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CONSUMPTION-SAVING DECISIONS WHEN THE HORIZON IS RANDOM

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1. Introduction

Most problems of allocation under uncertainty can be classified as either one-stage problems in which the source of uncertainty is a single random event, or multi-stage problems in which the source of uncertainty is an entire stochastic process. In the present essay we propose to study an allocation process which, in a sense, is intermediate in this classification: It is a multi-stage process in which uncertainty is introduced by a single random event, this event being the termination of the process. While it is true that an entire stochastic process can be defined so as to be equivalent to this event, the nature of the event allows the treatment of the problem to be quite different from the usual treatment of multi-stage problems involving uncertainty. In most such problems the notion of flexibility plays an important part. Roughly speaking, flexibility means that at each stage the decision maker acts according to a decision rule which takes into account the information gathered up to that stage. Such a flexible decision rule is often called a policy or a strategy. Present information is partly a product of past actions and the present action will in part determine future information. In our problem, however, flexibility has no role. It is possible for the decision maker to form a once-and-for-all decision rule (often called a plan or a program) which he would pursue without

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having to pause at each stage to take account of new information. The nature of flexible processes necessitates, in most cases, treatment by methods of dynamic programming. However, in our case the simpler informational structure of the process makes treatment by classical variational methods possible.

The problem which we propose to study here is one in which a consumer-unit allocates its resources over a lifetime, when the length of the lifetime is uncertain. Before entering the discussion of optimal behavior under this kind of uncertainty, we need, for purposes of reference, a summary of the main characteristics of optimal behavior in the analogous decision process with uncertainty altogether absent. A complete discussion of the perfect certainty case is contained in a paper [6] to which the present essay is a sequel. The notation and the basic assumptions of [6] will be retained.

Two types of utility functionals are considered, both of which are of the discounted-sum-of-utilities variety. First, what we shall refer to as the Fisher-constraint utility functional, to be used in a Fisher-type analysis, where lifetime savings enter as a constraint [2]. Second, a bequest-motive utility functional, to be investigated in cases where bequests are assumed to enter directly in the functional, thus rendering a savings-constraint unnecessary. We define the two functionals, denoting the former V and the latter U, as follows:

$$(1) \qquad V(c) = \int_0^T \alpha(t)\, g[c(t)]dt$$

(2) $$U(c) = \int_0^T \alpha(t)\, g[c(t)]dt + \beta(T)\varphi[S(T)] ,$$

where the definitions of the various symbols are:

T is the unit's horizon, a non-negative number.

$\alpha(t)$, $0 \leqq t \leqq T$ is a once differentiable function, assumed to be non-negative and to obey the normalization $\alpha(0) = 1$. $\alpha(t)$ is the <u>subjective discount function of consumption</u>.

$\beta(T)$ is a positive real number. However, in the presence of uncertainty T will be allowed to assume different values in some interval $[0, \bar{T}]$. In that case, $\beta(T)$ will be assumed to be a non-negative, once differentiable function on $[0, \bar{T}]$. It will be referred to as the <u>subjective discount function of bequests</u>.

$c(t)$ describes the unit's consumption level at time t. The function $c(t)$, $0 \leqq t \leqq T$, will be referred to as the <u>consumption plan</u>.

$S(t)$ is the unit's stock of <u>savings</u> at time t, and $S(T)$ is the unit's <u>bequests</u>.

g and φ are twice differentiable real functions. g is defined for all non-negative values and is strictly concave. φ is defined for all values and is concave. It is assumed that at least one of these functions is monotone increasing. In the Fisher-constraint case the monotone function is necessarily g.

S(t), the level of savings at time t, is further defined as the accumulation of some income flow, net of consumption outlays, compounded at some given rate of interest:

$$(3) \qquad S(t) = \int_0^t e^{\int_\tau^t j(x)dx} \left\{ m(\tau) - c(\tau) \right\} d\tau$$

where

$j(t)$ is the (instantaneous) rate of interest expected by the consumer-unit to prevail at time t.

$m(t)$ is the unit's receipts of income (other than interest) at time t. $m(t)$, $0 \leqq t \leqq T$, is the unit's income stream.

Two assumptions are implicit here, namely that $m(t)$ and $c(t)$ are measured in the same units and that $S(0)$, the initial asset-level, is equal to zero. Both of these assumptions are not essential in our discussion.

Consider first the Fisher-constraint case. The problem is:

$$(4) \qquad \max V(c)$$

$$\text{subject to:} \quad c(t) \geqq 0 \quad 0 \leqq t \leqq T$$

$$S(T) \geqq 0 .$$

Since g is assumed to be monotone increasing, the constraint on $S(T)$ will be fulfilled exactly, so that the problem reduces to

$$(5) \qquad \max \int_0^T \alpha(t)\, g[c(t)]\, dt$$

subject to: $c(t) \geqq 0 \qquad 0 \leqq t \leqq T$

$$\int_0^T e^{\int_t^T j(x)dx} \left\{ m(t) - c(t) \right\} dt = 0 .$$

The problem of attainment of this maximum in any given admissibility class will not be taken up. The present discussion will be confined to the formulation of necessary conditions which an extremal function must satisfy if it exists. The admissibility class implicit in the discussion is one of functions which are right-continuous on $[0, T]$. If an extremal exists in this class, then the strict concavity of g guarantees that this extremal will indeed provide a maximum.

Denote the extremal, i.e., the optimal consumption plan, by c^* . If c^* exists, then in intervals where the constraint $c(t) \geqq 0$ is ineffective, i.e., where c^* is interior, the following differential equation can be shown to hold:

$$(6) \qquad \dot{c}^*(t) = - \left[j(t) + \frac{\dot{\alpha}(t)}{\alpha(t)} \right] \frac{g'\left[c^*(t)\right]}{g''\left[c^*(t)\right]} .$$

A dot above the symbol denotes differentiation with respect to time. Equation (6) may be integrated to obtain a marginal utility condition applicable in intervals where the optimal plan c^* is interior:

$$(7) \qquad e^{-\int_t^T j(x)dx} \alpha(t)\, g'[c^*(t)] = k$$

where k is a constant of integration depending on the consumer's lifetime wealth.

Dropping the provision that t be taken in intervals where the optimal plan c^* is interior leads to the following modification of the marginal utility condition:

$$e^{-\int_t^T j(x)dx} \alpha(t)\, g'[c^*(t)] \leqq k \ , \qquad 0 \leqq t \leqq T \tag{8}$$

$$c^*(t) = 0 \quad \text{whenever} \quad < \text{ holds.}$$

Now let us turn to the bequest-motive case. Here the problem is

$$\max \left\{ \int_0^T \alpha(t)\, g[c(t)]dt + \beta(T)\varphi[S(T)] \right\} \tag{9}$$

$$\text{subject to:} \quad c(t) \geqq 0 \ , \quad 0 \leqq t \leqq T \ .$$

The remarks which were made in the case of the Fisher-constraint regarding the admissibility class and the existence of an optimal plan hold true in the unconstrained case as well.

It may be shown, once again, that in intervals where the optimal plan (if it exists) is interior, that is to say the constraint $c(t) \geqq 0$ is ineffective, the differential equation (6) must be satisfied. The optimal plan also satisfies the marginal utility condition (8), except that now the constant of integration k is expressed directly in terms of the marginal utility of bequests:

$$(10) \qquad e^{-\int_t^T j(x)dx}\alpha(t)g'[c^*(t)] \leqq \beta(T)\varphi'[S^*(T)], \quad 0 \leqq t \leqq T$$

$$c^*(t) = 0 \quad \text{whenever} \quad < \text{ holds,}$$

where $S^*(T)$ is the optimal level of bequests.

In both the constrained and the unconstrained cases, if an optimal plan c^* exists, then it can be shown to be continuous. Since we have already stated that in intervals where the optimal plan is interior it is also differentiable and therefore continuous, the above statement adds to our knowledge the fact that transitions to and from corner segments, where the constraint $c(t) \geqq 0$ is effective, are continuous and involve no jumps.

If we look back at the differential equation (6) and if we recall our concavity and monotonicity assumptions, we conclude immediately that

$$(11) \qquad \operatorname{sgn}\left[\dot{c}^*(t)\right] = \operatorname{sgn}\left[j(t) + \frac{\dot{\alpha}(t)}{\alpha(t)}\right]$$

where sgn denotes the sign function, and t is taken in intervals where c^* is interior. Equation (11), together with the knowledge that $c^*(t)$ is continuous, makes it possible to deduce the general shape of the optimal plan without being given prior specification of the functions g and φ.

We turn now to the question of what happens to optimal behavior when T, the horizon, is not a fixed number but a random variable distributed on some interval $[0, \bar{T}]$.

When T is random, two of the quantities which determine the features of the allocation process are also random: The attainable level of utility, and total lifetime savings, or bequests. As a general tool for dealing with the fact that the attainable level of utility is random, we have at our disposal the Bernoulli hypothesis of maximization of expected utility. The Bernoulli hypothesis is derivable from a set of rather compelling axioms, and it is usually accepted as a reasonably sound procedure. In cases where bequests enter directly in the utility functional, the Bernoulli hypothesis is all we need before we can proceed to a characterization of optimal behavior. However, in cases where a Fisher constraint of the type $S(T) \geqq 0$ is imposed, there remains the so-called feasibility problem to be solved: The constraint $S(T) \geqq 0$ restricts the choice of consumption plans c to a specific class, sometimes referred to as the feasible class. When $S(T)$ is random, membership of a plan c in the feasible class is a random event with an attached probability, and it is not clear in what sense one should optimize over such a class.

Ordinarily, the feasibility problem is treated in one of two ways: (a) The "decision-theoretical" way would be to define a penalty-function which would prescribe a penalty according to the extent to which the constraint is violated. This method of approach would then proceed to minimize expected penalty, as a part of the overall maximization process. (b) A procedure which corresponds to many of the standard rules in statistical inference would be to require that the constraint be met with a probability of α or more, where α is some preassigned number in the interval $[0, 1]$.

In our problem, the decision-theoretical penalty-function procedure would compel us, in effect, to insert bequests directly in the utility function, thus converting the problem into a member of the bequest-motive family of problems. It should be noticed, however, that our utility-of-bequests function, $\varphi[S(T)]$, in general does not only prescribe a penalty for violation of a bequests-constraint, but it also prescribes a reward for oversatisfaction of the constraint. If the notion of the penalty-function is applied in its usual narrower sense, φ must be restrained to satisfy the property

$$(12) \qquad \varphi'[x] = 0 \quad \text{for} \quad x \geqq 0 .$$

As for the second procedure, in which the solution is required to be feasible with a probability of α or more, the arbitrary manner in which α is chosen opens it to many objections. These objections are analogous in many ways to those raised recently in an article by K. J. Arrow [1] . In one case, however, this procedure is the natural one to apply: It is when violation of the constraint is a physically meaningless state of affairs. In such cases it is appropriate to require that the constraint be fulfilled with a probability of $\alpha = 1$. In our situation, a violation of the constraint $S(T) \geqq 0$ is not only physically meaningful, but even empirically reasonable under some circumstances. Nevertheless, in what follows we shall use a constraint which says that bequests must be nonnegative with probability one. This will be done merely to provide a framework for the introduction of insurance transactions into the picture.

2. <u>The Fisher-Constraint Imposed with Probability One</u>.

Let T , the consumer's horizon, be a random variable distributed on some interval $[0, \bar{T}]$ with a known probability distribution. Let this distribution be specified by a probability density function $\pi(t)$, $0 \leqq t \leqq \bar{T}$. Define the symbols $\Omega(t)$ and $\pi_t(\tau)$ as follows:

$$(13) \qquad \Omega(t) = \int_t^{\bar{T}} \pi(\tau)d\tau , \qquad 0 \leqq t \leqq \bar{T}$$

$$(14) \qquad \pi_t(\tau) = \frac{\pi(\tau)}{\Omega(t)} , \qquad 0 \leqq t \leqq \tau \leqq \bar{T} .$$

$\Omega(t)$ is the probability that the consumer will live to see time t , and $\pi_t(\tau)$ is the conditional probability density function of T , given that $T \geqq t$.

The utility functional, $V(c)$, is defined in (1), and we denote its expected value by $\bar{V}(c)$:

$$(15) \qquad \bar{V}(c) = EV(c) = \int_0^{\bar{T}} \pi(t) \int_0^t \alpha(\tau)\, g[c(\tau)]d\tau\, dt ,$$

and by a change in the order of integration we get

$$(16) \qquad \bar{V}(c) = \int_0^{\bar{T}} \Omega(t)\alpha(t)g[c(t)]dt .$$

We wish to determine the properties of that nonnegative consumption plan, c^* on $[0, \overline{T}]$ (if it exists), which maximizes $\overline{V}(c)$ subject to the statement that the Fisher-constraint shall hold with probability one:

$$(17) \qquad \text{Prob}\left\{ S(T) \geqq 0 \right\} = 1 \ .$$

This constraint is equivalent to the statement

$$(18) \qquad S(t) \geqq 0$$

whenever either $\pi(t) > 0$ or $t = \overline{T}$.

Because of our monotonicity assumption, we know that at $t = \overline{T}$ the constraint (18) is fulfilled exactly:

$$(19) \qquad S(\overline{T}) = 0 \ ,$$

and for other values of t , the constraint (18) may be rewritten as follows:

$$(20) \qquad \dot{S}(t) \geqq 0$$

whenever $\pi(t) > 0$ and $S(t) = 0$,

which, in view of the definition (3), is equivalent to

$$(21) \qquad c(t) \leqq m(t)$$

whenever $\pi(t) > 0$ and $S(t) = 0$.

The maximization problem may now be stated in the following way:

$$(22) \qquad \max \int_0^{\bar{T}} \Omega(t)\, \alpha(t) g[c(t)]dt$$

subject to (i) $c(t) \geqq 0$, $0 \leqq t \leqq \bar{T}$

(ii) $c(t) \leqq m(t)$ whenever $\pi(t) > 0$ and $S(t) = 0$

(iii) $S(\bar{T}) = 0$.

Assume for a moment that the constraints (i) and (ii) are both ineffective, so that the optimal plan, if it exists, is interior throughout the interval $[0, \bar{T}]$. Constraint (iii) can now be converted into a boundary condition, by expressing the function c in terms of the function S . The definition of $S(t)$ leads by differentiation with respect to t , to the equation

$$(23) \qquad c(t) = m(t) + j(t)S(t) - \dot{S}(t) ,$$

so the problem reduces to

$$(24) \qquad \max_S \int_0^{\bar{T}} \Omega(t)\alpha(t)g[m(t) + j(t)S(t) - \dot{S}(t)]dt ,$$

$$S(0) = S(\bar{T}) = 0 .$$

Applying the Euler condition and substituting $c^*(t)$ back inside $g[\cdot]$, one gets

$$(25) \qquad \Omega(t)\alpha(t)j(t)g'[c^*(t)] + \frac{d}{dt}\left\{ \Omega(t)\alpha(t)g'[c^*(t)] \right\} = 0 ,$$

which simplifies to

$$(26) \qquad \dot{c}^*(t) = -\left[j(t) + \frac{\dot{\alpha}(t)}{\alpha(t)} - \pi_t(t) \right] \frac{g'[c^*(t)]}{g''[c^*(t)]} .$$

If we now drop the momentary assumption that $c^*(t)$ is interior throughout, we are still in a position to state that if an interval exists in which $c^*(t)$ is interior, then in that interval the differential equation (26) must still hold. For any feasible plan which is interior in an interval $[t_o, t_1]$, a plan can be constructed which is also feasible and interior in $[t_o, t_1]$, the latter plan satisfying the differential equation (26) in $[t_o, t_1]$ and being no worse than the former.

Equation (26), when compared with its analogue, equation (6), in the case where uncertainty is absent, gives us an opportunity to make a statement about the effect of uncertainty of survival on impatience. Irving Fisher, when alluding to this question [2 , pp. 216-217] says:

> Uncertainty of human life increases the rate of preference for present over future income for many people, although for those with loved dependents it may decrease impatience.1/

1/ Fisher refers here to impatience in terms of a preference for present over future income. However, it is clear upon reading his discussion that by "income" in this context he has in mind a consumption-opportunity.

In the present section we are concerned with a consumer who does not have bequests in his utility functional, i.e., with a consumer with no "loved dependents," and we see that in intervals where the consumption plan is interior, the consumer is indeed more impatient than his counterpart under perfect certainty. In equation (6), the expression $-\dot{\alpha}(t)/\alpha(t)$ was the subjective discount factor. Now this expression is replaced by $-\dot{\alpha}(t)/\alpha(t) + \pi_t(t)$, an increase in the rate of subjective discount. As far as the consumer who has "no loved dependents" is concerned, Fisher's assertion is borne out in the following sense: The behavior of the consumer whose horizon is random will resemble that of a consumer who knows with certainty that his horizon is at $\overline{T}$, but who has an additional discount factor of $\pi_t(t)$, over and above $-\dot{\alpha}/\alpha$. There will be, however, one difference: The uncertain consumer is, in general, constrained by the requirement that he avoid getting himself into a position where his net assets would be negative. The certain-but-more-impatient consumer is not so constrained. This difference will, again in general, cause the behavior of the uncertain consumer to be different from that of the certain-but-more-impatient consumer even where this additional constraint [constraint (ii) in equation (22)] is ineffective.

We turn now to formulating the marginal utility condition for the present case. Such a condition can be obtained by integrating the differential equation (26), taking due account of the two inequality constraints. This procedure leads to

$$(27) \qquad e^{-\int_t^{\overline{T}} j(x)dx} \Omega(t)\alpha(t)\rho'[c^*(t)] \lesseqgtr K$$

$$c^*(t) = 0 \quad \text{if} \quad < \quad \text{holds}$$

$$c^*(t) = m(t) \text{ and } S^*(t) = 0 \quad \text{if} \quad > \quad \text{holds}$$

The optimal consumption plan, if it exists, consists of alternating segments of the following three types: (a) Corner segments where $c^*(t) = 0$, (b) corner segments where $c^*(t) = m(t)$, and (c) interior segments, where the differential equation (26) is obeyed. The marginal utility condition (27), in conjunction with our concavity assumption, can be shown to lead to the following theorem:

<u>If an optimal plan c^* exists and if m is a continuous function, then the optimal plan is continuous except possibly at points of transition into corner segments of type (b), where a downward discontinuity may occur.</u>[1]

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[1] The proof is analogous to the proof of the continuity theorem in [6], and it will not be repeated here.

It follows as a corollary that if $m(t)$ is positive everywhere in $[0, \overline{T}]$, then an interior segment always intervenes between any two corner segments of the optimal plan c^*.

Whenever $c^*(t)$ is interior, the differential equation (26) implies

$$(28) \qquad \operatorname{sgn}\left[\dot{c}^*(t)\right] = \operatorname{sgn}\left[j(t) + \frac{\dot{\alpha}(t)}{\alpha(t)} - \pi_t(t)\right] .$$

All of these facts, when taken together, permit us to attain a fairly detailed characterization of the optimal plan c^*. For example, suppose that the following situation prevails:

$$(29) \qquad \begin{aligned} &\pi(t) > 0\,, \quad 0 \leqq t \leqq \bar{T} \\ &m(t) = m\,, \quad 0 \leqq t \leqq \bar{T} \\ &\left[j(t) + \frac{\dot{\alpha}(t)}{\alpha(t)} - \pi_t(t) \right] \begin{cases} \geqq 0 & \text{for } 0 \leqq t \leqq t_o \\ \leqq 0 & \text{for } t_o \leqq t \leqq \bar{T} \end{cases} \\ &\text{for some } t_o \text{ in } [0, \bar{T}]\,. \end{aligned}$$

With these specifications, and with no information on the shape of the function g, we can make certain assertions on the behavior of the optimal plan $c^*(t)$, of which the following four are the most immediate:

(30) a. $c^*(0) \leqq m$

b. $c^*(\bar{T}) \geqq m$

c. If $c^*(t') = 0$ for some t', then $c^*(t) = 0$ for all t in $[0, t']$.

d. If $c^*(t'') = m$ and $S^*(t'') = 0$ for some t'', then $c^*(t) = m$ for all t in $[t'', \bar{T}]$.

By drawing a simple diagram one can convince oneself that the negation of any of these assertions is incompatible with the information which we have on the behavior of c^*.

Under the assumptions of this section, the uncertainty of survival has an effect both on the rate of saving at any point of time and on total lifetime savings. The two effects happen to be in opposite directions. At any moment of time, the uncertainty of survival tends to discourage the consumer from withholding current resources for future consumption. Over the entire lifetime, however, the uncertainty of survival causes the consumer to have positive expected savings, when in the absence of uncertainty his lifetime savings would be zero. The positive level of lifetime savings under uncertainty is, of course, unintentional and due only to the unpredictability of time of death. It would be natural in this situation to introduce annuities into the picture. We shall undertake to do so in Section 4.

3. Optimal Behavior in the Bequest-Motive Case.

Suppose now that the consumer's utility depends on bequests as well as on consumption and that, in particular, the utility functional is given by $U(c)$ of equation (2). We denote the expected utility functional by $\overline{U}(c)$:

$$(31) \qquad \begin{aligned} \overline{U}(c) &= EU(c) \\ &= \int_0^{\overline{T}} \pi(t) \left\{ \int_0^t \alpha(\tau) g[c(\tau)] d\tau + \beta(t)\varphi[S(t)] \right\} dt \\ &= \int_0^{\overline{T}} \left\{ \Omega(t)\alpha(t) g[c(t)] + \pi(t)\beta(t)\varphi[S(t)] \right\} dt \ . \end{aligned}$$

Our task is now to maximize $\overline{U}(c)$ subject only to the constraint $c \geq 0$. Assuming for a moment that this constraint is ineffective, we may investigate the variation of $\overline{U}$ as the function c varies, by writing

$$c = c^{*} + \epsilon x \tag{32}$$

where x is an arbitrary admissible function on $[0, \bar{T}]$. A necessary condition for c^{*} to be optimal is that the first variation of $\bar{U}$ vanish for all choices of x. The first variation of $\bar{U}$ is given by

$$\delta\bar{U} = \left.\frac{d\bar{U}}{d\epsilon}\right|_{\epsilon=0} \tag{33}$$

$$= \int_{0}^{\bar{T}} \left\{ \Omega(t)\alpha(t)g'[c^{*}(t)]x(t) - \pi(t)\beta(t)\varphi'[S^{*}(t)] \int_{0}^{t} e^{\int_{\tau}^{t} j(u)du} x(\tau)d\tau \right\} dt$$

$$= \int_{0}^{\bar{T}} x(t) \left\{ \Omega(t)\alpha(t)g'[c^{*}(t)] - \int_{t}^{\bar{T}} \pi(\tau)\, e^{\int_{t}^{\tau} j(u)du} \beta(\tau)\varphi'[S^{*}(\tau)]d\tau \right\} dt.$$

Setting $\delta\bar{U} = 0$ for all choices of x, one arrives at the following first-order condition for a maximum:

$$\Omega(t)\alpha(t)g'[c^{*}(t)] = \int_{t}^{\bar{T}} \pi(\tau)\, e^{\int_{t}^{\tau} j(u)du} \beta(\tau)\varphi'[S^{*}(\tau)]d\tau, \quad \text{for all } t \text{ in } [0,\bar{T}] . \tag{34}$$

Note that for $t = \bar{T}$, condition (34) is trivially fulfilled, so that a separate argument will have to be made to derive a condition for $t = \bar{T}$. For t other than $\bar{T}$, however, condition (34) can be divided by $\Omega(t)$, leading finally to

$$\alpha(t)g'[c^{*}(t)] = \int_{t}^{\bar{T}} \pi_{t}(\tau)\, e^{\int_{t}^{\tau} j(u)du} \beta(\tau)\varphi'[S^{*}(\tau)]d\tau, \text{ for all } t \text{ in } [0, \bar{T}) . \tag{35}$$

Condition (35) is clearly a marginal utility condition. It says that at time t, the marginal utility of consumption should be made equal to the conditional mean, given that the consumer survive to time t, of the properly discounted marginal utilities of bequests.

To derive a condition for $t = \bar{T}$ we make use of a limiting procedure. Since the right-hand-side of (35) is an arithmetic mean, it must satisfy the following inequalities:

$$(36) \qquad \max_{\tau \text{ in } [t,\bar{T}]} \left\{ e^{\int_t^\tau j(u)du} \beta(\tau)\varphi'[S^*(\tau)] \right\}$$

$$\geqq \int_t^{\bar{T}} \pi_t(\tau)\, e^{\int_t^\tau j(u)du} \beta(\tau)\varphi'[S^*(\tau)]d\tau$$

$$\geqq \min_{\tau \text{ in } [t,\bar{T}]} \left\{ e^{\int_t^\tau j(u)du} \beta(\tau)\varphi'[S^*(\tau)] \right\},$$

for all t in $[t, \bar{T})$.

As we now let t approach $\bar{T}$, we see that

$$(37) \qquad \int_t^{\bar{T}} \pi_t(\tau)\, e^{\int_t^\tau (j(u)du} \beta(\tau)\varphi'[S^*(\tau)]d\tau \to \beta(\bar{T})\varphi'[S^*(\bar{T})] .$$

We may now define the first order condition for $t = \bar{T}$ as the limit of the condition for t as $t \to \bar{T}$. Thus,

$$(38) \qquad \alpha(\bar{T})g'[c^*(\bar{T})] = \beta(\bar{T})\varphi'[S^*(\bar{T})]$$

is the desired condition. This condition is stated merely for the sake of completeness, since the probability that the consumer will reach $\bar{T}$ is equal to zero.

Restoring the constraint $c(t) \geqq 0$ into the analysis causes the following modification in the first-order condition:

(39)
$$\alpha(t)g'[c^*(t)] \leqq \int_t^{\bar{T}} \pi_t(\tau)\, e^{\int_t^\tau j(u)du} \beta(\tau)\varphi'[S^*(\tau)]d\tau \;, \quad 0 \leqq t < \bar{T} \;,$$

$c^*(t) = 0$ whenever $<$ holds.

Whenever the constraint $c(t) \geqq 0$ is ineffective, we may differentiate equation (35) to obtain a differential equation in $c^*(t)$. Observing that

(40)
$$\frac{d}{dt}\, \pi_t(\tau) = \pi_t(t)\, \pi_t(\tau) \;,$$

the differentiation of (35) with respect to t leads to

(41)
$$\dot{\alpha}(t)g'[c^*(t)] + \alpha(t)g''[c^*(t)]\dot{c}^*(t) = -\pi_t(t)\beta(t)\varphi'[S^*(t)] - [j(t)-\pi_t(t)]\,\alpha(t)g'[c^*(t)] \;,$$

which, in turn, can be written as follows:

(42)
$$\dot{c}^*(t) = -[j(t) + \frac{\dot{\alpha}(t)}{\alpha(t)} - \pi_t(t)]\,\frac{g'[c^*(t)]}{g''[c^*(t)]} - \frac{\pi_t(t)}{\alpha(t)}\,\frac{\beta(t)\varphi'[S^*(t)]}{g''[c^*(t)]} \;.$$

To equation (42) one may add an equation in $S^*(t)$, and thus obtain a system of two simultaneous differential equations. This additional equation is linear, and it is an immediate consequence of the definition of $S^*(t)$:

(43)
$$\dot{S}^*(t) = m(t) - c^*(t) + j(t)S^*(t) \;.$$

From the point of view of characterization of solution, the system (42)-(43) is not very promising, to say the least. Even when g and φ are of simple shapes, the solution of this system is very difficult indeed. As a result, little can be said in this case about the shape of the optimal plan c^* except that it is continuous.

If there is a point t in $[0, \bar{T}]$ at which $\pi(t) = 0$, then at that point the differential equation (42) is identical with its analogue in the perfect-certainty case (6). This, of course, does not mean that the optimal plan itself will resemble its perfect-certainty analogue even at points where $\pi(t) = 0$.

Equation (42) also shows that the uncertainty of survival affects impatience in two ways: On one hand, the consumer is less likely to withhold present resources for the sake of future consumption, because he knows that he may not live to see that future. On the other hand, an increase in present consumption is simultaneously a decrease in assets available for bequest in the event of death, and since bequests enter the consumer's system of preferences, the overall effect of uncertainty on impatience is ambiguous. This is indeed what Fisher expects the case to be. By rewriting equation (42) with a slight modification, we can point out the conditions under which uncertainty of survival would cause impatience to increase or decrease, as the case may be.

$$(44)\quad \dot{c}^*(t) = - \left[j(t) + \frac{\dot{\alpha}(t)}{\alpha(t)} \right] \frac{g'[c^*(t)]}{g''[c^*(t)]} + \frac{\pi_t(t)}{\alpha(t)} \frac{\alpha(t)g'[c^*(t)] - \beta(t)\varphi'[S^*(t)]}{g''[c^*(t)]}$$

We may interpret anything which causes $\dot{c}^*$ to increase (given the levels of c, j and S) as a decrease of impatience and anything which causes $\dot{c}^*$ to decrease as an increase of impatience. Given this interpretation, equation (44) says that at time t the consumer is more impatient than he would be in the absence of uncertainty if $\alpha(t)g'[c^*(t)] > \beta(t)\varphi'[S^*(t)]$, and that he is less impatient than he would otherwise be if $\alpha(t)g'[c^*(t)] < \beta(t)\varphi'[S^*(t)]$. Both of these situations are possible under the assumptions which have been made. The effect of uncertainty on the rate of saving depends on very much the same considerations, and its effect on total lifetime savings is, unfortunately, equally ambiguous. The attempt to assess this effect by introducing variations of different kinds in the probability distribution of T does not seem to bear fruit. More definitive results are, however, obtainable after the introduction of insurance.

4. Annuities, Life-Insurance, and the Fisher-Constraint.

Consider once again the consumer who is required to observe the Fisher constraint, $S(T) \geqq 0$, with probability one. Suppose, however, that an institution exists which, for a fee, would be willing to relieve the consumer of this predicament by undertaking to insure that the constraint be met at all times. The next few paragraphs describe how this would be achieved.

As we have seen, the requirement that the Fisher-constraint be met with probability one is equivalent to the statement

$$(45) \qquad c(t) \leqq m(t) \quad \text{whenever} \quad S(t) = 0 \quad \text{and} \quad \pi(t) > 0 .$$

Suppose now that at some point of time, say t_o , this constraint is effective. This means that at t_o the consumer would have liked to consume more by going into debt, but he cannot do so because in the event of death he would be leaving a legacy of debt which, the constraint says, he must not do. In reality, however, the consumer would usually have a way of circumventing this constraint by getting a life-insured loan. In other words, a financial institution will lend him the given amount and simultaneously sell him a life-insurance policy for the same amount. The consumer, in effect, pays an additional interest charge, in the form of a premium, which is computed from what is believed to be the consumer's probability of survival.

Now consider the possibility that at some other point of time, say t_1, the consumer's assets are actually positive. At t_1, then, the consumer is saving some of his resources for future consumption, and these resources are stored in the form of notes bearing an instantaneous rate of interest of $j(t_1)$. However, the consumer may not live to see the future for which he is saving. If he actually dies in the meantime, his savings become a legacy in which (as in all Fisher-constraint cases) he has no interest whatsoever. Once again, a financial institution, in most cases an insurance company, would be able to offer the consumer an arrangement in which both parties stand to gain. Under this arrangement, the consumer would deposit his savings at the financial institution and the institution would pay him a rate of interest which is higher than the market rate, provided that in the event of the consumer's death, the institution would be held free of any obligation. In effect, the consumer who enters such an arrangement is buying an annuity.

In the Fisher-constraint model, if annuities were available to the consumer, then he would invest his entire savings at each point of time in such annuities. The reason is obvious: With annuities, the consumer gets a higher rate of interest on his savings so long as he lives, and he pays for this excess interest in resources which would be left over after his death and which are therefore of no value to him.

For convenience of reference, let us now define two types of financial obligations which the consumer can either buy or sell: (a) "j notes," i.e., notes which, at time t, bear an instantaneous rate of interest of $j(t)$. (b) "j+ notes," i.e., notes which bear at time t an instantaneous rate of interest larger than $j(t)$ according to some actuarial schedule, but which are automatically canceled in the event of the consumer's death. Thus, taking a life-insured loan is a transaction in which the consumer <u>sells</u> j+ notes, and buying an annuity is a transaction in which the consumer <u>buys</u> j+ notes. In the Fisher-constraint case, the consumer's assets or liabilities would be entirely in the form of j+ notes, and none would be in the form of j notes. The difference between our j+ notes and realistic annuities or life-insured loans is mainly that the j+ notes are extremely short-term obligations, whereas life-insured loans and annuities frequently are not.

We now have to make some assumption concerning the rate of interest which j+ notes yield to their bearer. In other words, we have to make an assumption concerning the behavior of the financial institution with which the consumer would negotiate a life-insured loan or from which he would buy an annuity. We shall assume that the rate of interest on j+ notes is <u>actuarially fair</u>. In other words, we shall assume that the buyer of j+ notes requires the expected value of the interest payment at time t to be at a rate equal to the market rate, $j(t)$. Ordinarily, if the financial institution is "risk-averse," i.e., if it has a concave utility function and it maximizes expected utility, then it would demand an expected rate of interest which is higher than the market rate, and it would pay at a rate whose expected value is lower than the market rate. We make the assumption of an actuarially fair rate in order to avoid a detailed analysis of the decision process facing the financial institution.

Denote the rate of interest on j+ notes at time t by $r(t)$. Suppose 1 dollar's worth of j+ notes is bought at time t and then redeemed with interest at time $t + \Delta$, unless the consumer's death intervenes. The return (including principal) at time $t + \Delta$, if the consumer survives, is approximately equal to $1 + \Delta r(t)$, with the approximation approaching equality as $\Delta \to 0$. The assumption that $r(t)$ is actuarially fair means that it has to satisfy the equation:

$$(46) \qquad [1 + \Delta\, r(t)]\, \frac{\Omega(t + \Delta)}{\Omega(t)} \doteq 1 + \Delta\, j(t)$$

where $\doteq$ denotes approximation which tends to equality as $\Delta \to 0$. The ratio $\Omega(t + \Delta)/\Omega(t)$ is the conditional probability that the consumer survive to time $t + \Delta$, given that he had survived to time t. Isolating $r(t)$ in (46) leads to

$$(47) \qquad r(t) \doteq \frac{\Omega(t) - \Omega(t + \Delta)}{\Delta\, \Omega(t + \Delta)} + j(t)\, \frac{\Omega(t)}{\Omega(t + \Delta)} \quad .$$

The exact expression for $r(t)$ is now obtained by letting $\Delta \to 0$ in the right-hand-side of (47):

$$(48) \qquad r(t) = \pi_t(t) + j(t) \; .$$

As has already been mentioned, if $j+$ notes are available to him, the Fisher-constraint consumer will keep all his assets or liabilities in the form of $j+$ notes. The effective rate of interest for purposes of calculating this consumer's optimal plan is therefore $r(t)$, $0 \leqq t < \overline{T}$. It may be noted that if for some t, $\pi(t) = 0$ then at time t the consumer can hold liabilities in j notes. But this does not cause any complication because if $\pi(t) = 0$, the j note and the $j+$ note are one and the same thing.

Holding assets and liabilities exclusively in the form of $j+$ notes means that the Fisher-constraint is always exactly fulfilled as a result of the actions of the financial institution. Hence, the only constraint which has to be taken into account in the formation of the optimal consumption plan is the constraint $c(t) \geqq 0$. Whenever this constraint is ineffective, the optimal consumption plan is interior and it must satisfy the differential equation (26), with $r(t)$ replacing $j(t)$ as the rate of interest:

$$\text{(49)} \qquad \dot{c}^*(t) = -\left[r(t) + \frac{\dot{\alpha}(t)}{\alpha(t)} - \pi_t(t)\right] \frac{g'[c^*(t)]}{g''[c^*(t)]} \quad .$$

But in view of (48), this equation reduces to

$$\text{(50)} \qquad \dot{c}^*(t) = -\left[j(t) + \frac{\dot{\alpha}(t)}{\alpha(t)}\right] \frac{g'[c^*(t)]}{g''[c^*(t)]} \quad ,$$

which is identical with equation (6) of the perfect-certainty case. So far as the differential equation in the optimal plan is concerned, the introduction of annuities and life-insured loans is equivalent to removal of the uncertainty of survival. This is clearly not true, in general, for the constant of integration which would have to be evaluated in solving equation (50). It is, in fact, meaningless to speak of "the" constant of integration in the perfect-certainty case, because for each horizon T in $[0, \bar{T}]$, there would, in general, be a different constant of integration, depending on the corresponding lifetime wealth. A question which still remains to be answered is the following: What constraint would have to be used in the evaluation of this constant? Alternatively, what is the constraint which keeps the consumer who has access to $j+$ notes from an unbounded consumption stream? Clearly, the constraint is given by the requirement that at time $\bar{T}$ the consumer must plan to settle his accounts with the financial institution. The appropriate way of writing this constraint turns out to be

$$(51) \qquad \int_0^{\bar{T}} e^{-\int_0^t r(\tau)d\tau} \left\{ m(t) - c(t) \right\} dt = 0 ,$$

i.e., over the entire interval $[0, \bar{T}]$, the accumulated excess of consumption over earnings, discounted backwards to $t = 0$ at the actuarial rate $r(t)$, must be equal to zero. Constraint (51) is expressed in terms of a discounted sum rather than in terms of a compounded sum even though we have been using compounded sums all along. The reason for this is that the integral corresponding to (51), where compounding is used rather than discounting, does not exist. It is not without a detailed argument that we can justify the substitution of a discounted sum where the appropriate compounded sum does not exist. We shall attempt to give such a detailed argument in the next section.

In any case, it is interesting to note that constraint (51), upon substitution of $j(t) + \pi_t(t)$ for $r(t)$, reduces to:

$$\int_0^{\bar{T}} \Omega(t) e^{-\int_0^t j(\tau)d\tau} \left\{ m(t) - c(t) \right\} dt = 0 , \tag{52}$$

which, in turn, is equivalent to

$$E\left[\int_0^T e^{-\int_0^t j(\tau)d\tau} \left\{ m(t) - c(t) \right\} dt \right] = 0 , \tag{53}$$

where T is the random variable over which the expectation E is taken. In the perfect-certainty case, i.e., when T was a fixed number, we had the constraint $S(T) = 0$ which, for finite T, could have been written as follows:

$$\int_0^T e^{-\int_0^t j(\tau)d\tau} \left\{ m(t) - c(t) \right\} dt = 0 . \tag{54}$$

Now that T is a random variable, we find that the introduction of insurance and annuities in the system leads to the requirement that (54) hold on the average. Equation (53) also has a different interpretation: It says that if we view the transactions which the consumer makes with the insurance company as a single lifetime contract, the expected profits which the company will derive from the contract will be equal to zero. This is, of course, another expression of the fact that the rate of interest which the insurance company charges and pays is actuarially fair. The consumer's lifetime contract with the insurance company is one in which he offers to transfer to the company a stream of funds $m-c$ (that is, a flow of $m(t) - c(t)$ at time t) so long as he lives. The profit which the insurance company would make from such a contract, neglecting operating costs, is equal to the value of the stream of earnings, discounted for alternative use of resources. Therefore, given that the consumer dies, say, at time T , the profit which the company derives from the contract is

$$(55) \qquad \int_0^T e^{-\int_0^t j(\tau)d\tau} \left\{ m(t) - c(t) \right\} dt .$$

What condition (53) says is that the contracts which the consumer offers the company on terms acceptable to both are constrained so as to give rise to zero expected profits.

Suppose now that an economy exists where all the consumers satisfy the assumptions of this section. Lifetime consumer savings in such an economy are zero and insurance companies have zero profits. If, in addition, the population structure and lifetime incomes in this economy are stationary, then it can be shown that expected aggregate savings generated in the consumer sector of such an economy are zero at all times. However, if it is assumed that insurance companies lend at more than the actuarially fair rate and borrow at less than the actuarially fair rate, then expected aggregate savings generated in the consumer sector would always be positive. If the insurance companies are risk-averse in the usual sense of this word, the latter would indeed be the case.

5. The Bequest-Motive Case: A Portfolio Problem.

The usefulness of annuities and life insurance is clearly not restricted to the Fisher-constraint case. Let us return once again to the unconstrained case, in which the consumer's preferences depend directly on bequests as well as on consumption. In this situation, it will, for example, be advantageous to the consumer to buy an annuity if his current asset level exceeds what he considers to be an appropriate bequest in case of death. As for life insurance, in the Fisher-constraint case buying life insurance was necessarily in conjunction with a consumption-loan. In the bequest-motive case, however, there is room for the more common phenomenon, where the consumer buys life insurance to provide for his survivors in case of death. In the present section, then, we shall look for a characterization of optimal behavior in the case where bequests enter directly in the consumer's preferences, and where life insurance and annuities are available to him.

Once again, we shall refer to regular notes, bearing the market rate of interest $j(t)$ at time t, as "j notes" and to actuarial notes, which bear the rate of interest $r(t)$ at time t, as "j+ notes." We recall that j+ notes are automatically canceled in the event of the consumer's death, whereas j notes remain valid and in case of death they become a part of the consumer's bequest. Under the Fisher-constraint hypothesis, the consumer held all his assets (positive or negative) in the form of j+ notes. Under present circumstances, however, the consumer will, in general, hold his assets in the form of both j notes and j+ notes. The consumer's decision problem now consists of choosing, for each point of time t in $[0, \bar{T}]$, both the optimal consumption level and the optimal portfolio-mix, as between j notes and j+ notes.

As before, the quantity $S(t)$ will be used to denote the consumer's stock of j notes at time t. It will also be convenient to define two other quantities:

$Q(t)$ will denote the consumer's stock of $j+$ notes at time t.

$R(t)$ will denote the consumer's total assets at time t.

$S(t)$, $Q(t)$ and $R(t)$ are all in terms of dollars invested, so we can write:

$$(56) \qquad R(t) = S(t) + Q(t) \qquad \text{for all } t \text{ in } [0, \overline{T}] .$$

On the other hand, $R(t)$ is made up of the accumulation of income receipts minus consumption outlays, plus interest charges:

$$(57) \qquad R(t) = \int_0^t \left\{ m(\tau) - c(\tau) \right\} d\tau + \int_0^t j(\tau)\, S(\tau) d\tau + \int_0^t r(\tau)\, Q(\tau) d\tau .$$

Substituting from (56) in (57) leads to the elimination of Q and to the following recursive relationship in $R(t)$:

$$(58) \qquad R(t) = \int_0^t \left\{ m(\tau) - c(\tau) \right\} d\tau + \int_0^t r(\tau)\, R(\tau) d\tau - \int_0^t \pi_\tau(\tau)\, S(\tau) d\tau ,$$

which, in turn, is equivalent to

$$(59) \qquad R(t) = \int_0^t e^{\int_\tau^t r(u) du} \left\{ m(\tau) - c(\tau) - \pi_\tau(\tau)\, S(\tau) \right\} d\tau .$$

That (58) and (59) are equivalent can be seen by differentiation of both with respect to t.

It is important to recognize that, in the present case, $S(t)$ is no longer determined by the stream $\left\{ c(\tau),\ 0 \leqq \tau \leqq t \right\}$. Since $R(t)$ is in no way restricted, equation (59) does not constitute a restriction on $c(\tau)$ and $S(\tau)$. Indeed, c and S are the two decision variables in our problem. Consequently, the expected utility functional, $\bar{U}$, now depends on both:

$$(60) \qquad \bar{U}(c,S) = \int_0^{\bar{T}} \left\{ \Omega(t)\alpha(t)g[c(t)] + \pi(t)\beta(t)\varphi[S(t)] \right\} dt \ .$$

Our aim is to characterize the maximizing functions, c^* and S^*, if they exist.

Clearly, the maximization of (61) is not without a constraint at all. The functions c and S, though not locally constrained, are in fact globally constrained. It is reasonable to think of the global constraint as

$$(61) \qquad Q(\bar{T}) = 0 \ ,$$

which means that at time $\bar{T}$ the insurance company will refuse to write a life-insurance policy for the consumer, and that the consumer, for his part, will have no use for an annuity. This is signified in the model by the fact that as t approaches $\bar{T}$, $r(t)$, the rate of interest on j+ notes, tends to infinity. An infinite rate of interest means, as it usually does, that no transaction can take place.

The fact that $\pi_t(t)$, and with it $r(t)$, diverges as $t \to \bar{T}$ is clear whenever $\pi(\bar{T}) > 0$, because $\pi_t(t) = \pi(t)/\Omega(t)$. However, when $\pi(\bar{T}) = 0$, the divergence of $\pi_t(t)$ has to be demonstrated. We shall do so now for the case where $\pi(t)$ is monotone decreasing in the neighborhood of $\bar{T}$. If $\pi(\bar{T}) = 0$, then the case when $\pi(t)$ is monotone decreasing in the neighborhood of $\bar{T}$ will certainly cover those probability density functions which can reasonably occur in statistics of mortality. To prove the divergence of $\pi_t(t)$, let a quantity $\pi(\tau_t)$ be defined as follows:

$$(62) \qquad \pi(\tau_t) = \max_{\tau \text{ in } [t,\bar{T}]} \pi(\tau) .$$

Then, by the mean value theorem:

$$(63) \qquad \Omega(t) \leqq (\bar{T} - t)\, \pi(\tau_t) \qquad \text{for all } t .$$

It therefore follows that

$$(64) \qquad \frac{\pi(t)}{\Omega(t)} \geqq \frac{1}{\bar{T} - t} \; \frac{\pi(t)}{\pi(\tau_t)} \qquad \text{for all } t .$$

But now, if $\pi(t)$ is monotone decreasing in the neighborhood of $\bar{T}$, then $\pi(t)/\pi(\tau_t) = 1$ in that neighborhood, so that

$$(65) \qquad \frac{\pi(t)}{\Omega(t)} \geqq \frac{1}{\bar{T} - t} \qquad \text{for } t \text{ near } \bar{T} ,$$

i.e., $\pi_t(t)$ diverges as $t \to \bar{T}$. Since the rate of interest on $j+$ notes becomes infinite as t approaches $\bar{T}$, one has to require that the consumer's net holdings of $j+$ notes at time $\bar{T}$ be zero. This is precisely what condition (61) says. Rewriting (61) in view of (56) yields:

$$(66) \qquad R(\bar{T}) = S(\bar{T}) ,$$

which, in view of (59), becomes

$$S(\bar{T}) = \int_0^{\bar{T}} e^{\int_t^{\bar{T}} r(\tau)d\tau} \left\{ m(t) - c(t) - \pi_t(t)S(t) \right\} dt \quad . \tag{67}$$

At first blush, this condition looks like a bona fide constraint. However, a second look reveals that, in general, the integral in (67) does not exist.

It is interesting to note that the situation here is analogous to that discussed in a controversy between A. P. Lerner [3] and P. A. Samuelson [5] a few years ago. There an economy of infinite duration and no capital was considered, in which each generation consumes more than it produces by receiving from younger generations transfers which exceed those it had made to older generations. Here we encounter a similar phenomenon in a world in which the horizon is finite but random. If the consumer is required to settle his accounts with the insurance company at time $\bar{T}$, then he can consume more than he earns and in fact sustain any consumption plan whatever. He does this by borrowing with life insurance at an ever increasing rate of interest, with the complete certainty (probability one) that when the time comes to settle his debts he would be safely dead. In short, the integral in (67) does not exist because a constraint does not exist.

In order to remedy the situation, one must agree that the constraint which requires the consumer to settle his accounts with the insurance company should apply to some stage in the consumer's lifetime which he has a positive probability of reaching, say $\bar{T} - \epsilon$.[1] With this in mind, it is possible to derive a constraint which retains the convenience of saying $\bar{T}$, while actually

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[1] Alternatively, one could assume positive probability mass at $\bar{T}$.

meaning $\overline{T} - \epsilon$. To start out, we write the constraint

$$Q(\overline{T} - \epsilon) = 0 \qquad \text{for some } \epsilon > 0 . \tag{68}$$

By (56):

$$R(\overline{T} - \epsilon) = S(\overline{T} - \epsilon) \tag{69}$$

and, by (59),

$$S(\overline{T} - \epsilon) = \int_0^{\overline{T}-\epsilon} e^{\int_t^{\overline{T}-\epsilon} r(\tau)d\tau} \left\{ m(t) - c(t) - \pi_t(t)S(t) \right\} dt . \tag{70}$$

Multiplying both sides by $e^{-\int_0^{\overline{T}-\epsilon} r(\tau)d\tau}$ leads to

$$e^{-\int_0^{\overline{T}-\epsilon} r(\tau)d\tau} S(\overline{T}-\epsilon) = \int_0^{\overline{T}-\epsilon} e^{-\int_0^t r(\tau)d\tau} \left\{ m(t) - c(t) - \pi_t(t)S(t) \right\} dt . \tag{71}$$

We may safely assume that $S(\overline{T}-\epsilon)$ is finite for all ϵ , because otherwise an optimal solution (c^*, S^*) does not exist, and we are seeking conditions which have to hold given that an optimal solution exists. Thus, by picking ϵ small enough, the left-hand-side of (71) can be made arbitrarily small. The right-hand-side of (71) can be made arbitrarily close to

$$\int_0^{\overline{T}} e^{-\int_0^t r(\tau)d\tau} \left\{ m(t) - c(t) - \pi_t(t)S(t) \right\} dt . \tag{72}$$

So, a constraint which approximates (71) for small ϵ is:

(73) $$\int_0^{\overline{T}} e^{-\int_0^t r(\tau)d\tau} \left\{ m(t) - c(t) - \pi_t(t)S(t) \right\} dt = 0 ,$$

which is the desired constraint. Clearly, (73) does <u>not</u> approximate (71) for ϵ actually equal to zero, because then (71) is not defined.

In light of the fact that

(74) $$r(t) = j(t) - \frac{d}{dt} \log \Omega(t) ,$$

constraint (73) reduces to:

(75) $$\int_0^{\overline{T}} \Omega(t)\, e^{-\int_0^t j(\tau)d\tau} \left\{ m(t) - c(t) - \pi_t(t)S(t) \right\} dt = 0 .$$

Our problem is now to maximize (60) subject to (75) and also subject to $c(t) \geqq 0$. Let the optimal functions, if they exist, be denoted c^* and S^* as usual, and let

(76) $$c(t) = c^*(t) + \epsilon_1 x(t)$$
$$S(t) = S^*(t) + \epsilon_2 y(t)$$

for some arbitrary functions x and y on $[0, \overline{T}]$. Let us now form the Lagrangian for our maximization problem, denoting it by $L(c,S)$

$$L(c,S) = \int_{0}^{\bar{T}} \Big[\Omega(t)\alpha(t)g[c(t)] + \pi(t)\beta(t)\varphi[S(t)] + \lambda\Omega(t)e^{-\int_{0}^{t} j(\tau)d\tau} \Big\{ m(t) - c(t) - \pi_t(t)S(t) \Big\} \Big] dt \; , \tag{77}$$

which in view of (76) is a function of ϵ_1 and ϵ_2 . A first-order condition for a maximum, ignoring for the moment the constraint $c(t) \geqq 0$, is obtained by setting

$$\left.\frac{\partial L}{\partial \epsilon_1}\right|_{\epsilon_1=\epsilon_2=0} = 0 \tag{78}$$

$$\left.\frac{\partial L}{\partial \epsilon_2}\right|_{\epsilon_1=\epsilon_2=0} = 0$$

identically, for all choices of functions x and y . This procedure yields the following two equations:

$$\Omega(t)\alpha(t)g'[c^*(t)] = \lambda\Omega(t)\, e^{-\int_{0}^{t} j(\tau)d\tau} \; , \tag{79}$$

$$\pi(t)\beta(t)\varphi'[S^*(t)] = \lambda\pi(t)\, e^{-\int_{0}^{t} j(\tau)d\tau} \; , \tag{80}$$

and in both cases obvious cancellations can be made, so that the final pair of conditions is:

(81) $$\alpha(t)g'[c^*(t)] = \lambda e^{-\int_0^t j(\tau)d\tau}$$

(82) $$\beta(t)\varphi'[S^*(t)] = \lambda e^{-\int_0^t j(\tau)d\tau} ,$$

both holding for t in $[0, \overline{T}]$, (81) holding only for t where c^* is interior and (82) holding only for t where $\pi(t) > 0$.

A problem arises when $\pi(t) = 0$. In that case, equation (80) is trivially satisfied and equation (82) can no longer be derived from it. But this is entirely according to expectation. If $\pi(t) = 0$, the quantity $S(t)$, which is the consumer's bequests in case of death, ceases to be of consequence. Hence, whenever $\pi(t)$ happens to be zero, $S(t)$ may be picked arbitrarily. For the sake of consistency, we fix $S^*(t)$ to satisfy equation (82) also when $\pi(t) = 0$.

Conditions (81) and (82) can, of course, be merged into one marginal utility condition:

(83) $$\alpha(t)g'[c^*(t)] = \beta(t)\varphi'[S^*(t)] ,$$

which says that the marginal utility of consumption at time t must be equal to the marginal utility of bequests at time t .

Conditions (81) and (83) have to be adjusted to allow for the constraint $c(t) \geqq 0$. This is done introducing the inequality $\leqq$ in both and requiring that $c^*(t)$ be zero whenever $<$ holds.

By differentiating (81) and (82) with respect to t, we obtain two differential equations, in c^* whenever it is interior, and in S^* :

(84) $$\dot{c}^*(t) = -\left[j(t) + \frac{\dot{\alpha}(t)}{\alpha(t)}\right] \frac{g'[c^*(t)]}{g''[c^*(t)]}$$

(85) $$\dot{S}^*(t) = -\left[j(t) + \frac{\dot{\beta}(t)}{\beta(t)}\right] \frac{\varphi'[S^*(t)]}{\varphi''[S^*(t)]} .$$

Equation (84) is once again identical with the differential equation governing optimum consumption when no uncertainty is present.

Equations (84) and (85) make it possible to describe in detail both the optimal consumption plan c^* and the optimal bequest plan S^* in any given example. It is interesting to note that the possibility of insurance makes it possible to determine c^* and S^* independently, up to a constant of integration.

6. Conclusion

In an earlier essay [6] an attempt was made to study the differences between the implications of the Fisher-constraint hypothesis and those of the bequest-motive hypothesis in a world of complete certainty. It may therefore be appropriate to conclude the present essay with the following question: How does the behavior of the consumer in the face of uncertainty of survival reflect on the relative plausibility of the two hypotheses? The answer which the foregoing discussion seems to give to this question goes back to Marshall's Principles [4, p. 228]:

> But were it not for family affections, many who now work hard and save carefully would not exert themselves to do more than secure a comfortable annuity for their own lives.
>
> ...That men labour and save chiefly for the sake of their families and not for themselves, is shown by the fact that...in this country alone twenty millions a year are saved in the form of insurance policies and are available only after the death of those who save them.

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