu)^{1-\beta_1}(1+\eta\mu)^{\beta_1}}\right] e^{[\sigma_{1e}+\theta_1]t}\,\delta_1(t)^{\beta_1}$$

(26) $$Q_{1k}(t) = \frac{1}{(1+\mu)^{1-\alpha_1}(1+\eta\mu)^{\alpha_1}}\, e^{[\sigma_{1k}+\theta_1]t}\,\delta_1(t)^{\alpha_1}$$

(27) $$q_1(t) = \left(\frac{1-\beta_1}{1-\alpha_1}\right)\left(\frac{1+\mu}{1+\eta\mu}\right)^{\beta_1-\alpha_1} \eta^{\beta_1}\, e^{(\sigma_{1c}-\sigma_{1k})t}\,\delta_1(t)^{\beta_1-\alpha_1}$$

$$(28) \qquad r_1(t) = \alpha_1 \left[\frac{1+\mu}{1+\eta\mu}\right]^{-(1-\alpha_1)} e^{\sigma_{1k}t} \, \delta_1(t)^{-(1-\alpha_1)} \, q_1(t)$$

$$(29) \qquad w_1(t) = (1-\alpha_1) \left[\frac{1+\mu}{1+\eta\mu}\right]^{\alpha_1} e^{\sigma_{1k}t} \, \delta_1(t)^{\alpha_1} \, q_1(t)$$

$$(30) \qquad Y_1(t) = \frac{1}{\epsilon(1-s_1)} \, Q_{1c}(t)$$

$$(31) \qquad X_1(t) = \frac{1}{\lambda} \left[\frac{(1-\epsilon)(1-s_1)}{s_1 + (1-\epsilon)(1-s_1)}\right] Q_{1k}(t)$$

$$(32) \qquad C_1(t) = \frac{Q_{1c}(t)}{\epsilon}$$

$$(33) \qquad \dot{K}_1(t) = \left[1 - \frac{1}{\lambda}\left\{\frac{(1-\epsilon)(\lambda-s_1)}{s_1 + (1-\epsilon)(1-s_1)}\right\}\right] Q_{1k}(t)$$

Country 2

$$(34) \qquad K_2(t) = K_2(0) + \left\{\frac{(1-\epsilon)(1-s_1)}{\lambda s_1 - (1-\lambda)(1-\epsilon)(1-s_1)}\right\} \left\{K_1(t) - K_1(0)\right\}$$

$$(35) \qquad Q_{2c}(t) = e^{(\sigma_{2c}+\theta_2)t} \, K_2(t)^{\beta_2}$$

$$(36) \qquad Y_2(t) = \frac{(1-\epsilon)(1-s_1)}{\epsilon_2} \, Y_1(t)$$

$$(37) \qquad X_2(t) = s_2 \, Q_{2c}(t)$$

$$(38) \qquad p_2(t) = \left(\frac{1-\epsilon}{\epsilon}\right) \frac{Q_{1c}(t)}{Q_{2c}(t)}$$

$$(39) \qquad r_2(t) = \beta_2 \frac{(1-\epsilon)}{\epsilon} \quad \frac{Q_{1c}(t)}{K_2(t)}$$

$$(40) \qquad w_2(t) = (1-\beta_2)\left(\frac{1-\epsilon}{\epsilon}\right) e^{-\theta_2 t} \quad Q_{1c}(t)$$

<u>Intercountry Comparisons</u>

$$(41) \qquad \frac{Y_2(t)}{Y_1(t)} = \frac{(1-\epsilon)(1-s_1)}{s_2}$$

$$(42) \qquad \frac{C_2(t)}{C_1(t)} = \frac{(1-\epsilon)(1-s_2)}{s_2}$$

$$(43) \qquad \frac{p_2(t)}{q_1(t)} = \left\{\frac{(1-\epsilon)(1-s_1)}{s_1 + (1-\epsilon)(1-s_1)}\right\} \frac{Q_{1k}(t)}{s_2 Q_{2c}(t)}$$

$$(44) \qquad \frac{r_2(t)}{r_1(t)} = \left\{\frac{\beta_2(1-\epsilon)\ \eta\mu}{\beta_1\ s_2\ \epsilon\ (1+\eta\mu)}\right\} \frac{K_1(t)}{K_2(t)}$$

$$(45) \qquad \frac{w_2(t)}{w_1(t)} = \left\{\frac{(1-\beta_2)(1-\epsilon)\mu}{(1-\beta_1)\ s_2\epsilon(1+\mu)}\right\} e^{(\theta_1-\theta_2)t}$$

3. Some Results

1) Let us first examine the possible range of values for the parameter λ. It will be recalled that λ is the proportion of country 2's import bill paid by its own exports. In other words, country 2 receives foreign aid equivalent to $(1-\lambda)$ times the value of its imports. Thus a value of $\lambda > 1$ will mean, as we noted earlier, that country 2 aids country 1. There is no logical reason to preclude this possibility and we shall not place any upper bound on λ. We see from equation (33) that if λ falls short of $\underline{\lambda}$ given by

$$(46) \qquad \underline{\lambda} = \frac{(1-\epsilon)(1-s_1)}{s_1 + (1-\epsilon)(1-s_1)}$$

$\dot{K}_1(t)$ becomes negative. This means, in our model, that country 1, in aiding country 2, exports not only its entire output of newly produced capital but also part of its existing stock of old capital. It is natural to require that this possibility is excluded and we shall therefore require that $\lambda \geq \underline{\lambda}$. It is clear that the lower the value of $\underline{\lambda}$, the easier it is for country 1 to let country 2 to pay a smaller proportion of its import bill through its exports.

2) It is easily seen from (46) that $\underline{\lambda}$ decreases as either s_1 or ϵ increases. The first of these two results can be explained as follows: From (23) and (26) one can show that as s_1 increases, *ceteris paribus*, the existing stock of capital in country 1 at any point of time as well as the

flow of newly produced capital goods increase. Naturally the effect is to make the problem of subsidizing the import bill of country 2 through export of capital goods from country 1 easier. It is not easy to provide a simple enough explanation to the second result. An increase in ϵ means *ceteris paribus*, an increase in the demand for domestically produced consumer goods in country 1, and a corresponding decrease in the demand for consumer goods imported from country 2. However, the effect of an increase in ϵ on the capital stock of country 1 at any point in time cannot be unambigously determined.

3) From the definition $\Phi(\lambda)$ it is clear that it is an increasing function of λ. From (23) we note that $K_1(t)$, the stock of capital in country 1 at any time t, is larger for a larger value of $\Phi(\lambda)$. Putting these two together we can state that as λ (the proportion of the import bill of the capital importing country 2 met by its export earnings) increases the stock of capital in the capital exporting country 1 at each point of time t increases. The increase in λ, through its effect on total capital stock, increases also the flow of output of both consumer and capital goods in country 1. However, the effect of an increse in λ on the stock of capital in country 2 is not unambiguous. This effect, as can be seen from (34), consists of the sum of two terms: (a) a positive effect due to the positive effect of a larger λ on $K_1(t)$ and (b) a negative effect arising from the fact that a larger λ means that *ceteris paribus*, only a smaller addition to the stock of capital $K_2(t)$ country 2 at any t, can be financed out of given export bill. The net effect may be positive or negative.

4) From (41) and (42) we note that the incomes and consumption expenditures in one country bear a constant proportion to their corresponding values in the other country. This means that these expenditures in two countries have the same rate of growth over time. Since labor force in each country is assumed to grow exponentially over time, income and consumption expenditures per worker in the country with the faster growing labor force will decline steadily over time as a proportion of their corresponding values for the other country. It may be worth pointing out that these proportions do not depend upon the factor endowments or rates of technical change. In other words difference in these factors between countries are compensated by appropriate price adjustments. The parameters that do influence the proportions are (a) s_1 which affects the volume of the stock and flow of capital goods in country 1 and (b) $\epsilon(s_2)$ which affects the demand (supply) in country 1 for consumer goods exported by country 2. Thus an increase in s_1 or ϵ or s_2 decreases the income (or consumption expenditure) of country 2 as a proportion of that for country 1.

5) From (31) it is clear that country 1 exports a constant proportion of its current output of capital goods to country 2.

6) From (23) it is clear that asymptotic rate of growth of capital stock in country 1 is g . This does not depend either on the saving ratio s_1 or the parameters of the production function of the consumer goods industry. This result is well known in the literature on two-sector models of economic growth. However, as we noted in result (1) above the level of the stock of capital depends upon the saving ratio s_1 . Another implication of

this result on growth rate of capital stock is that in the absence of technical progress in the capital goods industry, capital labor ratios (aggregate and industry wise) approach constant values as time goes to infinity. This can be seen from (22) and (24).

7) Using (22), (25) and (26) it is easily seen that the asymptotic rate of growth of the output of consumer and capital goods in country 1 are respectively $\frac{\beta_1 \sigma_{1k}}{1-\alpha_1} + \sigma_{1c} + \theta_1$ and g. The first of these two rates is also the asymptotic rate of growth of income in both countries. It is interesting to note that the common asymptotic rate of growth of income in both countries depends only on the parameters relating to the capital exporting country.

8) We note from (27) that the asymptotic rate of growth of $q_1(t)$ the price of a unit of capital good in country 1 in terms of a unit of consumer good produced in that country is $\sigma_{1c} - \left(\frac{1-\beta_1}{1-\alpha_1}\right) \sigma_{1k}$. Thus $q_1(t) \to \infty$, a positive constant or zero according as $(1-\alpha_1)\, \sigma_{1c}$ is $>$, $=$ or $<$ $(1-\beta_1)\, \sigma_{1k}$. As one would expect the asymptotic rate of growth of $q_1(t)$ is an increasing function of σ_{1c} and a decreasing function of σ_{1k}.

9) We can deduce from (34) that the stock of capital in country 2 will approach asymptotically, a constant proportion of the stock of capital in country 1, the common asymptotic rates of growth of the two stocks being $\frac{\sigma_{1k}}{1-\alpha_1} + \theta_1$. The asymptotic rate of growth of $Q_{2c}(t)$, the output of

consumer goods produced by country 2 is $\sigma_{2c} + \theta_2 (1-\beta_2) + \beta_2 \left[\frac{\sigma_{1k}}{1-\alpha_1} + \theta_1\right]$.

The relative price of a unit of output of country 2 in terms of a unit of consumer goods produced by country 1 has an asymptotic growth rate of

$$\left[\frac{\beta_1 \sigma_{1k}}{1-\alpha_1} + \sigma_{1c} + \theta_1\right] - \left[\sigma_{2c} + \theta_2(1-\beta_2) + \beta_2 \left\{\frac{\sigma_{1k}}{1-\alpha_1} + \theta_1\right\}\right]$$

or $\frac{\sigma_{1k}}{1-\alpha_1} (\beta_1-\beta_2) + \sigma_{1c} - \sigma_{2c} + (1-\beta_2)(\theta_1-\theta_2)$

This is seen from (38).

10) The terms of trade of country 2 i.e., $\left\{\frac{p_2(t)}{q_1(t)}\right\}$ (the relative price of a unit of its exports in terms of a unit of its imports) has an asymptotic rate of growth of $(1-\beta_2)\left\{\theta_1 - \theta_2 + \frac{\sigma_{1k}}{1-\alpha_1}\right\} - \sigma_{2c}$.

Thus $\frac{p_2(t)}{q_1(t)}$ will approach ∞, a positive constant or zero as time goes to infinity according as $(1-\beta_2)\, g$ is $>$ or $=$ or $<$ $(1-\beta_2)\theta_2 + \sigma_{2c}$. Dividing through by $(1-\beta_2)$ we can say that

$$\lim_{t\to\infty} \frac{p_2(t)}{q_1(t)} = \left.\begin{matrix}\infty \\ \text{constant} \\ 0\end{matrix}\right\} \Longleftrightarrow g \begin{matrix} > \\ = \\ < \end{matrix} g'$$

where $g' = \frac{\sigma_{2c}}{1-\beta_2} + \theta_2$. Now g' would have been the asymptotic rate of growth of output (Q_{2c}) of country 2 if the terms of trade were fixed. With variable terms of trade, this rate becomes $g'' = (1-\beta_2)\, g' + \beta_2\, g$. Now $(g'' - g')$ has the same sign as $(g - g')$. Hence the actual asymptotic rate of growth of Q_{2c}, namely g'' , exceeds the potential rate of growth g' with fixed terms of trade if and only if the actual terms of trade improve over time. In the special case of no technical change in either country, the above result implies that the terms of trade of the country with the faster rate of growth of labor force will deteriorate over time.

11) It can be seen from (23), (34) and (44) that the competitive rate of rental on capital stock in country 2 relative to that in country 1 approaches a constant as time goes to infinity. It can be shown that this asymptotic ratio of rentals per unit of capital stock, increases, ceteris paribus, as either (a) λ increases or (b) s_1 increases or (c) s_2 decreases. These results can be explained as follows: A larger λ implies a smaller import of capital goods by country 2 from a given quantity of its exports. This in turn implies smaller rate of investment and total capital stock in country 2, thus raising the rental rate in country 2 asymptotically in relation to the rental rate in country 1. The increase in s_1 increases the capital stock in country 1 relative to that in country 2. An increase in s_2 has the opposite effect.

12) It can be seen from (45) that the competitive wage rate in the country with faster rate of growth of labor force falls over time relative to the wage rate in the other country.

4. The Model With Foreign Investment

In the model presented in Section 2, one country made a free grant of capital to the other. In this section we shall modify the mechanism of international capital flow as follows: we shall consider country 2 as relatively underdeveloped. Part of the capital stock of country 2, denoted by $K_{2f}(t)$ is assumed to be owned by investors in country 1. These investors receive $r_2(t)\ K_{2f}(t)$ as income on their investment where $r_2(t)$ as will be recalled is the instantaneous rate of rental on a unit of capital stock in country 2. The rate of change of $K_{2f}(t)$, will be made to depend on (a) the stock $K_{2f}(t)$ and (b) the spread between the rates of return to investment in the two countries. We shall continue to maintain the restriction that only currently produced capital goods and the consumer goods of country 2 move between countries. In other words transfer of a part of the existing stock of capital from one country to the other is ruled out. Let us denote by $A(t)$ the value of the change in $K_{2f}(t)$. Thus $A(t)$ represents the net capital flow from country 1 to country 2. Most of the equations (1)-(21) describing the model continue except for the following modifications:

(1)' $$Y_1(t) \equiv C_1(t) + q_1(t)\ \dot{K}_1(t)$$

(13)' $$Y_2(t) \equiv C_2(t) + q_1(t)\ \dot{K}_2(t)$$

(21)' $$p_2(t)X_2(t)+A(t) \equiv q_1(t)\ X_1(t) + r_2(t)\ K_{2f}(t)$$

Two new variables $A(t)$ and $K_{2f}(t)$ have entered the model. The following two additional equations complete the model:

$$(47) \qquad A(t) = q_1(t)\,\dot{K}_{2f}(t)$$

$$(48) \qquad \dot{K}_{2f}(t) = \rho\left[\frac{r_2(t)}{q_1(t)} - \frac{r_1(t)}{q_1(t)}\right]K_{2f}(t)$$

It is easy to see by the income identities (1) and (13) are changed. In the present model the trade between the two countries is balanced and hence net trade balance drops out of the income identities. The new balance of payments constraint says (if seen from country 2's view point) that imports from and factor payments to country 1 equal exports to and net foreign investment by the same country. Equation (47) states that the value of the change in capital stock owned by country 1 in country 2 equals net investment flow from country 1 to country 2. Equation (48) states that the rate of change in foreign and capital stock in country 2 is directly proportional to the level of that stock and the excess of the rate of return per unit of investment in country 2 over that in country 1.

It was not possible to obtain an explicit solution for this model covering all initial conditions. However, it is shown in Appendix B that a particular (and unique) balanced growth solution exists. It is again an open question whether in fact every other solution of the model converges (in a suitably defined sense) to this balanced growth solution. This solution has a number of properties which are very similar to the asymptotic properties of the model

CFDP No. 164 - Correction to
Model of Section 4

The income identities, to be consistent with the definitions given in Section 2 must read:

$$Y_1(t) \equiv C_1(t) + q_1(t)\,\dot{K}_1(t) + A(t)$$

$$Y_2(t) \equiv C_2(t) + q_1(t)\,\dot{K}_2(t) - A(t)$$

Equations (1)' and (2)' do not therefore correspond to incomes, but to total expenditures on consumption and investment. Needless to say that the results of this section are valid only when savings in either country are a constant proportion of total expenditures rather than of total income. I wish to thank Professor Arthur Okun for drawing my attention to this.

described in Section 2. We shall discuss in what follows this and other properties of this balanced growth solution.

Let g and η have the same definition as in Section 2. Let $\hat{u}$ be that root of the quadratic $Au^2 + Bu + C = 0$ which lies in

$$\left(0\ ,\ \frac{\alpha_1 s_1}{g[\alpha_1 s_1 + \beta_1(1-s_1)\epsilon]}\right) \quad \text{where}$$

$$A = g^2\left[\alpha_1 s_1 + \beta_1(1-s_1)\epsilon + \rho\,\beta_2\left\{\alpha_1(1-\epsilon)(1-s_1) - \frac{(1-s_2)}{s_2}\left\{\alpha_1 s_1 + \beta_1(1-s_1)\epsilon\right\}\right\}\right]$$

$$B = \alpha_1 g\left[\rho\left\{\alpha_1 s_1 + \beta_1(1-s_1)\epsilon + s_1\beta_2\frac{(1-s_2)}{s_2}\right\} - s_1\right] \text{ and}$$

$C = -\alpha_1^2\,\rho\,s_1$. The existence of such a $\hat{u}$ is shown in Appendix B.

The balance growth solution in the following:

$$K_1(t) = \hat{K}_1 e^{gt} \quad \text{where } \hat{K}_1 = \frac{\alpha_1 s_1 \eta\,\hat{u}^{\frac{1}{1-\alpha_1}}}{g(1-\eta)(1-s_1)\beta_1\epsilon\,\hat{u} + \alpha_1\eta s_1} \tag{49}$$

$$K_2(t) = \hat{K}_2 e^{gt} \quad \text{where } \hat{K}_2 = \frac{\hat{u}^{\frac{\alpha_1}{1-\alpha_1}}}{g} - \hat{K}_1\left[\frac{\alpha_1\eta s_1 + (1-s_1)\beta_1\epsilon}{\alpha_1\,\eta\,s_1}\right] \tag{50}$$

$$K_{2f}(t) = \hat{K}_{2f}\,e^{gt} \quad \text{where } \hat{K}_{2f} = \frac{g\hat{u}}{s_1}\left\{\frac{(1-\epsilon)(1-s_1)\,\hat{K}_1 - s_1\hat{K}_2}{\alpha_1 - g(1-\frac{1}{\rho})\,\hat{u}}\right\} \tag{51}$$

$$\left.\begin{aligned} \delta_{1k}(t) &= \hat{u}^{\frac{1}{1-\alpha_1}} e^{(g-\Theta_1)t} = \left\{\hat{u}\, e^{\sigma_{1k}t}\right\}^{\frac{1}{1-\alpha_1}}, \quad \delta_{1c}(t) = \eta\, \delta_{1k}(t) \\ \delta_1(t) &= \hat{K}_1\, e^{(g-\Theta_1)t} = \hat{K}_1\, e^{\frac{\sigma_{1k}}{1-\alpha_1}t} \end{aligned}\right\} \tag{52}$$

$$\left.\begin{aligned} L_{1k}(t) &= \hat{L}_{1K}\, e^{\Theta_1 t}, \quad L_{1c}(t) = \hat{L}_{1c}\, e^{\Theta_1 t} \qquad \text{where} \\ \hat{L}_{1k} &= \frac{K_1 - \eta\, \hat{u}^{\frac{1}{1-\alpha_1}}}{(1-\eta)\, \hat{u}^{\frac{1}{1-\alpha_1}}} \quad \text{and} \quad \hat{L}_{1c} = \frac{\hat{u}^{\frac{1}{1-\alpha_1}} - \hat{K}_1}{(1-\eta)\, \hat{u}^{\frac{1}{1-\alpha_1}}} \end{aligned}\right\} \tag{53}$$

$$Y_1(t) = \left[(1-\alpha_1)\, \hat{u}^{\frac{\alpha_1}{1-\alpha_1}} + g\epsilon(1-\beta_1)\, \hat{K}_1\right] \frac{\eta^{\beta_1}\, \hat{u}^{\frac{\beta_1-\alpha_1}{1-\alpha_1}}}{\epsilon}\, e^{\left\{\sigma_{1c}+\Theta_1+\beta_1\frac{\sigma_{1k}}{1-\alpha_1}\right\}t} \tag{54}$$

$$\frac{Y_2(t)}{Y_1(t)} = \frac{s_1}{s_2}\, \frac{\hat{K}_2}{\hat{K}_1}, \quad \frac{C_1(t)}{C_2(t)} = \frac{(1-s_2)\, s_1}{(1-s_1)\, s_2}\, \frac{\hat{K}_2}{\hat{K}_1} \tag{55}$$

$$q_1(t) = \left(\frac{1-\beta_1}{1-\alpha_1}\right) \eta^{\beta_1}\, \hat{u}^{\frac{\beta_1-\alpha_1}{1-\alpha_1}}\, e^{\left\{\sigma_{1c} - \frac{(1-\beta_1)}{1-\alpha_1}\sigma_{1k}\right\}t} \tag{56}$$

$$r_1(t) = \frac{\alpha_1 q_1(t)}{\hat{u}}, \quad w_1(t) = (1-\alpha_1)\, \hat{u}^{\alpha_1}\, e^{\frac{\sigma_{1k}}{1-\alpha_1}t} \tag{57}$$

$$(58) \qquad \frac{r_2(t)}{r_1(t)} = 1 + \frac{g}{\rho}\,\frac{\hat{u}}{\alpha_1}\,, \quad \frac{w_2(t)}{w_1(t)} = \frac{g\left\{s_2(1-\epsilon)(1-s_1)\hat{K}_1 + s_1(1-s_2)\hat{K}_2\right\}}{s_1 s_2 (1-\alpha_1)\,\hat{u}^{\alpha_1/1-\alpha_1}}\, e^{(\theta_1-\theta_2)t}$$

$$(59) \qquad \frac{p_2(t)}{q_1(t)} = \left\{\frac{s_2(1-\epsilon)(1-s_1)\,\hat{K}_1 + s_1(1-s_2)\hat{K}_2}{s_1 s_2\,\hat{K}_2}\right\}^{\beta_2} g\, e^{(1-\beta_2)(g-g')t} \quad \text{where} \quad g'$$

$$= \frac{\sigma_{2c}}{1-\beta_2} + \theta_2$$

It is shown in Appendix B that the values of $\hat{K}_1$, $\hat{K}_2$, $\hat{L}_{1k}$, $\hat{L}_{1c}$ as given above are non-negative. With some restrictions on the parameters it is possible to ensure that the $\hat{K}_{2f}$ associated with the balanced growth solution is economically meaningful i.e., it lies in $[0, \hat{K}_2]$. Let us now proceed to a discussion of the properties of the solution given by (49)-(58).

It is possible to view the role of the parameter ρ in the present model as similar to a certain extent to the role of the parameter λ in the model of Section 2. It can be verified that the solution of the present model for $\rho = 0$ is the same as the asymptotic solution of the earlier model for $\lambda = 1$. This is as it should be since $\rho = 0$ implies (for the balanced growth solution) that there is no capital flow between the two countries and trade is balanced. The case $\lambda = 1$ has the same interpretation in the model of Section 1. The behavior of the solution (49)-(58) as ρ varies is of some interest. It can be shown that as ρ (the factor by which a given spread between the rates of return per unit investment in the two countries gets magnified into capital flow) increases, $\hat{u}$ decreases. This in turn implies, as can be shown from (49), (50), (52) and (53) that both $\delta_{1k}(t)$ and $L_{1k}(t)$ increase for all t. In other words an increase in ρ means a decrease in the rate of output of capital goods in country 1. It can also be shown that the stock of capital in country 2 at any t increases relative to that in country 1. As could be expected this shift in the ratio of capital stocks results in a decrease in the rental per unit of capital in country 2 relative to that in country 1. This is seen from (58). From (55), it can be shown that both income and consumption in country 2 increase relative to their values in country 1.

The behavior of the balanced growth solution of the present model is essentially the same as the asymptotic behavior of the model of Section 2. Conclusions (4) - (10) and (12) of Section 2 continue to hold true. Conclusion (11) of Section 2 needs modification. There an increase in the savings ratio s_2 of country 2 resulted in a decrease in the asymptotic value of the ratio of rental per unit of capital in country 2 to that in country 1. In the present solution an increase in s_2 has an exactly opposite effect -- it increases the ratio of rentals. This conclusion is surprising and no intuitive explanation of it is obvious.

APPENDIX A

Let us solve the system of equations representing the model of Section 2.

From equations (11) and (12) and using the definitions of the factor ratio variables we get:

$$\frac{1-\beta_1}{\beta_1}\ \delta_{1c}(t) = \frac{1-\alpha_1}{\alpha_1}\ \delta_{1k}(t)$$

(A1) or
$$\delta_{1c}(t) = \eta\ \delta_{1k}(t)$$

where
$$\eta = \frac{\beta_1(1-\alpha_1)}{\alpha_1(1-\beta_1)}$$

Using (A1) in (12) we get

(A2)
$$q_1(t) = \left(\frac{1-\beta_1}{1-\alpha_1}\right)\eta^{\beta_1}\ \delta_{1k}(t)^{\ \beta_1-\alpha_1}\ e^{(\sigma_{1c}-\sigma_{1k})t}$$

Using (2), (4) and (21) in (1) we obtain

(A3)
$$q_1(t)\ [\dot{K}_1(t) + (1-\lambda)\ X_1(t)] = \frac{s_1}{1-s_1}\ C_1(t) = \frac{s_1}{(1-s_1)}\ \frac{1}{\epsilon}\ Q_{1c}(t)$$

Now from (5) and (7) we get

$$\dot{K}_1(t) + X_1(t) = e^{\sigma_{1k}t}\ L_{1k}(t)\ \delta_{1k}(t)^{\ \alpha_1} \qquad \text{or}$$

$$q_1(t)\left[\dot{K}_1(t) + (1-\lambda)\, X_1(t)\right] = q_1(t)\, e^{\sigma_{1k}t}\, L_{1k}(t)\, \delta_{1k}(t)^{\alpha_1} - \lambda q_1(t)\, X_1(t)$$

$$= q_1(t)\, e^{\sigma_{1k}t}\, L_{1k}(t)\, \delta_{1k}(t)^{\alpha_1} - p_2(t)\, X_2(t) \quad \text{using (21)}$$

$$= q_1(t)\, e^{\sigma_{1k}t}\, L_{1k}(t)\, \delta_{1k}(t)^{\alpha_1} - \frac{1-\epsilon}{\epsilon}\, Q_{1c}(t) \quad \text{using (3) and (4)}$$

Using (A3) this can be simplified to yield

$$\text{(A4)} \quad \left[\frac{s_1 + (1-\epsilon)(1-s_1)}{\epsilon(1-s_1)}\right] Q_{1c}(t) = q_1(t)\, e^{\sigma_{1k}t}\, L_{1k}(t)\, \delta_{1k}^{\alpha_1}$$

Substituting for $Q_{1c}(t)$ and $q_1(t)$ from (6) and (A2) respectively and using (A1) we get

$$\left[\frac{s_1 + (1-\epsilon)(1-s_1)}{\epsilon(1-s_1)}\right] e^{\sigma_{1c}t}\, L_{1c}(t)\, \eta^{\beta_1}\, \delta_{1k}(t)^{\beta_1}$$

$$= \left[\left(\frac{1-\beta_1}{1-\alpha_1}\right) \eta^{\beta_1}\, \delta_{1k}(t)^{\beta_1-\alpha_1}\, e^{(\sigma_{1c}-\sigma_{1k})t}\right] e^{\sigma_{1k}t}\, L_{1k}(t)\, \delta_{1k}(t)^{\alpha_1}$$

Canceling out terms common to both sides we have

$$\left[\frac{s_1 + (1-\epsilon)(1-s_1)}{\epsilon(1-s_1)}\right] L_{1c}(t) = \left(\frac{1-\beta_1}{1-\alpha_1}\right) L_{1k}(t) \qquad \text{or}$$

$$\text{(A5)} \qquad L_{1c}(t) = \mu L_{1k}(t) \quad ,$$

where $$\mu = \frac{\epsilon(1-\beta_1)(1-s_1)}{(1-\alpha_1)[s_1 + (1-\epsilon)(1-s_1)]}$$

Using (A5) in (9) and (10) we get

$$L_{1c}(t) = \left(\frac{\mu}{1+\mu}\right) e^{\theta_1 t}, \quad L_{1k}(t) = \left(\frac{1}{1+\mu}\right) e^{\theta_1 t} \tag{A6}$$

We can rewrite (8) as follows:

$$\delta_{1k}(t)\, L_{1k}(t) + \delta_{1c}(t)\, L_{1c}(t) = \delta_1(t)\, L_1(t) \tag{A7}$$

Using (10), (A1) and (A6) in (A7) we obtain

$$\left(\frac{1+\eta\mu}{1+\mu}\right) \delta_{1k}(t) = \delta_1(t) = e^{-\theta_1 t} K_1(t) \tag{A8}$$

From (5) and (7) we get

$$\dot{K}_1(t) = Q_{1k}(t) - X_1(t)$$

$$= Q_{1k}(t) - \frac{p_2(t)\, X_2(t)}{\lambda q_1(t)} \qquad \text{[because of (21)]}$$

$$= Q_{1k}(t) - \left(\frac{1-\epsilon}{\epsilon}\right) \frac{Q_{1c}(t)}{\lambda q_1(t)} \qquad \text{[using (3) and (4)]} \tag{A9}$$

Using (6) and (7) we can rewrite (12) as

$$(1-\beta_1) \frac{Q_{1c}(t)}{L_{1c}(t)} = q_1(t)\, (1-\alpha_1) \frac{Q_{1k}(t)}{L_{1k}(t)} \tag{12'}$$

Hence $$\frac{Q_{1c}}{q_1(t)} = \left(\frac{1-\alpha_1}{1-\beta_1}\right)\frac{L_{1c}(t)}{L_{1k}(t)} \quad Q_{1k}(t)$$

$$= \frac{(1-\alpha_1)\mu}{(1-\beta_1)} Q_{1k}(t) \qquad \text{(A10)} \qquad \text{[using (A6)]}$$

Using (A10) in (A9) we get

$$\dot{K}_1 = \left[\;\; - \quad \frac{(1-\epsilon)1-\alpha_1)\mu}{\epsilon(1-\beta_1)\lambda}\right] Q_{1k}(t)$$

$$= \left[1 - \quad \frac{(1-\epsilon)(1-\alpha_1)\mu}{\lambda\epsilon(1-\beta_1)}\right] e^{\sigma_{1k}t} L_{1k}(t)\ \delta_{1k}(t)^{\alpha_1}$$

$$= \Phi(\lambda)\ e^{\sigma_{1k}t} \quad \frac{1}{1+\mu} \quad e^{\theta_1 t} \left[\frac{e^{-\theta_1 t} \quad K_1(t)(1+\mu)}{1+\eta\mu}\right]^{\alpha_1} \qquad \text{[using (A6) and (A8)]}$$

where $$\Phi(\lambda) = \left[1 - \frac{(1-\epsilon)(1-\alpha_1)\mu}{\lambda\epsilon(1-\beta_1)}\right] = \left[1 - \frac{(1-\epsilon)(1-s_1)}{\lambda\left\{s_1 + (1-\epsilon)(1-s_1)\right\}}\right]$$

Hence

$$\text{(A 11)} \quad K_1(t)^{-\alpha_1}\,\dot{K}_1(t) = \left[\frac{\Phi(\lambda)}{(1+\mu)^{1-\alpha_1}(1+\eta\mu)^{\alpha_1}}\right] e^{[\sigma_{1k} + (1-\alpha_1)\theta_1]t}$$

Equation (A 11) is the fundamental differential equation of our international economy. Solving (A 11) we get

$$\text{(A12)} \quad K_1(t) = \left[K_1^{1-\alpha_1}(0) + \frac{(1-\alpha_1)\ \Phi(\lambda)}{\left\{\sigma_{1k}+\theta_1(1-\alpha_1)\right\}(1+\mu)^{1-\alpha_1}(1+\eta\mu)^{\alpha_1}}\left\{e^{[\sigma_{1k}+(1-\alpha_1)\theta_1]t} - 1\right\}\right]^{\frac{1}{1-\alpha_1}}$$

Let us define

$$\text{(A13)}\qquad \bar{K}_1^{1-\alpha_1}(0) = \frac{(1-\alpha_1)\,\Phi(\lambda)}{\left\{\sigma_{1k} + \Theta(1-\alpha_1)\right\}(1+\mu)^{1-\alpha_1}(1+\eta\mu)^{\alpha_1}}$$

Then we can rewrite (A12) as follows:

$$\text{(A14)}\quad K_1(t) = \left[\left\{K_1^{1-\alpha_1}(0) - \bar{K}_1^{1-\alpha_1}(0)\right\} + \left\{\bar{K}_1(0)\, e^{\left\{\left(\frac{1}{1-\alpha_1}\right)\sigma_{1k} + \Theta_1\right\}t}\right\}^{1-\alpha_1}\right]^{\frac{1}{1-\alpha_1}}$$

Using (A14) we can solve for the rest of the variables of our model:

$$\text{(A15)}\quad \delta_1(t) = \left[\left\{\delta_1^{1-\alpha_1}(0) - \bar{\delta}_1^{1-\alpha_1}(0)\right\} e^{-\Theta_1(1-\alpha_1)t} + \bar{\delta}^{1-\alpha_1}(0)\, e^{\sigma_{1k}t}\right]^{\frac{1}{1-\alpha_1}}$$

<u>Country 1</u>

$$\text{(A16)}\qquad \left.\begin{aligned} \delta_{1k}(t) &= \left(\frac{1+\mu}{1+\eta\mu}\right)\delta_1(t) \\ \delta_{1c}(t) &= \left[\frac{\eta(1+\mu)}{(1+\eta\mu)}\right]\delta_1(t) \end{aligned}\right]$$

$$\text{(A17)}\qquad Q_{1c}(t) = \left[\frac{\mu\eta^{\beta_1}}{(1+\mu)^{1-\beta_1}(1+\eta\mu)^{\beta_1}}\right] e^{(\sigma_{1c} + \Theta_1)t}\, \delta_1^{\beta_1}(t)$$

$$\text{(A18)}\qquad Q_{1k}(t) = \left[\frac{1}{(1+\mu)^{1-\alpha_1}(1+\eta\mu)^{\alpha_1}}\right] e^{(\sigma_{1k} + \Theta_1)t}\, \delta_1^{\alpha_1}(t)$$

(A19) $$q_1(t) = \left(\frac{1-\beta_1}{1-\alpha_1}\right)\left(\frac{1+\mu}{1+\eta\mu}\right)^{\beta_1-\alpha_1} \eta^{\beta_1} e^{(\sigma_{1c}-\sigma_{1k})t} \delta_1(t)^{\beta_1-\alpha_1}$$

(A20) $$w_1(t) = (1-\beta_1)\, e^{\sigma_{1c}t} \delta_{1c}(t)^{\beta_1} = (1-\beta_1) \frac{Q_{1c}(t)}{L_{1c}(t)}$$

(A21) $$r_1(t) = \beta_1\, e^{\sigma_{1c}t} \delta_{1c}^{-(1-\beta_1)} = \beta_1 \frac{Q_{1c}(t)}{L_{1c}(t)} \delta_{1c}(t)^{-\beta_1}$$

(A22) $$C_1(t) = \frac{1}{\epsilon} Q_{1c}(t)$$

(A23) $$q_1(t)\, X_1(t) = \frac{1}{\lambda} p_2(t)\, X_2(t) = \frac{1}{\lambda} \quad \frac{1-\epsilon}{\epsilon} Q_{1c}(t)$$

(A24) $$\left.\begin{aligned} X_1(t) &= \frac{\mu}{\lambda}\left(\frac{1-\epsilon}{\epsilon}\right)\left(\frac{1-\alpha_1}{1-\beta_1}\right)\left[\frac{1}{(1+\mu)^{1-\alpha_1}(1+\eta\mu)^{1-\alpha_1}}\right] e^{(\sigma_{1k}+\theta_1)t} \delta_1(t)^{\alpha_1} \\ &= \frac{\mu}{\lambda}\left(\frac{1-\epsilon}{\epsilon}\right)\left(\frac{1-\alpha_1}{1-\beta_1}\right) Q_{1k}(t) \\ &= \frac{1}{\lambda}\left[\frac{(1-\epsilon)(1-s_1)}{s_1+(1-\epsilon)(1-s_1)}\right] Q_{1k}(t) \end{aligned}\right]$$

(A25) $$\left.\begin{aligned} Y_1(t) &= \left(\frac{1}{1-s_1}\right) C_1(t) \\ &= \left[\frac{1}{\epsilon(1-s_1)}\right] Q_{1c}(t) \end{aligned}\right]$$

(A26) $$\dot{K}_1 = Q_{1k} - X_1(t) = \left[1 - \frac{1}{\lambda}\left\{\frac{(1-\epsilon)(1-s_1)}{s_1 + (1-\epsilon)(1-s_1)}\right\}\right] Q_{1k}(t)$$

Country 2

$$\dot{K}_2(t) = X_1(t) = \left\{ \frac{(1-\epsilon)\ (1-s_1)}{\lambda s_1 - \ (1-\lambda)(1-\epsilon)(1-s_1)} \right\} \dot{K}_1(t)$$

(A27) $$K_2(t) = K_2(0) + \left\{ \frac{(1-\epsilon)\ (1-s_1)}{\lambda s_1 - (1-\lambda)(1-\epsilon)(1-s_1)} \right\} \left\{ K_1(t) - K_1(0) \right\}$$

(28) $$\left. \begin{aligned} Y_2(t) &= C_2(t) + p_2(t)\ X_2(t) = (1-s_2)\ Y_2(t) + p_2(t)\ X_2(t) \\ &= \frac{1}{s_2}\ p_2(t)\ X_2(t) = \frac{\lambda}{s_2}\ \ q_1(t)\ X_1(t) \\ &= \frac{\lambda}{s_2}\ \frac{(1-\epsilon)(1-s_1)}{\lambda}\ Y_1(t) = \frac{(1-\epsilon)(1-s_1)}{s_2}\ Y_1(t) \end{aligned} \right]$$

(29) $$Q_{2c}(t) = e^{[\sigma_{2c} + \Theta_2(1-\beta_2)]t}\ K_2^{\beta_2}(t)$$

(30) $$X_2(t) = s_2\ Q_{2c}(t)$$

(A31) $$\left. \begin{aligned} p_2(t) &= \frac{p_2(t)\ X_2(t)}{X_2(t)} = \frac{\lambda q_1(t)\ X_1(t)}{K_2(t)} \\ &= \left(\frac{1-\epsilon}{\epsilon}\right)\ \frac{Q_{1c}(t)}{s_2 Q_{2c}(t)} \end{aligned} \right]$$

(A32) $$w_2(t) = (1-\beta_2)\ p_2(t)\ \frac{Q_{2c}(t)}{L_{2c}(t)} = (1-\beta_2)\ e^{-\Theta_2 t}\ p_2(t)\ Q_{2c}(t)$$

(A33) $$r_2(t) = \frac{\beta_2 p_2(t) Q_{2c}(t)}{K_{2c}(t)}$$

Intercountry Comparisons

(A34) $$\frac{Y_2(t)}{Y_1(t)} = \frac{(1-\epsilon)(1-s_1)}{s_2}$$

(A35) $$\frac{C_2(t)}{C_1(t)} = \frac{(1-s_2)(1-\epsilon)}{s_2}$$

(A36) $$\frac{p_2(t)}{q_1(t)} = \lambda\ \frac{X_1(t)}{X_2(t)} = \left\{ \frac{(1-\epsilon)\ (1-s_1)}{s_1 + (1-\epsilon)(1-s_1)} \right\} \frac{Q_{1k}(t)}{s_2 Q_{2c}(t)}$$

Using (A6), (A16), (A20), (A31) and (A32) we get

(37) $$\frac{w_2(t)}{w_1(t)} = \left(\frac{1-\beta_2}{1-\beta_1}\right)\left(\frac{1-\epsilon}{s_2\epsilon}\right)\left(\frac{\mu}{1+\mu}\right) e^{(\theta_1-\theta_2)t}$$

Using (A6), (A16), (A21), (A31) and (A33) we get

(38) $$\frac{r_2(t)}{r_1(t)} = \left(\frac{\beta_2}{\beta_1}\right)\left(\frac{(1-\epsilon)}{s_2\epsilon}\right)\left(\frac{\eta\mu}{1+\eta\mu}\right) \frac{K_1(t)}{K_2(t)}$$

From (A14) and (A27) it is clear that

$$\lim_{t\to\infty} \frac{K_1(t)}{K_2(t)} = \frac{\lambda s_1 - (1-\lambda)(1-\epsilon)(1-s_1)}{(1-\epsilon)(1-s_1)}$$

$$= \left(\frac{\lambda - \underline{\lambda}}{\underline{\lambda}}\right) \quad \text{using (46)}$$

Hence

(A39) $$\lim_{t\to\infty} \frac{r_2(t)}{r_1(t)} = \frac{\beta_2}{\beta_1}\left(\frac{(1-\epsilon)}{s_2\epsilon}\right)\left(\frac{\eta\mu}{1+\eta\mu}\right)\left(\frac{\lambda}{\underline{\lambda}} - 1\right)$$

Since μ and $\underline{\lambda}$ do not involve s_2 and λ it is clear that this limit is a decreasing function of s_2 and an increasing function of λ.

APPENDIX B

It will be recalled that equations (2)-(12), (14)-(20) of the model described in Section 2 continue to hold for the model of Section 3. Equations (1), (13) and (21) are replaced by (1)', (13)' and (21)' of page 20. Two additional equations (47) and (48) complete the model of Section 3. It is seen that the following equations developed in Appendix 1 continue to hold:

$$\text{(B1)} \qquad \delta_{1c}(t) = \eta\ \delta_{1k}(t) \quad \text{where} \quad \eta = \frac{\beta_1(1-\alpha_1)}{\alpha_1(1-\beta_1)}$$

$$\text{(B2)} \qquad q_1(t) = \left(\frac{1-\beta_1}{1-\alpha_1}\right) \eta^{\beta_1}\, e^{(\sigma_{1c}-\sigma_{1k})t}\, \delta_{1k}(t)^{\beta_1-\alpha_1}$$

Using (B1), (8a), (9) and (10) we get

$$\text{(B3)} \qquad L_{1c} = \left\{\frac{\delta_{1k}(t) - \delta_1(t)}{(1-\eta)\delta_{1k}(t)}\right\} e^{\theta_1 t}, \qquad L_{1k}(t) = \left\{\frac{\delta_1(t)-\eta\ \delta_{1k}(t)}{(1-\eta)\delta_{1k}(t)}\right\} e^{\theta_1 t}$$

Using (B3), (5), (7) and (16) we obtain

$$\text{(B4)} \qquad \dot{K}_1(t) + \dot{K}_2(t) = Q_{1k}(t) = e^{(\sigma_{1k}+\theta_1)t} \left\{\frac{\delta_1(t) - \eta\ \delta_{1k}(t)}{(1-\eta)\ \delta_{1k}(t)}\right\} \delta_{1k}(t)^{\alpha_1}$$

Let us define

$$\text{(B5)} \qquad u(t) = e^{\theta_1(1-\alpha_1)t}\ \delta_{1k}^{(1-\alpha_1)t}$$

Using (B5) we can rewrite (B4) as follows:

$$\text{(B6)} \qquad \dot{K}_1(t) + \dot{K}_2(t) = \left\{ \frac{K_1(t) - \eta\, u(t)^{\frac{1}{1-\alpha_1}}}{(1-\eta)\, u(t)} \right\} e^{(1-\alpha_1)gt} \quad \text{where} \quad \varepsilon = \frac{\sigma_{1k}}{1-\alpha_1} + \theta_1$$

Using (1)', (2) and (4) we can show that

$$\dot{K}_1(t) = \frac{s_1}{1-s_1} \; \frac{Q_{1c}(t)}{\epsilon q_1(t)} = \frac{s_1\, e^{\sigma_{1c}t} L_{1c}(t)\, \delta_{1c}(t)^{\beta_1}}{\epsilon(1-s_1)\, q_1(t)} \quad \text{(using (6))}$$

$$= \frac{s_1\, e^{\sigma_{1c}t}\, \eta^{\beta_1}\, \delta_{1k}(t)^{\beta_1}\, L_{1c}(t)}{\epsilon(1-s_1) \cdot \frac{1-\beta_1}{1-\alpha_1}\, \eta^{\beta_1}\, e^{(\sigma_{1c}-\sigma_{1k})t}\, \delta_{1k}(t)^{\beta_1-\alpha_1}} \quad \text{(using B2)}$$

$$= \left\{ \frac{s_1 \alpha_1 \eta\, e^{\sigma_{1k}t}\, \delta_{1k}(t)^{\alpha_1}}{(1-s_1)\, \beta_1\, \epsilon} \cdot \right\} \left\{ \frac{\delta_{1k}(t) - \delta_1(t)}{(1-\eta)\, \delta_{1k}(t)} \right\} e^{\theta_1 t} \quad \text{(using B3 and the definition of } \eta \text{)}$$

$$\text{(B7)} \qquad = \left\{ \frac{s_1\, \alpha_1\, \eta}{(1-s_1)\, \beta_1 \epsilon} \right\} \left\{ \frac{u(t)^{\frac{1}{1-\alpha_1}}\; K_1(t)}{(1-\eta)\, u(t)} \right\} e^{(1-\alpha_1)gt} \quad \text{(using B5)}$$

From (11) we get
$$\frac{r_1(t)}{q_1(t)} = \alpha_1\, e^{\sigma_{1k}(t)}\, \delta_{1k}(t)^{-(1-\alpha_1)}$$

$$= \frac{\alpha_1\, e^{(1-\alpha_1)gt}}{u(t)} \qquad \text{using (B5)}$$

From (19) we get $r_2(t) = \dfrac{\beta_2 p_2(t) Q_{2c}(t)}{K_2(t)}$. In view of (13)', (14) and (15) we can assert that:

$$p_2(t)\, Q_{2c}(t) = p_2(t)\, X_2(t) + \left(\frac{1-s_2}{s_2}\right) q_1(t)\, K_2(t)$$

$$= \left\{\frac{(1-\epsilon)(1-s_1)}{s_1}\right\} q_1(t)\dot{K}_1(t) + \left(\frac{1-s_2}{s_2}\right) q_1(t)\, \dot{K}_2(t)$$

Hence $\dfrac{r_2(t)}{q_1(t)} = \beta_2 \dfrac{p_2(t)Q_{2c}(t)}{q_1(t)K_2(t)} = \beta_2 \dfrac{(1-\epsilon)(1-s_1)\dot{K}_1(t)}{s_1 K_2(t)} + \beta_2\left(\dfrac{1-s_2}{s_2}\right) \dfrac{\dot{K}_2(t)}{K_2(t)}$

Substituting the expressions for $\dfrac{r_1(t)}{q_1(t)}$ and $\dfrac{r_2(t)}{q_1(t)}$ in (48) we get

$$\text{(B8)} \quad \dot{K}_{2f}(t) = \rho \left[\beta_2 \frac{(1-\epsilon)(1-s_1)\dot{K}_1(t)}{s_1 K_2(t)} + \beta_2\left(\frac{1-s_2}{s_2}\right) - \frac{\alpha_1 e^{(1-\alpha_1)gt}}{u(t)}\right] K_{2f}(t)$$

Using (16), (47) in (21)' and substituting for $p_2(t)\, X_2(t)$ and $r_2(t)$ from the above derivations we get:

$$\left\{\frac{(1-\epsilon)(1-s_1)}{s_1}\right\} q_1(t)\dot{K}_1(t) + q_2(t)\dot{K}_{2f}(t) = q_1(t)\dot{K}_2(t)$$

$$+ \beta_2 q_1(t) \left\{\frac{(1-\epsilon)(1-s_1)\dot{K}_1(t)}{s_1\, K_2(t)} + \frac{1-s_2}{s_2} \quad \frac{\dot{K}_2(2)}{K_2(t)}\right\} K_{2f}(t)$$

Assuming $q_1(t) \neq 0$ we can rewrite the above equation as

$$\text{(B9)} \qquad \left\{ \frac{(1-\epsilon)(1-s_1)}{s_1} \right\} \dot{K}_1(t) + \dot{K}_{2f}(t) = \dot{K}_2(t)$$

$$+ \beta_2 \left\{ \frac{(1-\epsilon)(1-s_1)\dot{K}_1(t)}{s_1 K_\epsilon(t)} + \left(\frac{1-s_2}{s_2} \right) \frac{\dot{K}_2(t)}{K_2(t)} \right\} K_{2f}(t)$$

Equations (B6), (B7), (B8) and (B9) are the four simultaneous differential equations which determine $K_1(t)$, $K_{2f}(t)$ and $u(t)$. Once these are determined the remaining variables can be solved for by substitution in appropriate equations. It is clear how one could construct a solution to this system of differential equations starting from given initial values for $K_1(t)$, $K_2(t)$ and $K_{2f}(t)$. For, we can solve for $\dot{K}_1(t)$, $\dot{K}_2(t)$ and $\dot{K}_{2f}(t)$ uniquely in terms of $K_1(t)$, $K_2(t)$, $K_{2f}(t)$ and $u(t)$ using (B6) - (B8). Substituting in (B9) we can then solve for $u(t)$ in terms of $K_1(t)$, $K_2(t)$ and $K_{2f}(t)$. It can be shown that either (a) <u>no</u> nonnegative solution or (b) an unique nonnegative solution for $u(t)$ exists. If we do have a nonnegative $u(t)$, we are left with equations which determine $\dot{K}_1(t)$, $\dot{K}_2(t)$ and $\dot{K}_{2f}(t)$ uniquely in terms of $K_1(t)$, $K_2(t)$ and $K_{2f}(t)$. Thus given any initial values for $K_1(t)$, $K_2(t)$ and $K_{2f}(t)$ the future time paths of these variables are uniquely determined; assuming that at no t we run into a situation where either no negative $u(t)$ exists or some other variable of the system violates the nonnegativity constraints imposed by the problem.

It is thus possible in principle to obtain (using a computer) the unique solution of the system when it exists. However it was not possible to obtain the same solution analytically. We therefore searched for a possible solution in which, if the initial conditions were appropriate, the variables $K_1(t)$, $K_2(t)$ and $K_{2f}(t)$ will grow exponentially at the same rate. From an inspection of (B6) - (B9) it is obvious that this common rate of growth has to be g. Therefore, the following solution was tried:

$$K_1(t) = \hat{K}_1 e^{gt}, \quad K_2(t) = \hat{K}_2 e^{gt}, \quad K_{2f}(t) = \hat{K}_{2f} e^{gt} \quad \text{and}$$

$u(t) = \hat{u} e^{(1-\alpha_1)gt}$. This solution will be economically meaningful only if $\hat{K}_1$, $\hat{K}_2$, $\hat{u} \geqq 0$ and $0 \leqq \hat{K}_{2f} \leqq \hat{K}_2$. Substitution in (B6) - (B9) gives us the following equations to determine $\hat{K}_1$, $\hat{K}_2$, $\hat{K}_{2f}$ and $\hat{u}$.

$$\text{(B10)} \qquad g[\hat{K}_1 + \hat{K}_2] = \left\{ \frac{\hat{K}_1 - \eta \hat{u}}{(1-\eta)\hat{u}} \right\}^{\frac{1}{1-\alpha_1}}$$

$$\text{(B11)} \qquad g \hat{K}_1 = \frac{\alpha_1 s_1 \eta}{(1-s_1)\beta_1 \epsilon} \left\{ \frac{\hat{u}^{\frac{1}{1-\alpha_1}} - \hat{K}_1}{(1-\eta)\hat{u}} \right\}$$

(B12) $$g = \rho\left[\frac{\beta_2(1-\epsilon)(1-s_1)g}{s_1\hat{K}_2}\hat{K}_1 + \frac{\beta_2(1-s_2)g}{s_2} - \frac{\alpha_1}{\hat{u}}\right]$$

(B13) $$\left\{\frac{(1-\epsilon)(1-s_1)g}{s_1}\right\}\hat{K}_1 + g\,\hat{K}_{2f} = g\,\hat{K}_2 + \beta_2\left[\frac{(1-\epsilon)(1-s_1)g}{s_1}\frac{\hat{K}_1}{\hat{K}_2} + \frac{1-s_2}{s_2}g\right]\hat{K}_{2f}$$

From (B11) we get

(B14) $$\hat{K}_1 = \frac{\alpha_1 s_1 \eta\,\hat{u}^{\frac{1}{1-\alpha_1}}}{g\,(1-\eta)(1-s_1)\,\beta_1\epsilon\,\hat{u} + \alpha_1 s_1\eta}$$

Using (B14) in (B10),

(B15) $$\hat{K}_2 = \frac{1}{g}\left\{\frac{(1-s_1)\beta_1\epsilon + \alpha_1 s_1}{(1-\eta)}\right\}\eta\hat{u}^{\frac{\alpha_1}{1-\alpha_1}} - \frac{\hat{K}_1}{g\,\hat{u}}\left\{\frac{(1-s_1)\beta_1\,\epsilon + \alpha_1\eta s_1}{(1-\eta)}\right\}$$

$$= \frac{\hat{u}}{g}^{\frac{\alpha_1}{1-\alpha_1}} - \left\{\frac{(1-s_1)\beta_1\epsilon + \alpha_1\eta s_1}{\alpha_1\eta s_1}\right\}\hat{K}_1$$

Hence $$\frac{\hat{K}_2}{\hat{K}_1} = \frac{g(1-\eta)(1-s_1)\beta_1\,\epsilon\,\hat{u} + \alpha_1 s_1\eta}{g\,\alpha_1\eta s_1\,\hat{u}} - \frac{(1-s_1)\beta_1\,\epsilon + \alpha_1\,\eta\,s_1}{\alpha_1\eta s_1}$$

$$= \frac{\alpha_1 s_1 - g\left\{(1-s_1)\beta_1\epsilon + \alpha_1 s_1\right\}\hat{u}}{g\,\alpha_1 s_1\,\hat{u}}$$

or

$$\text{(B16)} \qquad \frac{\hat{K}_1}{\hat{K}_2} = \left\{ \frac{g\, \alpha_1\, s_1\, \hat{u}}{\alpha_1 s_1 - g\left\{(1-s_1)\beta_1 \epsilon + \alpha_1 s_1\right\} \hat{u}} \right\}$$

Substituting (B16) in (B12) we get:

$$g = \rho \left[\frac{\beta_2(1-\epsilon)(1-s_1)\, g^2\, \alpha_1\, \hat{u}}{[\alpha_1 s_1 - g\ \ (1-s_1)\beta_1\ \epsilon + \alpha_1 s_1\ \ \hat{u}]} + \frac{\beta_2(1-s_2)}{s_2} g - \frac{\alpha_1}{\hat{u}} \right]$$

or $A\,\hat{u}^2 + B\,\hat{u} + C\,\hat{u} = 0$ where

$$A = g^2 \left[\left(\alpha_1 s_1 + \beta_1(1-s_1)\epsilon\right) + \rho\, \beta_2 \left\{ \alpha_1(1-\epsilon)(1-s_1) - \frac{(1-s_2)}{s_2}\left\{\alpha_1 s_1 + \beta_1(1-s_1)\epsilon\right\}\right\} \right]$$

$$B = g\, \alpha_1 \left[\rho \left\{ (1-s_1)\beta_1\ \epsilon + \alpha_1 s_1 + s_1 \beta_2 \frac{(1-s_2)}{s_2} \right\} - s_1 \right]$$

$$C = -\alpha_1^2\, \rho\, s_1$$

It is clear from (B16) that $\frac{\hat{K}_1}{\hat{K}_2} \geqq 0$ for $\hat{u} \geqq 0$ whenever $\hat{u}$

$\leq \frac{\alpha_1 s_1}{g[(1-s_1)\beta_1 \epsilon + \alpha_1 s_1]}$. From (B14) it is obvious that $\hat{K}_1 \geq 0$ for $\hat{u} > 0$

if $0 \leq \eta \leq 1$. If $\eta > 1$ $\hat{K}_1 > 0$ for $\hat{u} \geq 0$ only if

$\hat{u} \leq \dfrac{\alpha_1 s_1 \eta}{(\eta-1)g[(1-s_1)\beta_1\epsilon + \alpha_1 s_1]}$. For $\eta > 1$ it is obvious that $\dfrac{\eta}{\eta-1} > 1$.

Hence if we can show that there exists a root $\hat{u}$ (of $Au^2 + Bu + C = 0$)

satisfying $\hat{u} \geq 0$ and $\hat{u} \leq \dfrac{\alpha_1 s_1}{g\,[(1-s_1)\beta_1\epsilon + \alpha_1 s_1]}$ both $\hat{K}_1$ and $\hat{K}_2$

will be nonnegative. Let us consider now $f(u) = Au^2 + Bu + C$. Now $f(0) = C$

$= -\alpha_1^2 \rho\, s_1 < 0$. $f\left\{\dfrac{\alpha_1 s_1}{g[(1-s_1)\beta_1\epsilon + \alpha_1 s_1]}\right\} = \dfrac{\alpha_1^2 \rho\, \beta_2 s_1^2(1-\epsilon)(1-s_1)}{(1-s_1)\beta_1\epsilon + \alpha_1 s_1} > 0$.

Hence there exists a $\hat{u}$ such that $f(\hat{u}) = 0$ and $0 \leq \hat{u} < \dfrac{\alpha_1 s_1}{g[(1-s_1)\beta_1\epsilon + \alpha_1 s_1]}$.

Since $f(u)$ is a quadratic and has a real root, the other root must also be real. Now $C < 0$, and $\hat{u} \geq 0$. Hence the other root is positive if $A < 0$ and negative if $A > 0$. However it is clear that in either case

$f'(\hat{u}) = 2A\hat{u} + B > 0$.

We still have to examine whether $\hat{K}_{2f}$ lies between 0 and $\hat{K}_2$. Now from (B12) and (B13) we get

(B17) $$\hat{K}_{2f} = \frac{g[(1-\epsilon)(1-s_1)\,\hat{K}_1 - s_1 K_1 - s_1 \hat{K}_2]}{s_1[\,\dfrac{\alpha_1}{\hat{u}} - g\,(1-\dfrac{1}{\rho})]}$$

It is not easy to determine whether for all relevant values of the parameters of our model (B17) will yield an economically meaningful value for $\hat{K}_{2f}$ i.e., a value in $[0, \hat{K}_2]$. However, we shall show that for positive values of ρ sufficiently small we do get economically meaningful values. We note that for all values of ρ in $[0, 1]$ the denominator of (B17) is positive. Hence for any ρ in $[0, 1]$ in order for $\hat{K}_{2f} > 0$ we must have

$$\frac{\hat{K}_1}{\hat{K}_2} \geq \frac{s_1}{(1-\epsilon)(1-s_1)} .$$

It is easily verified that $\hat{K}_2 - \hat{K}_{2f}$

$$= \frac{s_1 \hat{K}_2 \left[\frac{\alpha_1}{\hat{u}} + \frac{g}{\rho}\right] - g(1-\epsilon)(1-s_1)\hat{K}_1}{s_1\left[\frac{\alpha_1}{\hat{u}} - g\left(1 - \frac{1}{\rho}\right)\right]}$$

This means that for $0 \leqq \rho \leqq 1$, $\hat{K}_2 - \hat{K}_{2f} \geqq 0$ if and only if

$$\frac{\hat{K}_1}{\hat{K}_2} \leqq \frac{s_1 \left[\frac{\alpha_1}{\hat{u}} + \frac{g}{\rho}\right]}{g(1-\epsilon)(1-s_1)} = \frac{s_1[\alpha_1 \rho + g\hat{u}]}{u\ \rho\ g(1-\epsilon)(1-s_1)} .$$

Hence if we can ensure that $\dfrac{s_1}{(1-\epsilon)(1-s_1)} \leqq \dfrac{\hat{K}_1}{\hat{K}_2} \leqq \dfrac{s_1[\alpha_1\ \rho + g\hat{u}]}{\hat{u}\ \rho\ g(1-\epsilon)(1-s_1)}$

We would have ensured that $0 \leqq \hat{K}_{2f} \leq \hat{K}_2$.

Let us go back to the quadratic which determined $\hat{u}$. This was $f(u) = Au^2 + Bu + C = 0$. Now A, B and C are functions of ρ. It is easily seen from the defintions of these coefficients, they are continuous functions of ρ and have certain values $A(0)$, $B(0)$, and $C(0)$. As we let $\rho \to 0$ the quadratic approaches $A(0)u^2 + B(0)u + C(0)$. The roots of $f(u) = 0$ approach the roots of $A(0)u^2 + B(0)u + C(0) = 0$ as $\rho \to 0$. It is seen that $C(0) = 0$. Hence the roots of $A(0)u^2 + B(0)u + C(0) = 0$ are $u = 0$

and $u = -\dfrac{B(0)}{A(0)} = \dfrac{\alpha_1 s_1}{g[\alpha_1 s_1 + \beta_1(1-\epsilon)s_1]}$. It is clear that the root $\hat{u}$

which we used above approaches the nonzero limit. This can be shown as follows:

$$f(\hat{u}) = A\hat{u}^2 + B\hat{u} + C = 0 .$$

Hence $\dfrac{d\hat{u}}{d\rho} = -\left(\dfrac{\frac{dA}{d\rho}\hat{u}^2 + \frac{dB}{d\rho}\hat{u} + \frac{dC}{d\rho}}{2A\hat{u} + B}\right)$. As we indicated above

$2A\hat{u} + B > 0$. Now $\dfrac{dA}{d\rho} = g^2 B_2\left[\alpha_1(1-\epsilon)(1-s_1) - \dfrac{(1-s_2)}{s_2}\left(\alpha_1 s_1 + \beta_1(1-s_1)\,\epsilon\right)\right]$

$$\frac{dB}{d\rho} = g\,\alpha_1\left[(1-s_1)B_1\epsilon + \alpha_1 s_1 + s_1\,\frac{\beta_2(1-s_2)}{s_2}\right] \text{ and } \frac{dC}{d\rho} = -\,\alpha_1^2 s_1$$

Hence $\frac{dA}{d\rho}\hat{u}^2 + \frac{dB}{d\rho}\hat{u} + \frac{dC}{d\rho} = \frac{1}{\rho}f(\hat{u}) + \frac{g}{\rho}\hat{u}\left[\alpha_1 s_1 - g\left\{\alpha_1 s_1 + \beta_1(1-s_1)\epsilon\right\}\hat{u}\right]$

$$> 0 \quad \text{since } f(\hat{u}) = 0 \text{ and } \hat{u} < \frac{\alpha_1 s_1}{g[\alpha_1 s_1 + \beta_1(1-s_1)\epsilon]}$$

Hence $\frac{d\hat{u}}{d\rho} < 0$. This implies that as $\rho \to 0$, $\hat{u}$ (> 0) increases and cannot reach a zero limit. Let us define $\hat{u}(0) = \dfrac{\alpha_1 s_1}{g[\alpha_1 s_1 + \beta_1(1-\epsilon)s_1]}$.

Now from (B16) it is clear that

$$\frac{\hat{K}_1}{\hat{K}_2} = \frac{g}{\hat{u}(0)} \cdot \left\{\frac{\hat{u}}{\hat{u}(0) - \hat{u}}\right\}. \quad \text{Hence as } \rho \to 0, \ \frac{\hat{K}_1}{\hat{K}_2} \to \infty .$$

Therefore for small enough values of ρ $\dfrac{\hat{K}_1}{\hat{K}_2} \geqq \dfrac{s_1}{g(1-\epsilon)(1-s_1)}$. It can be shown that for values of ρ too small, the upperbound on $\dfrac{\hat{K}_1}{\hat{K}_2}$ will be violated. If there exists a value $\hat{\rho}$ for ρ for which $\dfrac{\hat{K}_1}{\hat{K}_2} = \dfrac{s_1}{g(1-\epsilon)(1-s_1)}$

then for values of ρ in some interval with $\hat{\rho}$ as an upperbound and some positive values as lower bound, $\frac{\hat{K}_1}{\hat{K}_2}$ will be within the required bounds. We shall assume that such a $\hat{\rho}$ exists.

It remains to be shown that $\frac{d\hat{u}}{ds_2} > 0$. This can be proved in a way similar to the proof of $\frac{d\hat{u}}{d\rho} < 0$.