COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

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APPENDIX

to

"On the Concept of Optimal Economic Growth"

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1. Notations

Instantaneous discount rate $\rho = -\sigma$

Exponential growth rate of labor force λ

	At time t — absolute	At time t — per worker	Integrated over time (per worker)
Consumption flow	X_t	x_t	
Capital stock	Z_t	z_t	
Labor force	$L_t = L_o e^{\lambda t}$		
Production function	$F(Z, L)$	$f(z)$	
Utility		$u(x)$	U, V, W

Derivatives with respect to time are denoted by dots, $\dot{z}_t = \frac{dz_t}{dt}$,

other derivatives by dashes, $f'(z) = \frac{df}{dz}$, $u'(x) = \frac{du}{dx}$.

$\hat{}$ denotes optimal paths and their asymptotic levels.

$\equiv$ denotes equality by definition.

Formulae in the article are numbered (1), (2), ... , in the Appendix ($\underline{1}$), ($\underline{2}$), ...

2. Assumptions

(a) = (8) $L_t = L_o e^{\lambda t}$ where $0 < \lambda < f'(0)$, for all $t \geqq 0$,

(b) = (9) $F(Z, L) = L f(\frac{Z}{L}) = L f(z)$ for all $L > 0$, $Z \geqq 0$,

(c) $f(0) = 0$; $f'(z) > 0$, $f''(z) < 0$ for $0 \leqq z$,

(d) = (11a) for each $\lambda > 0$ such that $0 < \lambda < f'(0)$ there is a $\bar{z}_\lambda > 0$ such that $f(\bar{z}_\lambda) = \lambda \bar{z}_\lambda$ (the subscript λ of $\bar{z}_\lambda$ is omitted in what follows) ,

(e) $u'(x) > 0$, $u''(x) < 0$ for $0 < x < \infty$.

3. Some implications of feasibility

Given the initial stock z_o of capital per worker, the attainable set of growth paths (x_t, z_t) is now given, in terms of per-worker variables, by the requirements that, for all $t \geqq 0$,

(1)
- (a) $x_t > 0$, $z_t > 0$,
- (b) z_t is continuous,
- (c) z_t , x_t and $\dot{z}_t$ are differentiable to the right,
- (d) $\dot{x}_t$ is continuous to the right

(2) $x_t + \dot{z}_t = f(z_t) - \lambda z_t \equiv g(z_t)$, say,

(3) $\qquad z_o$ is prescribed, where $0 < z_o < \bar{z}$.

The <u>feasible set</u> is the union of all attainable sets with $0 < z_o < \bar{z}$.

We note that, by assumption (c), both the feasible set and the attainable sets are convex, and that the function $g(z)$ defined in (2) is strictly concave. Since $g(z)$ vanishes for $z = 0$ and for $z = \bar{z}$, it reaches its maximum $\hat{x}$ in a unique point $\hat{z}$, so that

(4) (a) $\quad \hat{x} \equiv g(\hat{z}) > g(z)$ for all $z \neq \hat{z}$, where $0 < \hat{z} < \bar{z}$,

(b) $\quad g'(z) > g'(\hat{z}) = 0 > g'(z^*)$ whenever $0 \leqq z < \hat{z} < z^* \leqq \bar{z}$.

From (1a), (2) we have

(5) $$\dot{z}_t < x_t + \dot{z}_t = g(z_t)$$

and hence for all feasible paths, using (1a), (3), and the fact that $g(z) > 0$ only for $0 < z < \bar{z}$,

(6) $$0 \leqq z_t \leqq \bar{z} \text{ for all } t \geqq 0 .$$

4. A basic inequality and one application

The concavity assumption (e) of $u(x)$ implies that,

$$(\underline{7}) \qquad u(x) - u(x^*) \leqq u'(x^*) \cdot (x-x^*) \quad \text{for all} \quad x\ ,\ x^*\ ,$$

and the concavity of $g(z)$ implied in Assumption (c) and ($\underline{2}$) that

$$(\underline{8}) \qquad g(z) - g(z^*) \leqq g'(z) \cdot (z-z^*) \quad \text{for all} \quad z\ ,\ z^*\ .$$

We shall make many comparisons of utility integrals for feasible growth paths (x_t, z_t) and (x_t^*, z_t^*), based on ($\underline{7}$) and on either ($\underline{4}$) or ($\underline{8}$). To avoid repetition we state this comparison here in its most general form, where $0 \leqq T < T^* \leqq \infty$, and ρ is as yet unspecified.

$$(\underline{9}) \quad \begin{cases} (\underline{9}a) \begin{cases} {}_{T}U^*_{T^*}(\rho) \equiv \int_T^{T^*} e^{-\rho t} \Big(u(x_t) - u(x_t^*)\Big)dt \leqq \int_T^{T^*} e^{-\rho t}\ u'(x_t^*)(x_t - x_t^*)dt = \\ = \int_T^{T^*} e^{-\rho t} u'(x_t^*)\Big(g(z_t) - g(z_t^*) - \dot z_t + \dot z_t^*\Big)dt = \\ = \int_T^{T^*} e^{-\rho t} u'(x_t^*)\Big(g(z_t) - g(z_t^*)\Big)dt - \Big[e^{-\rho t}u'(x_t^*)(z_t - z_t^*)\Big]_T^{T^*} + \end{cases} \\ + \int_T^{T^*} \left[\frac{d}{dt}\Big(e^{-\rho t}u'(x_t^*)\Big)\right](z_t - z_t^*)\ dt \leqq \\ \leqq \int_T^{T^*} e^{-\rho t}\Big[u'(x_t^*)\Big(g'(z_t^*) - \rho\Big) + u''(x_t^*)\ \dot x_t^*\Big](z_t - z_t^*)dt \\ - \Big[e^{-\rho t}u'(x_t^*)(z_t - z_t^*)\Big]_T^{T^*}\ . \end{cases}$$

If $T^* = \infty$ the validity of (9) depends on convergence of the integrals involved.

One application of (9) will be used repeatedly. We define a <u>bulge</u> in a growth path (x_t, z_t) as an interval $[T, T^*]$ such that

$$(\underline{10}) \begin{cases} (\underline{a}) & 0 \leqq T < T^* < \infty\,, \quad z_T = z_{T^*} = z^*\,, \text{ say, and} \\ (\underline{b}) & \text{either } z^* \leqq \hat{z}(\rho) \text{ and } z_t < z^* \text{ for } T < t < T^* \\ & \text{or } z^* \geqq \hat{z}(\rho) \text{ and } z_t > z^* \text{ for } T < t < T^*\,, \end{cases}$$

where $\hat{z}(\rho)$ is defined by (27) or

$$(\underline{11}) \qquad g'\left(\hat{z}(\rho)\right) = \rho$$

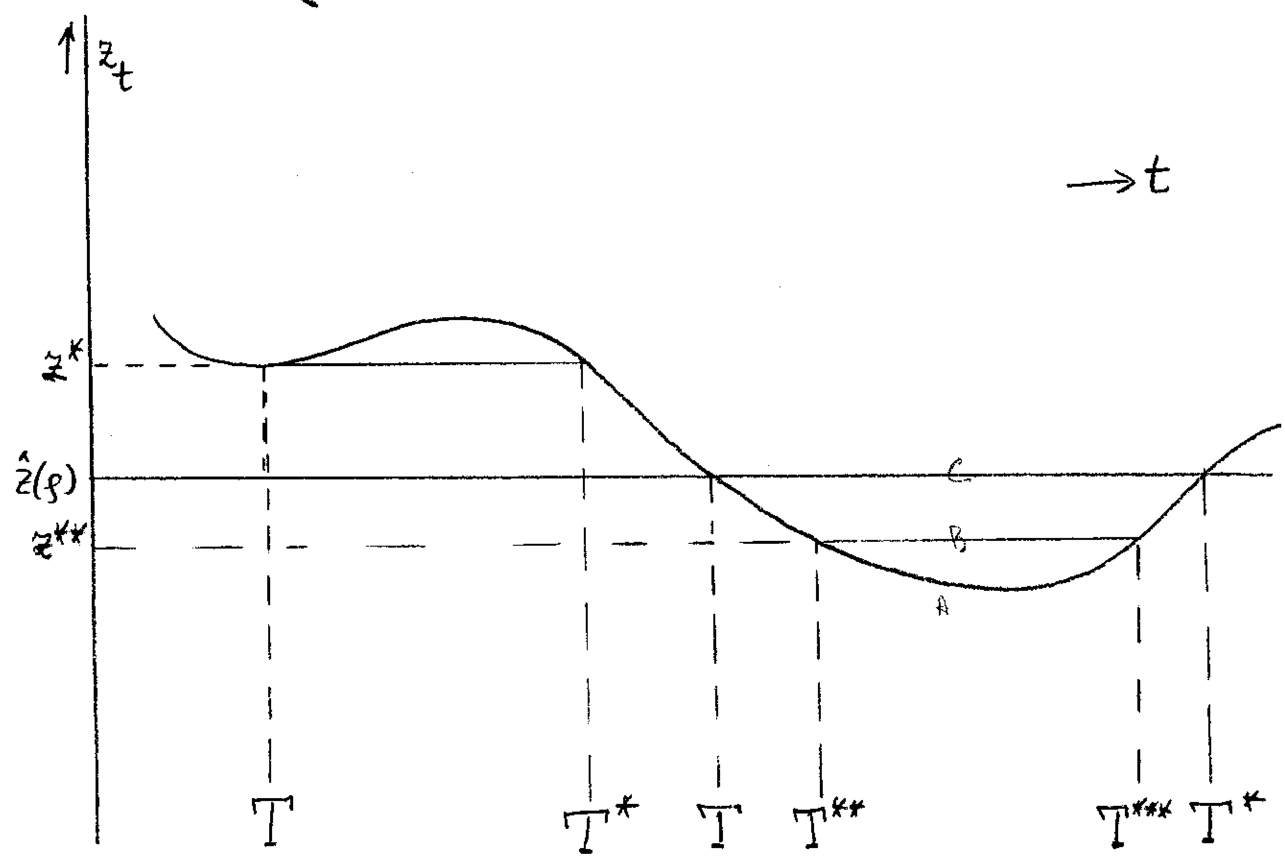


Fig. 11

Figure 11 shows z_t for a path with two bulges, both denoted $[T, T^*]$. The effect on the utility integral of "straightening out" a bulge is found from (9) by taking $z_t^* = z^*$, $x_t^* = x^* \equiv g(z^*)$, and satisfies

$$(\underline{12}) \qquad \int_T^{T^*} e^{-\rho t}\Big(u(x_t) - u(x^*)\Big)dt \leqq u'(x^*)\Big(g'(z^*) - \rho\Big)\int_T^{T^*} e^{-\rho t}(z_t - z^*)dt < 0$$

if $z^* \neq \hat{z}(\rho)$, because in that case $g'(z^*) - \rho$ and $z_t - z^*$ are opposite in sign. If $z^* = z(\rho)$, and if for instance $z_t < z^*$ for $T < t < T^*$ as in the second bulge in Figure 11, we can by suitable choice of a number $z^{**} < \hat{z}(\rho)$ write the left hand member of (12) as the sum of two such integrals, one comparing $(x^*, z^*) = \Big(\hat{x}(\rho), \hat{z}(\rho)\Big)$ on $[T, T^*]$ with $\Big(x_t^{**}, z_t^{**}\Big)$ defined by $z_t^{**} \equiv \max\{z^{**}, z_t\}$, the other comparing (x^{**}, z^{**}) where $x^{**} \equiv g(z^{**})$ with (x_t, z_t) on an interval $[T^{**}, T^{***}]$ such that $T < T^{**} < T^{***} < T^*$. Since of these integrals the former is nonpositive, the latter negative, (12) is valid also if $z^* = \hat{z}(\rho)$. We thus have

Lemma 1: For any ρ, a path (x_t, z_t) optimal on any finite or infinite time interval cannot contain a bulge.

This conclusion, and the inequality (12) on which it is based, remain valid for $T^* = \infty$ and $\rho \geqq 0$ if the definition of a bulge is extended to read "(10b) and either (10a) or (10a')" ,

(10a') $0 \leqq T < T^* = \infty$, $\rho \geqq 0$, $z_T = z^*$, and if $\rho = 0$ then $\lim_{t \to \infty} z_t = z^*$,

as illustrated in Figures 12 and 13.

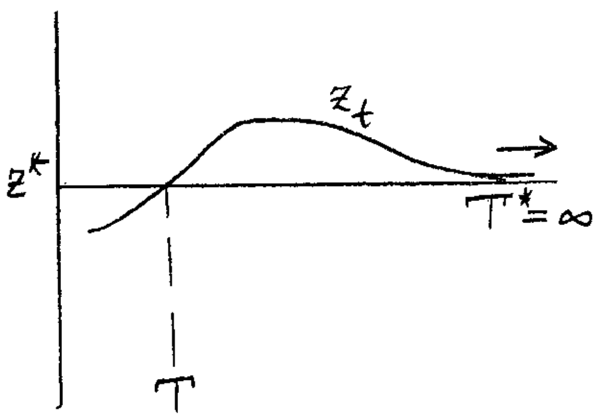
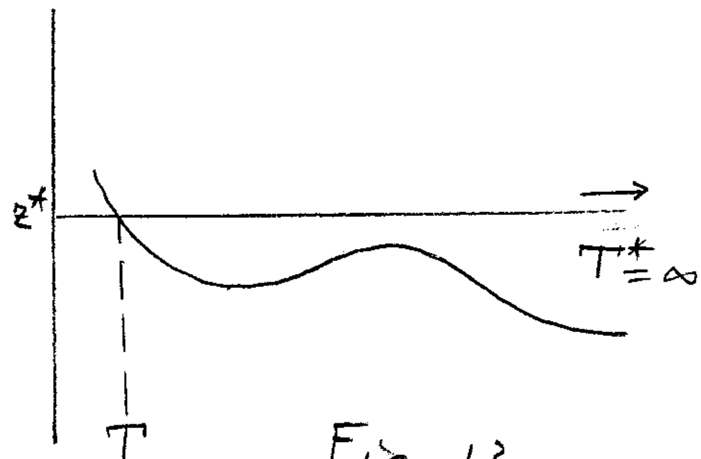


Fig 12



Fig, 13

5. Inferiority of indefinitely fluctuating paths if $\rho \leqq 0$.

We define the asymptotic range of the path (x_t, z_t) as the nonempty closed interval

(13) $$[\underline{\zeta}, \overline{\zeta}] , \quad \underline{\zeta} \equiv \lim_{T \to \infty} \inf_{t \geqq T} z_t , \quad \overline{\zeta} = \lim_{T \to \infty} \sup_{t \geqq T} z_t .$$

A positive length $\bar{\zeta} - \underline{\zeta}$ of the asymptotic range implies that z_t continues to fluctuate between any neighborhood of $\underline{\zeta}$ and any neighborhood of $\bar{\zeta}$, infinitely often, and for arbitrarily large t.

<u>Lemma</u> 2: <u>If</u> $\rho \leqq 0$ <u>and if</u> $\underline{\zeta} < \bar{\zeta}$ <u>for the attainable path</u> (x_t, z_t), <u>then there exists for each</u> $N > 0$ <u>an attainable path</u> $\left(x_t^*, z_t^*\right)$ <u>and a</u> $T_N > 0$ <u>such that</u>

$$(\underline{14}) \qquad U_T^*(\rho) \equiv \int_0^T e^{-\rho t}\left(u(x_t) - u(x_t^*)\right) dt \leqq -N \quad \text{for all} \quad T \geqq T_N .$$

For the proof of Lemma 2 we must strengthen ($\underline{12}$) to obtain a positive lower bound on the gain $|{}_T U_{T^*}^*(\rho)|$ associated with the "straightening out" of a bulge $[T, T^*]$. For this purpose we choose an interval $[z_*, z^*]$ such that

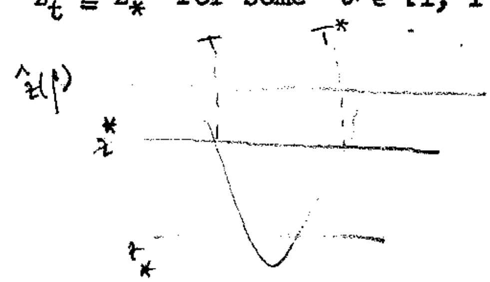
$$(\underline{15}) \quad (\underline{a}) \qquad \underline{\zeta} < z_* < z^* < \bar{\zeta} \quad \text{and either} \quad (\underline{b}) \quad \hat{z}(\rho) < z_* \quad \text{or} \quad (\underline{c}) \quad z^* < \hat{z}(\rho) ,$$

which is always possible. If for definiteness we assume ($\underline{15c}$), we have from Assumption (c)

$$(\underline{16}) \qquad g'(z) - \rho \geqq g'(z^*) - \rho \equiv \gamma > 0 \quad \text{for} \quad z_* \leqq z \leqq z^* .$$

Now z_t has infinitely many bulges $[T, T^*]$ with the properties

$$(\underline{17}) \qquad z_T = z_{T^*} = z^* , \quad z_t \leqq z_* \quad \text{for some} \quad t \in [T, T^*] .$$



Because of the continuity of z_t we can for each of these choose an interval $[\tau, \tau^*]$ such that, if we write $z^* - z_* \equiv 2\epsilon$,

(18) $$T < \tau < \tau^* < T^* \text{ and } z_\tau = z_* < z_t < z_{\tau^*} = z^* - \epsilon \text{ for } \tau < t < \tau^* .$$

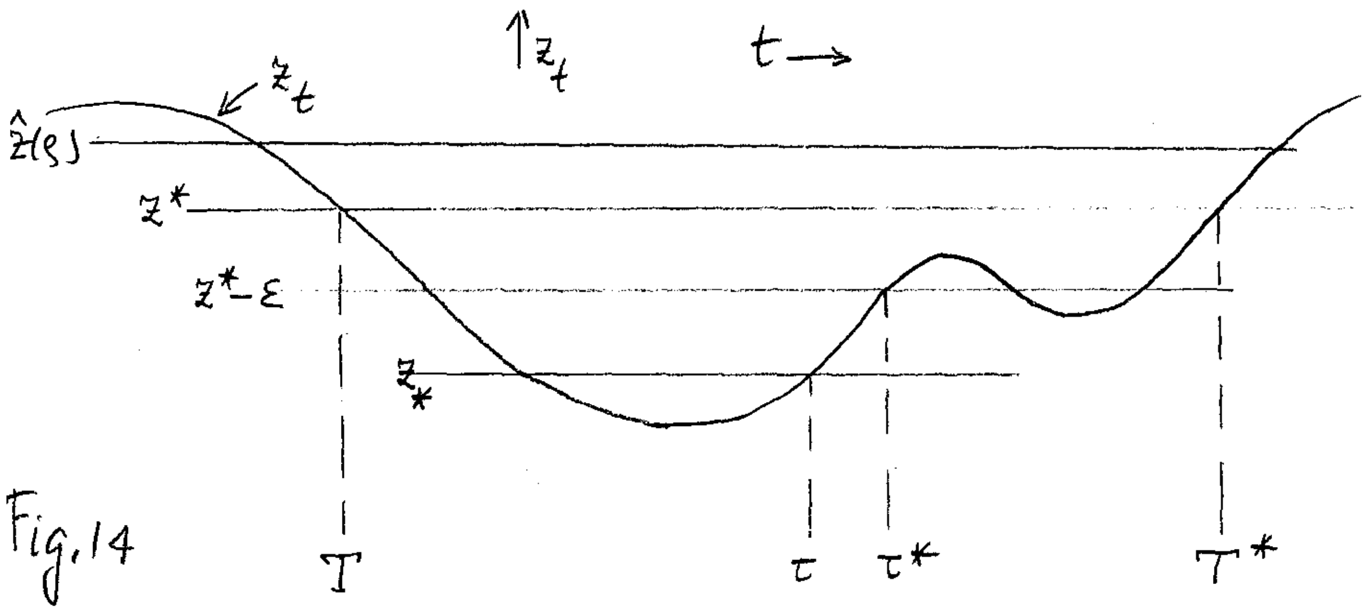


Fig. 14

The construction is illustrated in Figure 14. Since the <u>last</u> inequality in (12) holds also for all subintervals of $[T, T^*]$, we have from (18), (16), if $x^* \equiv g(z^*)$ and $\rho \leqq 0$,

$$_TU^*_{T^*}(\rho) \equiv \int_T^{T^*} e^{-\rho t}\Big(u(x_t) - u(x^*)\Big)dt < u'(x^*)\,\gamma \int_\tau^{\tau^*} (z_t - z^*)dt < - u'(x^*)\,\gamma\epsilon\,(\tau^* - \tau) ,$$

since $\epsilon > 0$ and $u'(x^*)\,\gamma > 0$. On the other hand, we have from (18), (5) with

$x_t > 0$, (2) and (4a) that

$$\epsilon = \int_{\tau}^{\tau^*} \dot{z}_t dt < \int_{\tau}^{\tau^*} g(z_t) dt < (\tau^* - \tau)\, \hat{x} ,$$

whence $\tau^* - \tau > \epsilon/\hat{x} > 0$ and

$$(19) \qquad {}_{T}U^*_{T^*}(\rho) < - u'(x^*)\, \gamma\, \epsilon^2/\hat{x} \equiv - \alpha^* < 0 .$$

Finally, if we define the feasible path (x^*_t, z^*_t) by (2) and

$$z^*_t \equiv \max \left\{ z_t, z^* \right\} ,$$

we have

$$U^*_{T^{**}}(\rho) \equiv \int_0^{T^{**}} e^{-\rho t}\Big(u(x_t) - u(x^*_t)\Big)\, dt \leqq - n_{T^{**}}\, \alpha^* , \quad \text{where} \quad \alpha^* > 0 ,$$

if $n_{T^{**}}$ denotes the number of bulges in $[0, T^{**}]$. But $\lim_{T \to \infty} n_T = \infty$ because there are infinitely many bulges in $[0, \infty]$.

The choice of T_N such that $n_{T_N} \geqq N/\alpha^*$ thus establishes Lemma 2 in case (15c) holds. The proof from (15 b) is similar.

6. Proofs for $\rho = 0$.

Proof of (A). In (9a) take $\rho = 0$, $z_t^* = \hat{z}$, so $x_t^* = \hat{x} = g(\hat{z})$. Then, if we write $u(\hat{x}) \equiv \hat{u}$, $u'(\hat{x}) \equiv \hat{u}'$,

$$(20) \qquad {}_T U_{T^*} \equiv \int_T^{T^*} \left(u(x_t) - \hat{u}\right) dt \leqq \hat{u}' \cdot (z_T - z_{T^*}) \leqq \hat{u}' \cdot \bar{z} ,$$

by (4), (6), regardless of T , T^* , hence also for $T = 0$.

Proof of (B). We distinguish three cases regarding the asymptotic range $[\underline{\zeta}, \bar{\zeta}]$ of the given path (x_t, z_t) .

Case (i), $\underline{\zeta} < \bar{\zeta}$. In this case we have from Lemma 2 and from (20) applied to (x_t^*, z_t^*) , for any $N > 0$,

$$U_T = \int_0^T \left(u(x_t) - u(x_t^*)\right) dt + \int_0^T \left(u(x_t^*) - \hat{u}\right) dt \leqq - N + \hat{u}' \cdot \bar{z}$$

for all $T \geqq T_N$. In this case, therefore, U_T diverges to $-\infty$ as $T \to \infty$.

Case (ii), $\underline{\zeta} = \bar{\zeta} \neq \hat{z}$. For definiteness assume $\bar{\zeta} < \hat{z}$ and let $\hat{z} - \bar{\zeta} \equiv 2\epsilon$.

Then $\epsilon > 0$ and there exists $T < \infty$ such that

(21) $$z_t \leqq \overline{\zeta} + \epsilon = \hat{z} - \epsilon \quad \text{for all} \quad t \geqq T .$$

If now in (9a) we take $\rho = 0$, $z_t^* = \hat{z}$, $x_t^* = \hat{x} = g(\hat{z})$ for all $t \geqq 0$, then,

(22) $$_TU_{T^*} \equiv \int_T^{T^*} \left(u(x_t) - \hat{u}\right)dt \leqq \hat{u}' \left[\int_T^{T^*} \left(g(z_t) - \hat{x}\right)dt + z_T - z_{T^*}\right] \leqq -\alpha(T^*-T) + \beta ,$$

where by (4)

$$\alpha \equiv \hat{u}' \cdot \left(\hat{x} - g(\hat{z}-\epsilon)\right) > 0 , \qquad \beta \equiv \hat{u}' \cdot \overline{z} .$$

Hence $U_{T^*} = U_T + {_TU_{T^*}}$ diverges to $-\infty$ as $T^* \to \infty$ in this case, and by similar reasoning in the case $\hat{z} < \underline{\zeta}$, hence in the entire Case (ii).

Case (iii), $\underline{\zeta} = \overline{\zeta} = \hat{z}$. In this case clearly

(23) $$\lim_{T\to\infty} z_T = \hat{z} .$$

It follows from the third member of (20) that

$$G_T \equiv U_T + \hat{u}' \cdot z_T$$

is a nonincreasing function of T . Hence G_T either possesses a limit for $T \to \infty$ or diverges to $-\infty$. In view of (23), the same must then be true for U_T .

This completes the proof of statement (B). In addition, we have found

Lemma 3: If $\rho = 0$, a necessary condition for eligibility of the path (x_t, z_t) is that (23) is satisfied.

Proof of (C). An optimal path $(\hat{x}_t, \hat{z}_t)$ is now defined as one that maximizes

$$(24) \qquad U \equiv \int_0^\infty \left(u(x_t) - \hat{u}\right) dt$$

on the attainable-and-eligible set. A beautifully simple procedure used by Ramsey in his slightly different problem can be adapted to the present problem as long as $\rho = 0$.

From Lemmas 1 and 3 we conclude that, in any optimal path, $\hat{z}_t$ exhibits a nondecreasing, constant, or nonincreasing approach to $\lim_{t\to\infty} \hat{z}_t = \hat{z}$ according as $z_0 < \hat{z}$, $= \hat{z}$ or $> \hat{z}$. This establishes the second and third sentences of statement (C) with the term "weakly monotonic" substituted for "strictly monotonic." Now consider an attainable-eligible path (x_t, z_t) for which

$$(25) \qquad z_t = z^* \neq \hat{z} \quad \text{for} \quad T \leqq t \leqq T^*, \quad \text{where} \quad T < T^* .$$

Then, along the lines of (22),

$$ {}_TU_{T^*} \equiv \int_T^{T^*} \left(u(x_t) - \hat{u}\right) dt \leqq \hat{u}'(T^*-T)\left(g(z^*) - \hat{x}\right) < 0$$

by (4 a). It follows that the path

$$(x_t^*, z_t^*) = \begin{cases} (x_t, z_t) & \text{for } 0 \leqq t < T , \\ (x_{t+T^*-T} , z_{t+T^*-T}) & \text{for } T \leqq t , \end{cases}$$

is likewise attainable, and indeed eligible and preferable to (x_t, z_t), because it achieves a utility

$$U^* = U_T^* + {}_TU^* = U_T + {}_{T^*}U > U_T + {}_TU_{T^*} + {}_{T^*}U = U .$$

Therefore (25) cannot occur in an optimal path.

It follows that, if $z_o \neq \hat{z}$, an optimal path shows a strictly monotonic approach to the value $z_T = \hat{z}$ for $0 \leqq t < T$, where $T \leqq \infty$. We shall call any eligible path with that property a superior path. To complete the proof of (C) we only need to show that for an optimal path $T = \infty$. This is best obtained as a corrolary of the proof of (D).

Proof of (D). For all superior paths we can now make a useful change of the variable of integration in (24) from t to z. Since, by (2), $z_t = \hat{z}$ for $t \geqq T$ implies $x_t = \hat{x}$, $u(\hat{x}_t) = \hat{u}$, we have for all superior paths, using (2),

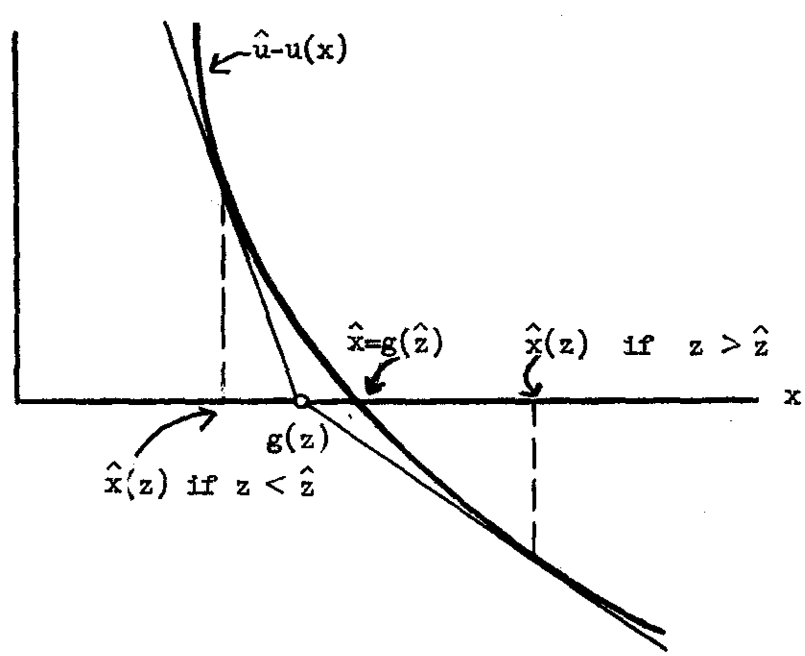
$$(26) \qquad U = \int_{z_o}^{\hat{z}} \frac{u\left(x_{t(z)}\right) - \hat{u}}{\dot{z}_{t(z)}} dz = \int_{z_o}^{\hat{z}} \frac{u\left(x(z)\right) - \hat{u}}{g(z) - x(z)} dz ,$$

where $t(z)$ denotes the inverse of z_t on $[0, T]$, and $x(z) \equiv x_{t(z)}$. The unknown function is now $x(z)$. The advantage from the change of variables lies in the fact that only $x(z)$ itself, and no derivative thereof, occur in the integrand in (26). Hence (26) is maximized on the set of superior paths if and only if $x(z)$ is given a value $\hat{x}(z)$ such that the integrand is maximized for every value of z. This requires $\hat{x}(z)$ for given z to be the solution for x of

$$(27) \qquad u'(x)\Big(g(z) - x\Big) = \hat{u} - u(x)$$

Figure 15 shows the determination of $x = \hat{x}(z)$ for the two cases $z < \hat{z}$ and $z > \hat{z}$. It is easily seen from the diagram or analytically that $\hat{x}(z)$ is unique, continuous, and strictly increasing for all z, and differentiable for $z \neq \hat{z}$. In particular,

Figure 15



(28) $$\hat{x}(z) < \hat{x}(\hat{z}) = \hat{x} = g(\hat{z}) < \hat{x}(z^*) \quad \text{if} \quad z < \hat{z} < z^* .$$

Once $\hat{x}(z)$ has been determined in the manner indicated, one reintroduces the time variable by

(29) $$t = \int_{z_o}^{z} \frac{dt}{dz}\, dz = \int_{z_o}^{z} \frac{dz}{g(z) - \hat{x}(z)} \equiv \hat{t}(z) .$$

The function $\hat{t}(z)$ and its inverse $\hat{z}_t$ are monotonic and differentiable with the proper range and domain in each case because, by (27), (28),

$$g(z) - x(z) = \frac{\hat{u} - u\left(\hat{x}(z)\right)}{u'\left(\hat{x}(z)\right)} \left\{ \begin{matrix} > \\ < \end{matrix} \right\} 0 \quad \text{if} \quad \left\{ \begin{matrix} 0 < z_o < \hat{z} \\ \hat{z} < z_o < \bar{z} \end{matrix} \right. .$$

Hence x_t is monotonic and differentiable. In order to see that $T = \infty$ whenever $z_o \neq \hat{z}$ note that, in a neighborhood of $z = \hat{z}$, $g(z) - g(\hat{z})$ is of second order in $(z - \hat{z})$ by (4 b), and of second order in $\left(\hat{x}(z) - \hat{x}\right)$ by the construction of Figure 15. Hence $\hat{x}(z) - \hat{x}$ and $z - \hat{z}$ are of the same order, and in (29) we have

$$\text{if} \quad z_o < \hat{z} , \quad \lim_{z \to \hat{z} - 0} t(z) = \infty ,$$

$$\text{if} \quad z_o > \hat{z} , \quad \lim_{z \to \hat{z} + 0} t(z) = \infty .$$

Therefore $T = \infty$. The proofs of (C) and (D) are thereby complete.

7. Proofs for $\rho > 0$ and for $\rho \geqq 0$.

Proof of (E). Let (x_t, z_t) be a feasible path with $x_t \geqq \underline{x} > 0$ for all t . In (9) we insert $x_t^* = \underline{x}$, $z_t^* = \underline{z} < \hat{z}$ such that $g(\underline{z}) = \underline{x}$. Then, if $u(\underline{x}) \equiv \underline{u}$, $u'(\underline{x}) \equiv \underline{u}'$, we have $\underline{u} \leqq u(x_t)$ and hence, for $0 \leqq T < T^* < \infty$,

$$0 \leqq \int_T^{T^*} e^{-\rho t} \left(u(x_t) - \underline{u}\right) dt \equiv {}_T V_{T^*}(\rho) - (\underline{u}/\rho)\left(e^{-\rho T} - e^{-\rho T^*}\right) \leqq$$

$$\leqq \underline{u}' \cdot |g'(\underline{z}) - \rho| \cdot (\overline{z}/\rho)\left(e^{-\rho T} - e^{-\rho T^*}\right) + \underline{u}'\overline{z}\left(e^{-\rho T} + e^{-\rho T^*}\right),$$

hence $\lim_{T, T^* \to \infty} {}_T V_{T^*}(\rho) = 0$ whenever $\rho > 0$.

Proofs of (F), (G).[1] These statements express, and provide economic interpretation for, the inequalities (9) if we take $T = 0$, $T^* \to \infty$, and if the "candidate-optimal" path $(\hat{x}_t, \hat{z}_t)$ is substituted for (x_t^*, z_t^*) . This is seen by reference to the definitions (21), (22) of the implicit prices p_t , q_t of the consumption good and of the use of the same good as capital good, respectively. The middle member of (25) in (G) is fitted in by taking the logical steps in (9) in a different sequence. The last term in the last member of (9) vanishes because both (x_t, z_t) and (x_t^*, z_t^*) are eligible and attainable. This is immediate if $\rho > 0$, and follows from Lemma 3 if $\rho = 0$.

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[1] Readers of this appendix are requested to substitute two pages, numbered 23, 24, found at the end of the appendix, for the pages bearing the same numbers in the article, CFDP 163, of which this is the appendix.

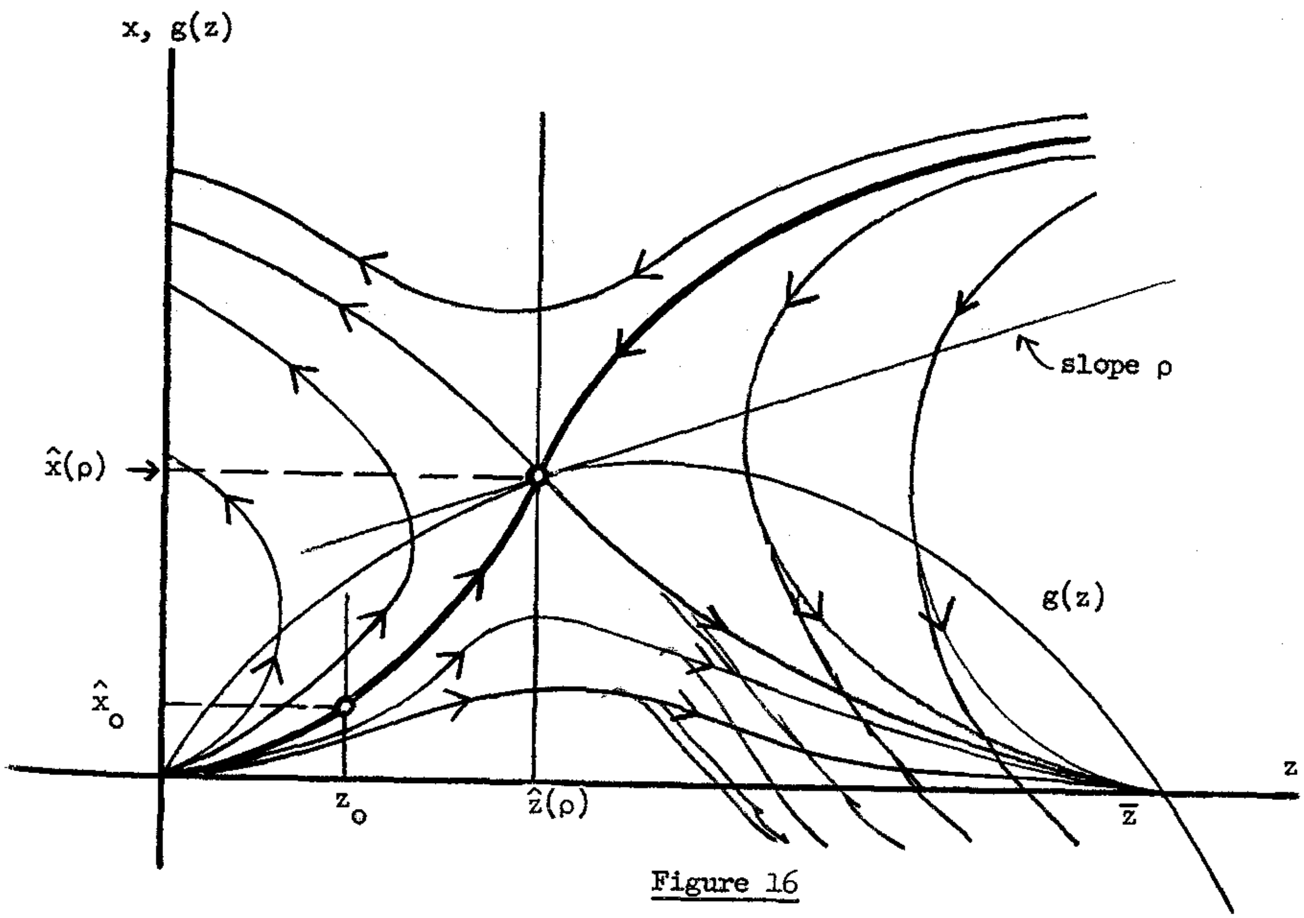
<u>Proofs of</u> (H), (I), (J). The sufficiency of the <u>Euler condition</u> (26), or

$$(\underline{30}) \qquad q_t + \dot{p}_t = u'(\hat{x}_t)\left(g'(\hat{z}_t) - \rho\right) + u''(\hat{x}_t)\,\dot{\hat{x}}_t = 0 \quad \text{for} \quad t \geqq 0 \ ,$$

for the optimality of the path $(\hat{x}_t, \hat{z}_t)$ if $\rho \geqq 0$ is an immediate implication of (F) and (G). We shall postpone the proof of its necessity until after we have studied the solution of the system of differential equations

$$(\underline{31}) \quad \begin{cases} (\underline{31}\ \underline{a}) \qquad \dot{z}_t = g(z_t) - x_t \\ \\ (\underline{31}\ \underline{b}) \qquad \dot{x}_t = -\dfrac{u'(x_t)}{u''(x_t)}\,(g'(z_t) - \rho) \end{cases} \qquad t \geqq 0 \ ,$$

obtained from ($\underline{2}$), ($\underline{30}$), on the entire halfaxis $0 \leqq t < \infty$, subject to the prescribed value of z_o.



<u>Figure 16</u>

Figure 16 sketches the trajectories of the point (x_t, z_t) starting from arbitrary initial values (x_o, z_o). Each trajectory is defined as the solution with $x(z_o) = x_o$ of the differential equation

$$\text{(\underline{32})} \qquad \frac{dz}{dx} = - \frac{u''(x)}{u'(x)} \cdot \frac{g(z) - x}{g'(z) - \rho} ,$$

obtained from (<u>31</u>) by elimination of t. If we prescribe only z_o and examine the trajectories for various x_o, all but one of the values of x_o lead to violation of one of the feasibility conditions, $x_t > 0$ or $z_t > 0$, at a finite time t, because the continuous right hand members of (31) vanish simultaneously only for $(x, z) = \left(\hat{x}(\rho), \hat{z}(\rho)\right)$. Now any part of any of the trajectories can occur as an optimal path with prescribed initial and final capital stocks z_o, z_T for a suitable finite horizon T. (In fact, this is true for $\rho \leqq 0$ as well as for $\rho > 0$, the case of Figure 16). However, only the special trajectory from the point $(\hat{x}_o, z_o)$ that ends in $\left(\hat{x}(\rho), \hat{z}(\rho)\right)$, shown as a heavier line, satisfies both (31) and the feasibility conditions for all $t \geqq 0$. The particular value $\hat{x}_o$ of x_o that corresponds to a given z_o is read off from that special trajectory. Time is reintroduced by solving (<u>31</u> <u>b</u>), say, with the solution $\hat{z}(x)$ of (<u>32</u>) defined by the initial value $\hat{z}(\hat{x}_o) = z_o$ inserted for z in (<u>31</u> <u>b</u>).

Denoting the resulting path by $(\hat{x}_t, \hat{z}_t)$, we note that, if $z_o < \hat{z}(\rho)$, the optimal initial consumption flow $\hat{x}_o$ leaves room for growth in the capital stock per worker, and both $\hat{x}_t$ and $\hat{z}_t$ increase with t to approach their asymptotic values $\hat{x}(\rho)$, $\hat{z}(\rho)$, respectively, as $t \to \infty$. If $z_o > \hat{z}$, both $\hat{x}_t$, $\hat{z}_t$ decrease, and approach the same asymptots from above. Finally, if $z_o = \hat{z}(\rho)$ we must have $\hat{x}_t = \hat{x}(\rho)$, $\hat{z}_t = \hat{z}(\rho)$ for all $t \geqq 0$.

In particular, if $\rho = 0$, (32) can be solved explicitly, by (27) above, which of course leads back to statement (D).

Since the path $(\hat{x}_t, \hat{z}_t)$ was (uniquely) derived from the sufficient condition (30) for optimality, it is clearly optimal. The proofs of (H), (I) and (J) can therefore be completed by showing that no other attainable and eligible path is optimal. To this end we note that, by Assumption (e), the $<$ sign applies in (7) whenever $x \neq x^*$. Now if (x_t, z_t) differs from $(\hat{x}_t, \hat{z}_t)$, we must have $x_t \neq \hat{x}_t$ for some t, because in the contrary case (2) and $z_o = \hat{z}_o$ would imply $z_t = \hat{z}_t$ for all t. But then, by the attainability condition (1 c), we have a strict inequality in (24) and, by (25), (26), the path (x_t, z_t) is not optimal.

8. Proof for $\rho < 0$.

We shall need the following lemma.

Lemma 4. *If $\varphi(x)$ is a positive and nonincreasing function of x defined for all $x > 0$, and if x_t is a positive integrable function of t on the interval $[T_1, T_2]$, $T_1 < T_2$, such that*

$$\int_{T_1}^{T_2} x_t \, dt \leqq (T_2 - T_1)\,\xi\,, \quad \text{where } \xi > 0\,, \tag{33}$$

then

$$\int_{T_1}^{T_2} \varphi(x_t)dt \geqq \tfrac{1}{2}(T_2 - T_1)\,\varphi(2\xi) \tag{34}$$

Proof: We define

$$\mu(x) \equiv \frac{1}{T_2 - T_1} \cdot \text{ measure of } \left\{ t \mid T_1 \leqq t \leqq T_2 \text{ and } x_t \leqq x \right\}$$

Then $\mu(0) = 0$, $\mu(\infty) = 1$, and, from (33) and the positiveness of x_t,

$$\xi \geqq \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} x_t dt = \int_0^\infty x \, d\mu(x) = \int_0^{2\xi} x \, d\mu(x) + \int_{2\xi}^\infty x \, d\mu(x) \geqq \tag{35}$$

$$\geqq 0 + 2\xi\left(1 - \mu(2\xi)\right)\,,$$

Likewise, from the nonincreasing property of $\varphi(x)$,

$$\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \varphi(x_t)dt = \int_0^{2\xi} \varphi(x)\, d\mu(x) + \int_{2\xi}^\infty \varphi(x)\, d\mu(x) \geqq \tag{36}$$

$$\geqq \varphi(2\xi)\,\mu(2\xi) + 0\,.$$

Since, from (35), $\mu(2\xi) \geqq \frac{1}{2}$, (36) implies (34) .

Proof of (K). We again distinguish the three cases with regard to the asymptotic range of z_t , used in the proof of (B).

Case (i), $\underline{\zeta} < \overline{\zeta}$. In this case statement (K) is equivalent to Lemma 2.

Case (ii), $\underline{\zeta} = \overline{\zeta} \equiv \zeta \neq \hat{z}$. For definiteness assume $\zeta < \hat{z}$ and let $\hat{z} - \zeta \equiv 3\epsilon$. Since now $\lim_{t\to\infty} z_t = \zeta$ we can choose T such that

$$(37) \qquad \hat{z} - 4\epsilon \leqq z_t \leqq z - 2\epsilon \quad \text{for} \quad t \geqq T ,$$

and at the same time large enough for there to exist an attainable path (x^*_t, z^*_t) on $[0, T]$ such that $z^*_T = z_T + \epsilon$. For $t \geqq T$ we choose (x^*_t, z^*_t) according to

$$(38) \qquad z^*_t = z_t + \epsilon , \quad \dot{z}^*_t = \dot{z}_t , \quad x^*_t = x_t + g(z^*_t) - g(z_t) \quad \text{for all} \quad t \geqq T .$$

Then, (x^*_t, z^*_t) is attainable throughout, and from (8), (37), for $t \geqq T$,

$$(39) \qquad x^*_t - x_t = g(z^*_t) - g(z_t) \geqq g'(z^*_t)(z^*_t - z_t) \geqq g'(\hat{z}-\epsilon).\epsilon \equiv \eta > 0 .$$

Hence, for $T < T^* < \infty$ and $-\rho \equiv \sigma > 0$,

$$(\underline{40}) \qquad {}_{T}W_{T^*}(-\sigma) \equiv \int_T^{T^*} e^{\sigma t}\left(u(x_t^*) - u(x_t)\right)dt \geqq e^{\sigma T} \int_T^{T^*} u'(x_t^*) \cdot (x_t^* - x_t)dt \geqq$$

$$\geqq \eta\, e^{\sigma T} \int_T^{T^*} u'(x_t^*)\, dt$$

On the other hand, by (2), (4 a), (37) ,

$$(\underline{41}) \qquad \int_T^{T^*} x_t^*\, dt = \int_T^{T^*} g(z_t^*)dt + z_T^* - z_{T^*}^* \leqq (T^* - T)\,\hat{x} + 2\epsilon \leqq (T^* - T)\xi$$

provided $T^* - T \geqq 1$ and $\xi \equiv \hat{x} + 2\epsilon$. It follows from (41) and Assumption (e) that x_t^* and $u'(x^*)$ when substituted for x_t and $\varphi(x)$ in Lemma 4 satisfy the premises of that lemma on the interval $[T, T^*]$. Hence, from (40), (34)

$${}_{T}W_{T^*}(-\sigma) \geqq \frac{1}{2}\eta \cdot u'(2\hat{x} + 4\epsilon) \cdot (T^* - T)\, e^{\sigma T}$$

from which (K) follows directly. The proof for $\hat{z} < \zeta$ is similar.

Case (iii), $\underline{\zeta} = \overline{\zeta} = \hat{z}$. For any $\epsilon > 0$, subject still to later choice, there now exists an integer T such that

$$(42) \qquad \hat{z} - \epsilon \leqq z_t \leqq \hat{z} + \epsilon \quad \text{for} \quad t \geqq T .$$

It will be useful to write ${}_T W_{T^*}(-\sigma)$ as the difference of two integrals

$$(\underline{43})\qquad {}_T W_{T^*}(-\sigma) = \int_T^{T^*} e^{\sigma t}\left(u(x_t^*) - \hat{u}\right) dt - \int_T^{T^*} e^{\sigma t}\left(u(x_t) - \hat{u}\right) dt = {}_T U^*_{T^*}(-\sigma) - {}_T U_{T^*}(-\sigma) \; .$$

Taking first the second term we have, from (7) with $x^* = \hat{x}$, ($\underline{2}$) and ($\underline{4}$ $\underline{a}$) ,

$$(\underline{44})\qquad \begin{aligned} {}_T U_{T^*}(-\sigma) &\leqq \hat{u}' \int_T^{T^*} e^{\sigma t}\left(g(z_t) - \hat{x} - \dot{z}_t\right) dt \leqq - \hat{u}' \int_T^{T^*} e^{\sigma t} \frac{d}{dt}(z_t - \hat{z}) dt = \\ &= - \hat{u}' \left[e^{\sigma T^*}(z_{T^*} - \hat{z}) - e^{\sigma T}(z_T - \hat{z}) - \sigma \int_T^{T^*} e^{\sigma t}(z_t - \hat{z}) dt \right] \leqq \\ &\leqq \hat{u}' \epsilon \left(e^{\sigma T^*} + e^{\sigma T} + (e^{\sigma T^*} - e^{\sigma T}) \right) = 2\, \hat{u}' \epsilon\, e^{\sigma T^*} \end{aligned}$$

For the first term in ($\underline{43}$) we choose an attainable path (x_t^*, z_t^*) which for $t \geqq T$ is given by

$$(\underline{45}) \begin{cases} (\underline{45}\ \underline{a})\quad T \leqq t < T^* - 1 , & z_t^* = \hat{z} + \eta , \quad \dot{z}_t^* = 0 , \quad x_t^* = g(\hat{z} + \eta) . \\ (\underline{45}\ \underline{b})\quad T^* - 1 \leqq t < T^* , & z_t^* = (T^* - t)(\hat{z} + \eta) + (t - T^* + 1) z_{T^*} , \quad \dot{z}_t^* = z_{T^*} - \hat{z} - \eta , \quad x_t^* = g(z_t^*) - z_{T^*} + \hat{z} + \eta , \\ (\underline{45}\ \underline{c})\quad T^* \leqq t , & z_t^* = z_t , \quad \dot{z}_t^* = \dot{z}_t , \quad x_t^* = x_t , \end{cases}$$

where $\eta \geqq 3\epsilon$ and the number $T^* > T + 1$ are still subject to later choice. In addition, T should be sufficiently large that, besides ($\underline{42}$), there exists an attainable path (x_t^* , z_t^*) on $[0, T]$ such that $z_T^* = \hat{z} + \eta$.

To obtain a lower bound on the first term in (43) we note that, in view of Assumption (c) and (46), there exist numbers $\eta_o > 0$ and $\gamma > 0$ such that, whenever $0 < \eta \leqq \eta_o$,

$$(46) \qquad |z - \hat{z}| \leqq \eta \text{ implies } 0 \leqq g(\hat{z}) - g(z) \leqq \gamma \eta^2 .$$

Hence, if $0 < \eta \leqq \eta_o$ and $u_o' \equiv u'\Big(g(\hat{z} + \eta_o)\Big)$, we have from (45 a) and (4 b), in analogy to (44)

$$(47) \quad {}_T U^*_{T^*-1}(-\sigma) \geqq u_o' \int_T^{T^*-1} e^{\sigma t}\Big(g(z+\eta) - g(\hat{z})\Big)\, dt \geqq - u_o' \gamma \eta^2 \sigma^{-1}\Big(e^{\sigma(T^*-1)} - e^{\sigma T}\Big) >$$

$$> - u_o' \gamma \eta^2 \sigma^{-1} e^{\sigma T^*} .$$

For $T^* - 1 \leqq t < T$, x_t^* varies and must be boxed in. If $0 < \epsilon \leqq \epsilon_o$, $0 < \eta \leqq \eta_o$, we have from (45 b), (4),

$$\underline{x}^* \equiv \min\Big\{g(\hat{z} - \epsilon_o) , g(\hat{z} + \eta_o)\Big\} - \epsilon_o < x_t^* \leqq \hat{x} + \epsilon_o + \eta_o \equiv \overline{x}^* ,$$

where ϵ_o, η_o are chosen small enough to make $\underline{x}^* > 0$. Then, because $u'(x)$ decreases with x,

$$\overline{u}' \equiv \max\Big\{ u'(\underline{x}^*), u_o' \Big\} \geqq u'(x_t^*) \geqq u'(\overline{x}^*) \equiv \underline{u}' > 0 \text{ for } T^* - 1 \leqq t < T .$$

We therefore have from (45 b), (46)

$$ (48) \quad {}_{T^*-1}U^*_{T^*}(-\sigma) \geqq \int_{T^*-1}^{T^*} e^{\sigma t}\, u'(x_t^*)\left(g(z_t^*) - g(\hat{z}) - \dot{z}_t^*\right)dt \geqq $$

$$ \geqq -\,\bar{u}'\,\gamma\,\eta^2\,e^{\sigma T^*} + \underline{u}'(\eta-\epsilon)\,e^{\sigma(T^*-1)} $$

Pulling together these inequalities we have, for any $T^{**} \geqq T^*$, from (45 c), (43), (44), (47), (48), since $\bar{u}' \geqq u_o' > \hat{u}'$,

$$ {}_{T}W_{T^{**}}(-\sigma) \geqq \left[-\,\bar{u}'\left(\gamma\,\eta^2(1+\sigma^{-1}) + 2\epsilon\right) + \underline{u}'(\eta-\epsilon)e^{-\sigma}\right]e^{\sigma T^*} \equiv A\,e^{\sigma T^*}\,, \quad \text{say}\,. $$

It is now possible, within the restrictions already imposed, to choose first η and then ϵ small enough to make $A > 0$, next to choose T to correspond to ϵ according to (42), and finally, given $N > 0$, to choose T^* large enough to make

$$ W_{T^{**}}(-\sigma) = W_T(-\sigma) + {}_{T}W_{T^*}(-\sigma) \;\geqq\; W_T(-\sigma) + Ae^{\sigma T^*} > N \quad \text{for all} \quad T^{**} \geqq T^*\,. $$