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PROPERTIES OF EFFICIENT ACCUMULATION PATHS IN A CLOSED PRODUCTION MODEL

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By

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I. INTRODUCTION

1.1. The examination of an abstract economy in which goods available at one time period are produced by the use of the same goods available in the previous period under constant returns to scale has been the subject of a continuously growing literature, which among others includes Dorfman, Samuelson, and Solow [1], Hicks [5], Morishima [12], Radner [14], Furuya and Inada [2], McKenzie [11], and Nikaido [13]. Interest has been centered on the examination of the behavior of "efficient" accumulation paths of finite or infinite duration.

1.2. Most of these contributions consider paths of finite duration. Briefly they have proved that -- under certain conditions -- efficient paths tend to move near a von Neumann path (of maximal balanced growth) for most of their duration. The corresponding theorems have been called "turnpike theorems" from the turnpike-like function of the von Neumann path.

The most general type of proof of the turnpike theorem which has so far been used is that of Radner[1] [14]. He showed that every efficient path cannot

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1. Morishima [12] and McKenzie [10] have independently established another version of the theorem using arguments based on the famous Solow and Samuelson [15] relative stability theorem. However the approach seems to be restricted to models with no joint production of commodities by each industry. The main objection to such an assumption is that durable capital goods cannot be considered. If such goods exist, joint production of commodities by each industry using capital goods necessarily emerges.

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be such that negative profits under the von Neumann prices subsist indefinitely. Thus there always exists a finite number of periods beyond which the path must be close to a path exhibiting zero profits under the von Neumann prices. If only the von Neumann process is profitable under the von Neumann prices then the turnpike theorem is proved in a very elementary way, which moreover does not place any other restrictive conditions on the technology (e.g., joint production is freely admitted). However, this single condition is very strong indeed. Essentially, it does not allow for decentralization of production in the economy. Consequently, if production is carried out by distinct production units, the Radner-type argument can only be used to show that any efficient path will approximate for most of its course a particular facet of the aggregate production set in which a von Neumann ray lies. This approach has been used by McKenzie [11]. It is only under additional assumptions that the efficient path will approach the von Neumann ray on that facet.

1.3. On the other hand, Furuya and Inada [2] examine efficient paths of infinite duration. Using a Radner-type argument they have shown that in a general von Neumann economy [16] any such path converges asymptotically to the von Neumann path. This global stability of the von Neumann path with respect to efficient paths of infinite duration corresponds to its turnpike property with respect to efficient paths of finite duration.

In the economy considered in [2] many accumulation paths are feasible from any initial position. However, if the von Neumann path is globally stable, then it is shown in [2] that only one efficient path starts from any initial position. Thus such an economy exhibits accumulation paths of the type postulated by Solow and Samuelson [15].

Most of the assumptions in [2] are not too restrictive. Their production system admits joint production of commodities, and various activities may produce every commodity. However, their assumption of strong superadditivity essentially implies that the aggregate production set is a strictly convex cone. But if production is carried out by distinct production units, then the aggregate production set may very well not be a strictly convex cone, even if each unit's production set has this property.[2,3] We may therefore conclude that a model like that in [14] or [2] does not allow for decentralization of production in the economy. Their model rather applies to an economy in which production is planned by a central bureau. This raises the question as to whether the results of [2] are valid in an economy in which production is carried out by many different production units each operating on the basis of its own production set.

1.4. In this paper we consider an economy with decentralized production explicitly introduced. The production system is composed of several

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2. E.g., suppose that a price system satisfies the profit conditions of competitive equilibrium and that more than one production unit is then operating at positive levels. In this case, if the production processes used by the units are not proportional to each other, then the dimensionality of the corresponding efficient facet of the aggregate production set in greater than one. Thus the aggregate production set is not a strictly convex cone.

3. This is the same objection as that raised with respect to Radner's model. Actually, the production system in both [14] and [2] is essentially the same.

production units each having its own technology, and is more general than that of McKenzie [11].

First, we examine the behavior of efficient paths of finite duration and we establish the turnpike theorem along the lines of the proof used by McKenzie [11].

Second, we consider efficient paths of infinite duration. A modification of the proof of the first part is sufficient for establishing the global stability of the von Neumann ray.

Finally, we consider the problem of the uniqueness of efficient paths of finite or infinite duration. Such paths are not necessarily unique without further restrictions on the technology. The question whether in a general closed production system with decentralized production efficient paths are uniquely determined by their initial position is not fully answered.

2. THE MODEL. PRELIMINARY RESULTS

2.1. The economy examined can best be described as a closed linear model of production (see Gale [3]). In each time period there exist n commodities, $i \in I = \{1, \ldots, n\}$.

The economy's production set T describes the technological possibilities of production of these commodities.

Definition 1: T is the set of pairs of n-dimensional commodity vectors $(-y^t, y^{t+1})$, $t \geqq 0$, such that the production of the output vector y^{t+1}, $y^{t+1} \geqq \Theta$, at $t+1$ from the input vector $-y^t$, $y^t \geqq \Theta$, at t is technically possible if and only if $(-y^t, y^{t+1}) \in T$. $T \subset R^n_t \times R^n_{t+1}$, $t \geqq 0$. [4]
T remains constant over time. Each $(-y^t, y^{t+1}) \in T$ is called a production process at t.

2.2. Assumptions:

T is assumed to have the following properties:

(T_1) T is a closed and convex cone in $R^n_t \times R^n_{t+1}$,
(Constant returns to scale and additivity);

(T_2) $(-y^t, y^{t+1}) \in T$ and $-y'^t \leqq -y^t$, $y^{t+1} \geqq y'^{t+1} \geqq \Theta$, implies that $(-y'^t, y'^{t+1}) \in T$, (Free disposal);

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4. R^n_t is a Euclidean n-dimensional space, the commodity space at t, $t \geqq 0$. We consider elements of R^n_t as column vectors, and elements of the dual space of R^n_t as row vectors.

We also use Ω to denote the positive orthant of the particular commodity space under consideration.

(T_3) $T \cap (\Theta, \Omega) = (\Theta, \Theta)$, (Impossibility of free production); and

(T_4) For every $i \epsilon I$, there exists $(-y^t, y^{t+1}) \epsilon T$ such that $y_i^{t+1} > 0$, (Possibility of production of every commodity).[5]

2.3. Technical efficiency:

Definition 2: A sequence $\{y^t\}_{t+0}^{H}$ of $H+1$ commodity vectors is a feasible accumulation path of duration H , $+\infty > H > 0$, given y^o , if $(-y^t , y^{t+1}) \epsilon T$, for all $t \epsilon [0 , H-1]$.

Such a path is also denoted by $(-y^o , y^H) \epsilon T^H$. T^H is the H period production set; it is easily seen that it has the same properties as T .

Definition 3: A sequence $\{y^t\}_{t=0}^{\infty}$ is a feasible accumulation path of infinite duration, given y^o , if $(-y^t , y^{t+1}) \epsilon T$ for all $t = 0, 1, \ldots,$ ad inf.

Definition 4: $(-y^o, y^H) \epsilon T^H$ is an efficient path for H periods if and only if there is no $(-y'^o, y'^H) \epsilon T^H$ such that $y'^o \leqq y^o$, $y^H \leq y'^H$. Let $\overline{T}^H$ be the set of all efficient paths for H periods.

Definition 5: A feasible path $\{y^t\}_{t=0}^{\infty}$ is an efficient path of infinite duration if and only if for any H the finite path consisting of the first $H+1$ terms of the sequence is efficient of duration H .[6]

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5. The above assumptions are the standard ones in the literature; see e.g., Karlin [6, p. 338]. We note that (T_2) and (T_4) imply that T has a non-empty interior in $R_t^n \times R_{t+1}^n$, $t \geqq 0$.

6. See Furuya and Inada [2]. We may note that under our present assumptions on T we may have an efficient path of duration H , which is not an efficient path of duration H' , for some H' , $0 < H' < H$. Such a path cannot be a part of an efficient path of infinite duration. On the other hand, if T has the property that any one of the inputs can produce some outputs in positive quantities (e.g, by complete or partial free storage), then the possibility noted above is excluded. Assumption (T_5) , which will be introduced in Section 2.10, also excludes the above possibility.

2.4. $T(y) = \left\{ y^{t+1} \mid (-y^t, y^{t+1}) \in T \ , \ y^t \leqq y \right\}$ is the <u>attainable production set</u>, given $y \geqq \Theta$. (T_1) - (T_4) imply that $T(y)$ is a closed, convex set in the nonnegative orthant of R^n_{t+1} , $T(\Theta) = \Theta$, and that there exists $y^{t+1} > \Theta$, $y^{t+1} \in T(y)$ for $y > \Theta$. (T_3) implies that $T(y)$ is bounded.

Similarly, $T^H(y) = \left\{ y^H \mid (-y^o, y^H) \in T^H \ , \ y^o \leqq y \right\}$ is the attainable H period production set, given $y \geqq \Theta$. $T^H(y)$, for H finite, is a bounded set in R^n_H .

Finally, $\bar{T}^H(y) = \left\{ y^H \in T^H(y) \mid y'^H \geq y^H \Longrightarrow y'^H \notin T(y) \right\}$ is the <u>efficient attainable H period set,</u> given $y \geqq \Theta$ [7] .

2.5. <u>A price vector</u> p^t , $p^t \geqq \Theta$, is an assignment of prices to the n commodities available at t , $t = 0, 1, 2, \ldots$.

The existence of an appropriate nonnegative, non-zero, <u>price system</u> $\left\{ p^t \right\}_{t=0}^{H}$, associated with each efficient path $\left\{ y^t \right\}_{t=0}^{H}$, is a well known result in the theory of efficient allocation of resources; see Koopmans [7, pp. 61-65, 80-85], Malinvaud [8], [9]. More explicitly, it can be shown

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7. $y^H \in \bar{T}^H(y)$ if and only if $(-y^o, y^H) \in \bar{T}^H$. Clearly, $\bar{T}^H(y)$ is the efficient envelope of [1, ch. 12] .

(see McKenzie [10, Lemma 1]) that for every $(-y^o, y^H) \in \bar{\bar{T}}^H$ there exists a sequence $p \equiv \{p^t\}_{t=0}^H$, $p^t \geq \Theta$, such that[8]

$$(1) \qquad -p^t y^t + p^{t+1} y^{t+1} = 0$$

$$-p^t y'^t + p^{t+1} y'^{t+1} \leqq 0 , \qquad t \in [0 , H-1] ,$$

for all $(-y'^o , y'^H) \in T^H$. If $p^H > \Theta$, the converse holds.[9]

The above lemma can be extended so as to apply to efficient paths of infinite duration. The proof is analogous to that in Malinvaud [9] and thus is not given here.

Lemma 1 : For every efficient path $\{y^t\}_{t=0}^{\infty}$, $y^o > \Theta$, there exists a sequence $p \equiv \{p^t\}_{t=0}^{\infty}$, $p^t \geq \Theta$, such that (1) is satisfied for all $t \geqq 0$ [10] .

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8. We note that $p^t = \Theta$ implies that $p^{t+1} = \Theta$. Since, $p \geq \Theta$, $p^o \geq \Theta$. However, in order to get $p^t \geq \Theta$, $t=1, \ldots, H$, we need $p^o y^o > 0$. This e.g., holds if $y^o > \Theta$.

9. It can be similarly shown that for every $y^H \in \bar{\bar{T}}^H(y^o)$ there exists $p^H \geq \Theta$ such that $p^H y^H = \alpha \geqq 0$, $p^H y'^H \leqq \alpha$, for all $y'^H \in \bar{\bar{T}}^H(y^o)$.

10. We only note that the additional assumption in [9] which is needed for the proof is implied by (T_1) and (T_4). $p^o y^o > 0$ is again needed for having $p^t \geq \Theta$, $t > 0$. Moreover, we must emphasize that $p^t \to \Theta$, or $p^t \to + \infty$, as $t \to + \infty$, are possible.

Remark 1: An efficient price path associated with an efficient path of (in-) finite duration is such that it maximizes the output value at any period H over all feasible paths $\{y'^t\}_{t=0}^{H}$, with $y'^o \leqq y^o$.

For $p^t y^t = p^o y^o$, for all $t > 0$, while $p^t y'^t \leqq p^o y'^o$. Thus $p^t y'^t \leqq p^t y^t$ for all $t \in [0, H]$, all H.

This simple property of any efficient path is repeatedly used below.

2.6. u-maximal accumulation paths of finite duration

The definition of an efficient path of finite duration implies that "social preferences" depend only on the final state of the path.[11]

We may formally introduce this preference function on the final state of the path as it is done in [14] and [13]:

Preferences are expressed by a real-valued function $u(y^H)$ defined on the nonnegative orthant of R^n_H. u is nonnegative, continuous and "quasi-homogeneous" [12] on Ω. Moreover, $y'^H > y^H \geqq \Theta$ implies $u(y'^H) > u(y^H)$.

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11. As Remark 1 shows, every $(-y^o, y^H) \in \overline{T}^H$ is a maximal solution of an appropriate linear program, in which y^o is given, and either p^H or the "desired proportions" among the components of y^H are exogenously specified. Conversely, if $p^H > \Theta$, any maximal solution to the above linear program is an efficient path.

Thus the objective of an accumulation program for H periods can be expressed as a preference on the final state of the program.

12. "Quasi-homogeneity" means that for y^H, $y'^H \in \Omega$, and $\lambda > 0$, $u(y^H) \geqq u(y'^H)$ if and only if $u(\lambda y^H) \geqq u(\lambda y'^H)$.

Definition 6: If $(-y^o, y^H) \in T^H$ is such that $u(y^H)$ is maximum over all $(-y'^o, y'^H) \in T^H$, with $y'^o \leqq y^o$, then $\{y^t\}_{t=0}^{H}$ is called a u-maximal path of duration H.[13]

If u is expressed by, (a) $u(y^H) = p^H y^H$, with $p^H > \Theta$ exogenously specified, or by (b) $u(y^H) = \max \{\lambda | y^H \geqq \lambda \bar{y}^H$, for $\bar{y}_i^H \geqq 0$ and $\sum_1^n \bar{y}_i^H = 1\}$, then u-maximal paths are efficient paths. Conversely, every efficient path is a u-maximal path, for an appropriately defined preference function.

2.7. Von Neumann Paths

Definition 7: For every $(-y^t, y^{t+1}) \in T$ we define the technological expansion factor of commodity i in the process by

$$(2) \qquad \rho_i(y^t, y^{t+1}) = \begin{cases} y_i^{t+1}/y_i^t & \\ \infty & \text{for} \\ \text{undefined} & \end{cases} \begin{cases} y_i^t > 0 \\ y_i^t = 0 \quad y_i^{t+1} > 0 \\ y_i^t = 0 , \quad y_i^{t+1} = 0 \end{cases}$$

$\rho(y^t, y^{t+1}) = \min_i \rho_i(y^t, y^{t+1})$ is the technological expansion factor of the process $(-y^t, y^{t+1})$.

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13. The existence of max $u(y'^H)$ over $T^H(y^o)$ is insured by the compactness of $T^H(y^o)$.

(3) $\rho^* = \sup \left\{ \rho(y^t, y^{t+1}) \mid (-y^t, y^{t+1}) \in T \ , \ y^t \geq \Theta \right\}$ is the <u>maximum technological expansion factor of T</u>.

It can be shown that $0 < \rho^* < +\infty$. Further, the following is a famous theorem:

<u>Theorem 1</u> (von Neumann): There exists $(-\tilde{y}^t, \tilde{y}^{t+1}) \in T$, with $\tilde{y}^t \geq \Theta$, and a price system $(p^t, p^{t+1}) \geq \Theta$ such that

(4)
$$\text{(a)} \quad \tilde{y}^{t+1} \geqq \rho^* \tilde{y}^t \ , \quad \text{and}$$

$$\text{(b)} \quad p^t = p^* \ , \quad p^{t+1} = \frac{1}{\rho^*} p^* \quad \text{and}$$

$$- p^* y^t + \frac{1}{\rho^*} p^* y^{t+1} \leqq 0 \quad \text{for all } (-y^t, y^{t+1}) \in T \ .$$

By (T_2), $(-\tilde{y}^t, \tilde{y}^{t+1}) \in T$ implies that there exists $(-y^*, \rho^* y^*) \in T$, with $y^* \geq \Theta$. $(-\tilde{y}^t, \tilde{y}^{t+1})$ is called a <u>maximum growth process</u>, whereas $(-y^*, \rho^* y^*)$ is called a <u>von Neumann process</u> (of maximum balanced growth). Also ρ^* and p^*, are called the <u>von Neumann factor of T</u> and a <u>von Neumann price vector</u>, respectively.[14]

For a proof of Theorem 1 see e.g., Gale [3, pp. 290-291]. Gale's theorem is actually stronger than Theorem 1. He also shows that (a) if there exist commodities which are not produced in any maximum growth process of T, then in every von Neumann price vector the prices of these commodities are zero, and (b) for every von Neumann price vector there exists a von Neumann process $(-y^*, \rho^* y^*) \in T$ for which $p^* y^* > 0$.

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14. The sets of maximum growth, and von Newmann processes, as well as the set of von Neumann price vectors, are closed convex cones.

The sequence $\{p^{*t}, y^{*t}\}_{t=0}^{H}$, $H \leqq +\infty$, with $p^{*t} \equiv \rho^{*-t} p^*$, and $y^{*t} \equiv \rho^{*t} y^*$, constitutes a von Neumann price-commodity path of duration H .[15]

2.8. Decentralization of Production

In our economy production is carried out by separate producers -- called industries -- each having its own technology. Furthermore, conditions are such that to each commodity there corresponds one industry capable of producing it (possibly along with other commodities). Thus n industries exist in the economy The ith industry operates under a production set T_i , $i \in I$, having the properties (T_1)-(T_4) of § 2.1. This identification of commodities with industries is accomplished by means of the following assumption:

(A_1) If $(-y^t, y^{t+1}) \in \overline{T}_i$, $i \in I$, then $\rho_i(y^t, y^{t+1}) > \rho_j(y^t, y^{t+1})$ for $j \neq i$ for which $\rho_j(y^t, y^{t+1})$ is defined.

Namely, in any process of the ith industry the technological expansion factor of the ith commodity is strictly higher than that of any other commodity.

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15. The von Neumann path $\{y^{*t}\}_{t=0}^{H}$ may not be an efficient path (i.e., whenever $\tilde{y}^{t+1} \geq \rho^* \tilde{y}^t$ holds). Additional assumptions are needed in order to insure that $\tilde{y}^{t+1} = \rho^* \tilde{y}^t$ holds.

(A_1) provides a simple means for a classification of all individual production processes of the closed linear production model of § 2.1. It is undoubtedly a restrictive assumption,[16] but still joint production of commodities is permitted.

We also assume that

$$(A_2) \qquad T = \sum_{1}^{n} T_i$$

Thus no external (dis-) economies among industries are present.

2.9. For any $(-y^t, y^{t+1}) \in T_i$, other than free disposal processes, we may write $(-y^t, y^{t+1}) = u_i^t(-a^i, b^i)$, where $b_i^i = 1$, $(-a^i, b^i) \in T_i$. Thus, in general $(-y^t, y^{t+1}) \in T_i$ implies that $(-y^t, y^{t+1}) \leqq u_i^t(-a^i, b^i)$, and $(-y^t, y^{t+1}) \in T$ implies that $(-y^t, y^{t+1}) \leqq \sum_{1}^{n} u_i^t(-a^i, b^i)$, for $(-a^i, b^i) \in T_i$, $b_i^i = 1$.[17]

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16. A particularly restrictive feature of (A_1) is that if any commodity other than the ith is produced in positive quantities in the ith industry, then its input into the process must also be positive. For otherwise $\rho_j(y^t, y^{t+1}) = +\infty$, $j \neq i$ and (A_1) is contradicted.

17. Of course $u_i(-a^i, b^i) + u_i'(-a'^i, b'^i) = u_i''(-a''^i, b''^i)$, with $u_i'' = u_i + u_i'$, $(-a''^i, b''^i) = \left(-\left(\frac{u_i}{u_i+u_i'} a^i + \frac{u_i'}{u_i+u_i'} a'^i\right), \left(\frac{u_i}{u_i+u_i'} b^i + \frac{u_i'}{u_i+u_i'} b'^i\right)\right)$ and $b_i''^i = 1$.

Thus for any $(-y^t, y^{t+1}) \epsilon T$ we have

$$(5) \quad \begin{aligned} B\,u^t &\geqq y^{t+1} \\ A\,u^t &\leqq y^t . \end{aligned}$$

$[A] \equiv [a^1, \ldots, a^n]$, and $[B] \equiv [b^1, \ldots, b^n]$, are nonnegative, n x n matrices, the <u>input and output matrices</u>, respectively, of the closed linear production model.[18] $u^t = (u_1^t, \ldots, u_n^t)$ is a nonnegative vector, the <u>activity vector</u>.[19]

2.10. We now assume that in our economy it is possible with any commodity to produce in a finite number of periods all commodities. Namely,[20]

(T_5) For any $y^o \geq \Theta$, there exists a finite number s such that

$(-y^o, y^s) \epsilon T^s$ with $y^s > \Theta$, (Possibility of production of every commodity by means of any commodity).

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18. Clearly, [A] and [B] are not constant matrices. For different production processes in T the corresponding input and output matrices in (5) may be different.

19. Similarly to footnote 17, if $Bu \geqq y^{t+1}$, $Au \leqq y^t$, and $B'u' \geqq y'^{t+1}$, $A'u^1 \leqq y'^t$, then $Bu + B'u' = B''(u + u') = B''u'' \geqq y^{t+1} + y'^{t+1}$, $Au + A'u' = A''u'' \leqq y^t + y'^t$. Thus again $b_i''^i = 1$, $i \epsilon I$.

20. See Furuya and Inada [2, p. 99].

(T_5) implies that for any $i \in I$ there exists $(-y^t, y^{t+1}) \in T$ for which $y_j^t = 0$, $j \neq i$, and $y_k^{t+1} > 0$ for at least one k, $n \neq i$.[21]

Let us reformulate Theorem 1 so that the decentralized character of production appears explicitly. Theorem 1 says that there exist processes $u_i^* (-a^{*i}, b^{*i}) \in T_i$, $i \in I$, and a price vector $p^* \geq \theta$ such that

$$(4') \qquad \begin{aligned} B^* u^* &\geqq \rho^* A^* u^* \\ p^* B &\leqq \rho^* p^* A \end{aligned} \qquad \text{holds,}$$

with $u^* \geq \theta$. Clearly $(-\tilde{y}^t, \tilde{y}^{t+1}) = (-A^* u^*, B^* u^*)$.

(T_5) enables us to strenghen the above theorem.

<u>Theorem 1'</u>: Under (T_1) - (T_5), Theorem 1 holds and moreover, (a) $p^* > \theta$, (b) $(-\tilde{y}^t, \tilde{y}^{t+1})$ is a von Neumann process, and (c) there exists a von Neumann process $(-y^*, \rho^* y^*)$ with $y^* > \theta$.

Proof: (a) By Theorem 1 $p^* \geq \theta$, and $p^* \tilde{y}^t = \rho^* p^* \tilde{y}^{t+1}$. Suppose that $p_i^* = 0$, $i \in J$, $J \subseteq I$, $I \not\subseteq J$. (T_5) implies that there exists a process $(-y^t, y^{t+1}) \in T$ such that $y_i^t = 0$, for all $i \notin J$, and $y_j^{t+1} > 0$ for at least one $j \in J$. Then $p^* y^{t+1} > \rho^* p^* y^t = 0$ which contradicts (4) or (4'). Thus $p^* > \theta$.

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21. (T_5) is a relatively weak assumption in a production system permitting joint production of commodities by each industry. It can be seen that (T_5) cannot hold in a Leontief production system.

Also, although (T_5) is expressed as a property of the economy's production set T, it must be considered along with (A_1) and (A_2). E.g., a not particularly strong sufficient condition for (T_5) is the following: For each $i \in I$, there is an industry j, $j \neq i$, with $(-y^t, y^{t+1}) \in T_j$, and $y_n^t = 0$, $k \neq i$, $y_n^{t+1} = 0$, $n \neq i, j$, and $y_i^{t+1} > 0$, $y_j^{t+1} > 0$.

(b) Since $p^* > \Theta$, $\tilde{y}^{t+1} = \rho^* \tilde{y}^t$ must hold.

(c) If every von Neumann process does not produce some of the commodities, then by Gale's stronger version of Theorem 1, $p^* > \Theta$ cannot be true. We thus see that

$$(4") \qquad \begin{aligned} B^* u^* &= \rho^* A^* u^* \\ p^* B &\leqq \rho^* p^* A \end{aligned}$$

holds, with $u^* \geq \Theta$, and $p^* > \Theta$. Furthermore, for at least one of u^* , $A^* u^* > \Theta$ is satisfied. In other words, with the addition of (T_5) every von Neumann path $\{y^{*t}\}_{t=0}^{H}$ -- at least one of which is a positive path -- is an efficient path, with which a positive price system $\{p^{*t}\}_{t=0}^{H}$ is associated.

3. CONVERGENCE TO THE VON NEUMANN FACET

3.1. Let C be a convex cone in R^n. If for $p \in R^n$, $p \neq \Theta$, $p \cdot x \leqq 0$ for all $x \in C$, the set of all $x \in C$ for which the equality holds is a facet, $F(p)$, of C.[22]

Thus if $(p^t, p^{t+1}) \geq \Theta$ is a price system and $-p^t y^t + p^{t+1} y^{t+1} \leqq 0$ for all $(-y^t, y^{t+1}) \in T$, the facet $F(p^t, p^{t+1})$ of T is the set of all $(-y^t, y^{t+1}) \in T$ for which the equality holds. $T(p^t, p^{t+1})$ is a closed, convex cone.

Definition 8: The von Neumann facet F^* is defined by $F^* = F(p^*, \frac{1}{\rho^*} p^*)$, where p^*, $\sum_{i=1}^{n} p_i^* = 1$, is in the relative interior of the closed, convex cone of the von Neumann price vectors.

3.2. We will next show that the most general property of efficient paths of finite or infinite duration is, respectively, their approximation to the von Neumann facet for most of their duration, or their asymptotic convergence to it. The reasoning leading to this conclusion is rather simple and it has been the basis of the Radner proof of the turnpike theorem. Namely, any production process which does not lie on the von Neumann facet is strictly

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22. Similarly, if $p \cdot x \leqq \alpha$ for all $x \in K$, a convex set in R^n, the set of all $x \in K$ for which the equality holds defines the facet $F(p)$ of K.

unprofitable under the von Neumann prices. However, an efficient path cannot be unprofitable under the von Neumann prices except for a finite number of periods not depending on the duration of the path.

3.3. Radner [14] shows that [23] for any thin neighboring cone of the von Neumann ray every u-maximal path $\{y^t\}_{t=0}^{H}$, starting at a common initial state, stays within the cone except for a finite number of periods not depending on the duration of the path.

It is not however indicated whether these periods can only be initial and terminal periods, or not. Nikaido [13] extends Radner's theorem and shows, under rather weak additional assumptions,[24] that every such u-maximal path lies entirely in the cone except for the k initial and k terminal periods of the path; $+\infty > k \geqq 0$.

3.4. For our purposes we can apply, with slight changes, the proofs given by Furuga and Inada [2, Theorem 4] or McKenzie [11, Theorem 1]. Moreover, we have exhibited a von Neumann process $(-y^*, \rho^* y^*)$ with $y^* > \Theta$. Finally, the last

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23. This of course holds under, among others, his main assumption that the von Neumann facet F^* is 1-dimensional.

24. Namely. (a) $y^* > \Theta$, where $(-y^*, \rho^* y^*)$ is the von Neumann process, and (b) $u(y^H)$ is such that $y^H > y'^H \geqq \Theta$ implies that $u(y^H) > u(y'^H)$.

of Nikaido's additional assumptions is satisfied for efficient accumulation paths, (see § 2.6 above). Thus we can use Nikaido's argument in the proof of the following theorem, which corresponds to Radner's theorem.[25]

<u>Theorem 2</u>: Let (T_1)-(T_5) be satisfied, and consider any efficient path $\{y^t\}_{t=0}^{H}$, with $y^o = \bar{y} \geq \theta$, for which its associated price path $\{p^t\}_{t=0}^{H}$ is such that $p^o y^o > 0$. Then for every $\epsilon > 0$ there is a finite positive number H_1 such that:

(a) if the path is of finite duration, then $d\big((-y^t, y^{t+1}), F^*\big) < \epsilon$ for $H_1 \leqq t \leqq H - H_1$, or

(b) if the path is of infinite duration, then $d\big((-y^t, y^{t+1}), F^*\big) < \epsilon$ for $H_1 \leqq t$.

In the interest of brevity we mention only some pertinent points about the proof: (a) The proof depends on the Radner lemma,[26] and it uses the property of efficient paths mentioned in Remark 1 above; (b) For efficient paths of infinite duration, we have seen that $p_i^t < +\infty$, $i \epsilon I$, and, with $p^o y^o > 0$, that $p^t \geq \theta$ for all $t > 0$. It is however still possible

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25. The norm of $y \in R^n$ is given by $\|y\| = \sum_{1}^{n} |y_i|$; the angle between any non-zero vectors in R^n is indicated by $d(y, y') = \left| \frac{1}{\|Y\|} Y - \frac{1}{\|Y'\|} Y' \right|$. It can be shown that d is a metric in $R^n - \{\theta\}$.

26. Namely, that for any $(-y^t, y^{t+1}) \epsilon T$ for which $d\big((-y^t, y^{t+1}), F^*\big) \geqq \epsilon > 0$ holds, $-(\rho^* - \delta) p^* y^t + p^* y^{t+1} \leqq 0$ is satisfied for a positive δ.

that $\| p^t \| \to 0$, or $\| p^t \| \to + \infty$, as $t \to + \infty$. This does not affect the proof of the second part of the theorem. Actually, for any $t \geqq 0$ we may normalize p^t so that $\| p^t \| = \sum_1^n p_i^t = 1$, as long as no intertemporal cost comparisons are made.

4. GENERALIZED FROBENIUS MATRICES

4.1. We have shown that the general property of efficient paths is their monotonic approximation or convergence to the von Neumann facet F^*. Their approximation or convergence to a von Neumann ray on this facet can be shown only under additional assumptions, whenever the dimension of F^* is higher than 1.

4.2. In order to do this we have to make a digression and introduce some new concepts.

Let C and D be linear homogeneous operators on R^n into R^n.

Definition 9: A scalar λ is an eigenvalue, and a non-zero vector $x \in R^n$ is a right eigenvector of (C, D) if

$$D\,x = \lambda\,C\,x\,. \tag{6}$$

Similarly, a non-zero vector $y \in R^n$ is a left eigenvector of $D - \lambda C$ if

$$y\,D = \lambda\,y\,C \tag{7}$$

Definition 10:[27] (C, D) is an F-operator if there exists a unique, simple, positive eigenvalue λ of (C, D) with positive right and left eigenvectors x, and y, respectively.

λ, x, and y, are called the F-eigenvalue and the right and left F-eigenvector of (C, D), respectively.

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27. This is a generalization of the concept of an F-matrix introduced by Uzawa [16].

4.3. We now present sufficient conditions under which a particular operator (C, D) is an F-operator. Let C, D be nxn nonnegative input and output matrices of a von Neumann production system. Let p be the von Neumann factor, and u , and p , be a maximal activity vector, and a von Neumann price vector, respectively. Then we may apply Theorem 1 and get

$$D\,u \geqq \rho\,C\,u\ , \quad \text{and} \quad p\,D \leqq \rho\,p\,C\ , \quad \text{with} \quad u \geq \theta\ , \quad p \geq \theta\ . \tag{8}$$

We have:

<u>Lemma 2</u>: if $d_{ii} > 0$ for all $i \in I$, and $d_{ij} \leqq \rho\, c_{ij}$ for all $i \neq j$, then the equalities hold in (8).

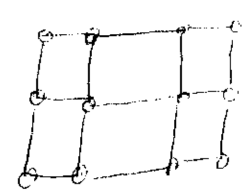
Proof: We consider only the case $D\,u \geqq \rho\,C\,u$. Suppose that there exists a maximal activity vector u such that

$$\sum_j^n d_{ij}\, u_j = \rho \sum_j^n c_{ij}\, u_j \qquad i \notin J\ ,$$

$$\sum_j^n d_{ij}\, u_j > \rho \sum_j^n c_{ij}\, u_j \qquad i \in J\ .$$

Now $J \subset I$, but $I \not\subset J$ for then the definition of ρ is contradicted. If there exists a maximal activity vector u' such that J is not empty, let u be the maximal activity vector such that J is maximal. Then we have

$$d_{ii}\, u_i = \sum_{j \neq i}^n (\rho c_{ij} - d_{ij})\, u_j + \rho\, c_{ii}\, u_i\ , \qquad i \notin J\ ,$$



$$d_{ii}\, u_i > \sum_{j \neq i}^n (\rho c_{ij} - d_{ij})\, u_j + \rho\, c_{ii}\, u_i\ , \qquad i \in J\ .$$

For each $j_o \in J$ we can take $(u_{jo} - \epsilon) \geqq 0$ instead of u_{jo} such that the inequalities are still satisfied. Then if there exists $i \notin J$ such that $(\rho c_{ijo} - d_{ijo}) u_{jo} > 0$, then by taking $u_{jo} - \epsilon$ instead of u_{jo} we will have $D\,u' = \rho\,C\,u'$ with one more inequality. But this contradicts our assumption about the selection of u. Hence $(\rho c_{ij} - d_{ij})\,u_j = C$ for all $i \notin J$, $j \in J$.

Let now $\bar{u} = (\Theta, u_j)$, where u_j, $j \in J$, is the corresponding component of u. Then we see that $D\,\bar{u} \geqq \rho\,A\,\bar{u}$ and that the equalities for $i \notin J$ have both sides equal to zero, whereas the inequalities for $i \in J$ are still satisfied. This again contradicts the definition of ρ, since ρ is not then maximum.||.

Lemma 3:[28] If $d_{ii} > 0$ for all $i \in I$, if $d_{ij} < \rho\,c_{ij}$ for $i \neq j$ if $c_{ij} > 0$, and if C is indecomposable, then

(a) u and p are positive and unique up to scalar multiplication;

(b) ρ is a simple root of the characteristic equation

$$\Delta(\rho) = |\,D - p\,C\,| = 0 \text{ ; and} \tag{9}$$

(c) no other eigenvalue of $D - \lambda\,C$ has a nonnegative right or left eigenvector.

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28. Lemma 2 and 3 (a) are generalizations of theorems 6 and 8 of Gale [3]. The proof of 3 (b) follows that in Gantmacher [4, p. 57].

Proof: (a) Suppose that u is such that $u_j = 0$ for $j \in J$, $J \subset I$, $I \not\subset J$. Then either $c_{ij} = 0$ for $i \in J$, $j \notin J$, which contradicts the indecomposability of C, or there exists $c_{i_o j_o} > 0$, $i_o \in J$, $j_o \notin J$. Since $d_{i_o j} < \rho\, c_{i_o j}$, $i_o \neq j$, if $c_{i_o j} > 0$, we have that $\sum_j c_{i_o j} u_j < \rho \sum_j c_{i_o j} u_j$. But this contradicts the equation system $D u = \rho C u$. Hence $u > \Theta$.

Now suppose that u', $u' \neq \mu u$, $\mu > 0$, is any other maximal activity vector. Let $\nu = \max \left\{ \nu \mid u \geqq \nu u' \right\}$. Then $u - \nu u' \geq \Theta$ is an activity vector. Moreover, $D(u - \nu u') = \rho\, C(u - \nu u')$ by Lemma 2. Hence $u - \nu u'$ is a maximal activity vector although it has at least one zero component. This contradicts the result established above.

(b) Consider the transpose matrix $E(\lambda) = \left(e_{ij}(\lambda)\right)$ of the cofactors of $[D - \lambda C]$. $[D - \lambda C]\, E(\lambda) = \Delta(\lambda)\, I = E(\lambda)\, [D - \lambda C]$, where $\Delta(\lambda) = | D - \lambda C |$. Then $[D - \rho C]\, E(\rho) = 0 = E(\rho)\, [D - \rho C]$, since $\Delta(\rho) = 0$. Since the eigenvector which corresponds to ρ is unique, $E(\rho)$ is a non-zero matrix. For its elements are non-zero multiples of the $(n-1) \times (n-1)$ minors of $\Delta(\rho)$, of which at least one must be non-zero. Moreover, if we consider any non-zero column of $E(\rho) = [e'(\rho), \ldots, e^n(\rho)]$, we see that $[D - \rho C]\, e^j(\rho) = \Theta$. Thus $e^j(\rho)$ is a right eigenvector of $D - \rho C$ corresponding to ρ. Since the eigenvector which corresponds to ρ is positive, all the components of $e^j(\rho)$ are non-zero and of the same sign. Similarly, considering any non-zero row of $E(\rho) = [e_1(\rho), \ldots, e_n(\rho)]$, we see that $e_i(\rho)[D - \rho C] = \Theta$, and hence that all the components of $e_i(\rho)$ are non-zero and of the same sign.

Thus all $e_{ij}(\rho)$ are non-zero and of the same sign.

Now $\Delta(\lambda) = \Sigma_\pi (\text{sgn } \pi)(d_{\pi(1)1} - \lambda c_{\pi(1)1})\ldots(d_{\pi(n)n} - \lambda c_{\pi(n)n})$, over all permutations π in $\{1, \ldots, n\}$. Also $\Delta'(\lambda) = \Delta_1(\lambda) + \ldots + \Delta_n(\lambda)$, where $\Delta_1(\lambda) = -\Sigma_\pi (\text{sgn } \pi) c_{\pi(1)1}(d_{\pi(2)2} - \lambda c_{\pi(2)2})\ldots(d_{\pi(n)n} - \lambda c_{\pi(n)n})$ $= -\sum_i c_{i1} e_{i1}(\lambda)$, etc. Thus $\Delta'(\lambda) = -\sum_{i,j} c_{ij} e_{ij}(\lambda)$. Hence $\Delta'(\lambda)|_{\lambda=\rho} = -\Sigma c_{ij} e_{ij}(\rho) \neq 0$, and ρ is a simple root of the characteristic equation $\Delta(\rho) = 0$.

(c) Suppose that there exists α such that $D x = \alpha C x$, with $x \geq \Theta$, and $x \neq \mu u$, $\lambda > 0$. If $\alpha = \rho$, part (b) is contradicted. Let then $\alpha \neq \rho$. But $p D x = \alpha p C x = \frac{\alpha}{\rho} p D x$. Hence $p D x = 0$. But $p D > \Theta$, since $d_{ii} > 0$, $d_{ij} \geqq 0$, and $p > \Theta$. Also $x \geq \Theta$. Hence $p D x > 0$ and we have a contradiction.

We see that under the conditions of Lemmas 2 and 3 the input and output operator characterizing a von Neumann production system is an F-operator. The von Neumann factor ρ is the F-eigenvalue, and u, p, are the (unique) right and left F-eigenvectors, of (C, D).

5. FURTHER SPECIFICATION OF THE MODEL

5.1 In this section some additional assumptions on the structure of the von Neumann facet F^* , as well as on the individual production sets are introduced. Our immediate purpose is to show that: (a) there exist production processes on F^* (with all industries participating) such that the corresponding input and output operator is an F-operator; and (b) F^* is spanned by a unique set of individual production processes (with all industries participating).

5.2. Consider again a maximum growth process $(-\tilde{y}^t, \tilde{y}^{t+1}) \in T$, the existence of which was established by Theorem 1. $(-\tilde{y}^t, \tilde{y}^{t+1}) = (- A u^*, B u^*)$, with $A \geq 0$, $B \geq 0$, and $u^* \geq \Theta$ is such that $B u^* \geqq \rho^* A u^*$.

We introduce

(T_6)[29] In any maximum growth process of T , if $u^*_k = 0$, $k \in I$, there exists at least one production process $(-a^k, b^k) \in T_k$ such that the resulting input and output matrices A and B have the following properties:

(a) For $i \neq j$, if $a^i_j > 0$, then $b^i_j < \rho^* a^i_j$;

(b) A is an indecomposable matrix.

---

29. (T_6) is a far more restrictive assumption than (A_1) . Although multi-product industries are still permitted, (T_6) requires that each industry specializes mainly in the production of its own commodity.

The consequences of the addition of (T_6) to (T_1)-(T_5) follow directly from Lemmas 2 and 3. (A, B) is an F-operator. Namely

(9) and
$$B\, u^* = \rho^* A\, u^* ,$$
$$p^* B = \rho^* p^* A ,$$

holds with $u^* > \Theta$, $p^* > \Theta$, (which are moreover unique). Furthermore, ρ^* is a unique and simple eigenvalue of (A, B). We conclude that all industries participate in any von Neumann process.

5.3. The production set of each industry is assumed to have the following property:

(T_7)[30] Let $(-y^t, y^{t+1})$, $(-y'^t, y'^{t+1}) \in T_i$, $i \in I$, and $(-y^t, y^{t+1}) \neq \mu\, (-y'^t, y'^{t+1})$, $\mu > 0$. Then, there exists $\left(-(y^t + y'^t) , y''^{t+1}\right) \in T_i$ with $y''^{t+1} \geq y^{t+1} + y'^{t+1}$, (Strong super-additivity).

With (T_7) we can show that the von Neumann process is unique.

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30. See Furuya and Inada [2]. In [2] this is considered as a property of the economy's production set. (T_7) essentially requires each industry's production set to be a strictly convex cone.

<u>Lemma 4:</u> The input and output matrices associated with any von Neumann process are unique.

Proof: Let $(-y^t, y^{t+1}), (-y'^t, y'^{t+1}) \epsilon T$ be any two maximum growth process. By (T_6) they are von Neumann processes; namely, $(-y^t, y^{t+1}) = (- A u^t , B u^t)$, $(-y'^t, y'^{t+1}) = (- A'u'^t, B' u'^t)$, $B u^t = \rho^* A u^t$, and $B' u'^t = \rho^* A' u'^t$, hold with u^t , $u'^t > \Theta$. Suppose now that $(- A, B) \neq (- A', B')$. Then $(-a^i, b^i) \neq (-a'^i, b'^i)$ for some $i \epsilon I$, and thus by (T_7) , there exists $(-\bar{y}^t, \bar{y}^{t+1}) \epsilon T$ with $\bar{y}^{t+1} = \bar{B} \bar{u}^t$, $\bar{y}^t = \bar{A} \bar{u}^t$ and $\bar{B} \bar{u}^t \geq B u^t + B' u'^t$, $\bar{A} \bar{u}^t = A u^t + A' u'^t$. Thus $\bar{B} \bar{u}^t \geq \rho^* \bar{A} \bar{u}^t$ holds. However, by (T_1)-(T_6) there exists $p^* > \Theta$ for which

$$(*) \quad \begin{array}{l} p^* B = \rho^* p^* A , \text{ and } p^* \tilde{b}^i \leqq \rho^* p^* \tilde{a}^i \text{ for any} \\ (-\tilde{a}^i , \tilde{b}^i) \epsilon T_i , i \epsilon I . \end{array}$$

On the other hand, $p^* \bar{B} \bar{u}^t > p^*(B u^t + B' u'^t) = \rho^* p^*(A u^t + A'u'^t)$ and this contradicts (*).|| .

The above lemma essentially shows that with the addition of (T_6) and (T_7) the von Neumann facet is spanned by a unique set of individual production processes. Let A^* and B^* be the corresponding input and output matrices. Then $F^* = \left\{(-y^t, y^{t+1}) \epsilon T \mid -y^t = - A^* u^t , y^{t+1} = B^* u^t , u^t \geqq \Theta \right\}$. We have thus shown that (a) the unique, positive von Neumann price vector p^* is such that

$$p^* b^{*i} = \rho^* p^* a^{*i}$$

(10)

$$p^* b^i < \rho^* p^* a^i \quad , \qquad \text{for all} \quad (-a^i, b^i) \neq (-a^{*i}, b^{*i}) \ , \quad i \in I \ ,$$

hold; and (b) the von Neumann process $(- A^* u^*, B^* u^*)$ is unique. The ray (y^*) is called the <u>von Neumann ray</u>.

6. CONVERGENCE TO THE VON NEUMANN RAY

6.1. The approximation of any efficient path to the von Neumann ray (y^*) on the von Neumann facet F^* will be now proved. The procedure which is used has been initiated by McKenzie in [11]. First, we examine sequences $\left\{w^t\right\}_{t=0}^{H}$ satisfying (12) below. Then, we extend the results established for these sequences to efficient paths moving within a thin neighboring cone of F^*.

6.2. By Theorem 2 every efficient path of finite duration moves within any neighboring cone of F^* except for a fixed number of initial and terminal periods. If F^* is 1-dimensional, then of course our purpose has been achieved. The approximation to (y^*) on F^* becomes a distinct problem whenever $\dim(F^*) > 1$, and it can be as high as n. Our analysis will be simplified if we assume that both projections of F^* into R^n_t and into R^n_{t+1} are n-dimensional, i.e., if we assume that

(T_8) A^* and B^* are nonsingular.[31]

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31. It is easily seen that no essential differences are thereby created. E.g., suppose that an accumulation path $\left\{y^t\right\}_{t=0}^{H}$ lies on F^*. Then

(*) $B^* u^t = y^{t+1}$, $A^* u^t = y^t$

holds for $t \in [0, H-1]$. It is clear that y^o is in the range of A^*, and that y^t, $t \in [1, H-1]$ is in the intersection of the ranges of A^* and B^*. No essential difference is made if either one is a proper subspace of R^n_t. With (T_8) we get from (*)

(**) $y^{t+1} = B^* A^{*-1} y^t$, $y^t \geqq \Theta$, for $t \in [0, H-1]$.

We immediately see that $B^* A^{*-1}$ is an F-operator on R^n onto R^n. From (9) and putting $y^* = A^* u^*$ we get

(11) $$B^* A^{*-1} y^* = \rho^* y^* \quad , \quad p^* B^* A^{*-1} = \rho^* p^* \quad .$$

Thus ρ^* is the F-eigenvalue, whereas $y^* = A^* u^*$, and p^* are its right and left F-eigenvectors, respectively.

6.2. We are interested in sequences $\{w^t\}_{t=0}^{H}$ in V^n [32] which satisfy the following equation[33]

(12) $$w^{t+1} = B^* A^{*-1} w^t \ , \quad t \in [0, H-1] \ , \quad w^t \in V^n, \ w^o \geqq \Theta$$

From (11) we see that (12) has a unique "balanced growth solution" $w^* = y^* > \Theta$.

It has been shown in [11] that any sequence satisfying (12) will approximate (w^*) for most of its duration if H is long enough, provided that an additional assumption is satisfied,[34] namely,

(T_9) ρ^* is different in absolute value than any other eigenvalue of $B^* A^{*-1}$.

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32. V^n is an n-dimensional vector space over the complex numbers. For $z \in V^n$, $z = x + i\,y$, with $x, y \in R^n$. The distance function d is well defined over $V^n - \{\Theta\}$. Also let $S^n \subset V^n$ be defined by $S^n = \{ z \mid x \geqq \Theta \ , \ y = \Theta \ ,$ and $\|z\| = 1 \}$.

33. Notice that no restrictions as to the signs of w^t, $t \in [1, H-1]$, are placed.

34. See also Uzawa [16], where infinite sequences of nonnegative vectors satisfying the above dynamic system are considered.

<u>Lemma 5</u>: Let $N(w^*)$ be a neighboring cone of (w^*). Then there exist $H_2, H_3 > 0$ and $\delta > 0$, such that for any $\left\{w^t(w^o)\right\}_{t=0}^{H}$, $w^t(w^o) \in V^n$, starting from any $w^o \geq \theta$ and satisfying (12) under (T_1)-(T_9), if $H > H_2$, $H > H_3$, and $d(w^h, S^n) < \delta$, then $w^t(w^o) \in N(w^*)$ for all t such that $H_2 \leqq t \leqq H - H_3$, where $\dfrac{H - H_2 - H_3}{H} \geqq 1 - \eta$, η arbitrary.

Proof:[35, 36] See McKenzie [11].

6.3. It can also be shown that, if an efficient path is within a neighboring cone of F^* for a finite number of periods, a sequence satisfying (12) can be found, which is near the path for these periods. This has been shown in [11] and is stated as follows:

<u>Lemma 6</u>: Let $(-y^o, y^H) \in \overline{T}^H$ for $H > 0$. For any $\delta > 0$ there exists $\epsilon > 0$ such that if $d((-y^t, y^{t+1}), F^*) < \epsilon$ for $t \in [0, H-1]$, there exists $\left\{w^t\right\}_{t=0}^{H}$, $w^t \in R^n_t$, satisfying (12), and $d(y^t, w^t) < \delta$ for $t \in [0, H]$. ϵ depends on H and δ.[37]

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35. We note that H_2, H_3 are not affected by increasing H. We also note that the same H_2, H_3 suffice for any $\left\{w^t\right\}_{t=0}^{H}$ satisfying the conditions of the lemma, i.e., that the approximation to (w^*) is uniform.

36. The idea of the proof is fairly simple. If a sequence $\left\{w^t\right\}_{t=0}^{H}$ which satisfies (12) is such that $d(w^H, S^n) < \delta$ and H is sufficiently large, then the influence of both those eigenvalues of $B^* A^{*-1}$ smaller in absolute value than ρ^* and those larger in absolute value than ρ^* is "small" during the time span from 0 to H except for some initial and terminal periods.

37. We note that a single ϵ suffices for all paths $(-y^o, y^H) \in \overline{T}^H$ satisfying the conditions of the lemma.

6.4. A combination of Theorem 2 along with the two previous lemmas insures the approximation to the von Neumann ray of any efficient path of finite duration, as well as the asymptotic convergence to it of any efficient path of infinite duration. We have:

<u>Theorem 3</u>: Let (T_1)-(T_9) be satisfied and consider any efficient path $\{y^t\}_{t=0}^{H}$ satisfying the conditions of Theorem 2. Then for every $\delta > 0$, there exist $H' > 0$, $H'' > 0$ such that:

(a) if the path is of finite duration then

$$d(y^t, y^*) < \delta \quad \text{for} \quad H' \leqq t \leqq H - H'', \quad \text{or}$$

(b) if the path is of infinite duration, then

$$d(y^t, y^*) < \delta \quad \text{for} \quad H' \leqq t .$$

H' and H'' depend only on δ and not on the duration of the path.

Proof: (a) The proof is carried out in three steps.

1) By Lemma 6, for any $\bar{H} > 0$, $\delta' > 0$, one finds $\epsilon > 0$ such that if $d((-y^t, y^{t+1}), F^*) < \epsilon$ for $t = H_1, \ldots, H_1 + \bar{H} - 1$, H_1 to be specified below, then $d(y^t, w^t) < \delta'$, $t = H_1, \ldots, H_1 + \bar{H}$ holds for a $\{w^t\}_{t=H_1}^{H_1+\bar{H}}$ satisfying (12)

2) By Theorem 2(a), for $\epsilon > 0$ there exists $H_1 > 0$ such that $d\big((-y^t, y^{t+1}), F^*\big) < \epsilon$ for $H_1 \leqq t \leqq H - H_1$ holds. Thus $d\big((-y^t, y^{t+1}), F^*\big) < \epsilon$ for $H_1 \leqq t \leqq H - H_1$ and $d(y^t, w^t) < \delta'$ for $H_1 \leqq t \leqq H_1 + \bar{H}$.

3) Consider now the sequence $\{w^t\}_{t=H_1}^{H_1+\bar{H}}$. We have that $w^{H_1} \geq \Theta$ and $d(w^{H_1+\bar{H}}, S^n) < \delta'$ since $d(w^{H_1+\bar{H}}, y^{H_1+\bar{H}}) < \delta'$ holds. By Lemma 5 for any $\delta'' > 0$ there exist $H_2 > 0$, $H_3 > 0$, $\delta' > 0$, such that if $d(w^{H_1+\bar{H}}, S^n) < \delta'$, then $d(w^t, w^*) < \delta''$ for $H_1 + H_2 \leqq t \leqq H_1 + \bar{H} - H_3$ holds. Choose $\bar{H} = H_2 + H_3$. Then for $t = H_1 + H_2$ $d(y^t, w^t) < \delta'$, $d(w^t, w^*) < \delta''$. Hence $d(y^{H_1+H_2}, w^*) < \delta' + \delta''$. For any $\delta > 0$ we can choose δ', δ'' so that $\delta' + \delta'' < \delta$. Therefore, for this $\delta > 0$, H_1 , H_2 , and H_3 are determined -- and they are finite -- and $d(y^{H_1+H_2}, w^*) < \delta$ holds.

We repeat the above steps (for the same δ) considering in turn the subsequences $\{y^t\}_{t=H_1+t'}^{H_1+H_2+H_3+t'}$, $t' = 1, 2, \ldots, H - 2H_1 - H_2 - 2H_3$, of $\{y^t\}_{t=0}^{H}$. We thus find different sequences, $\{w^t\}_{t=H_1+t'}^{H_1+H_2+H_3+t'}$,

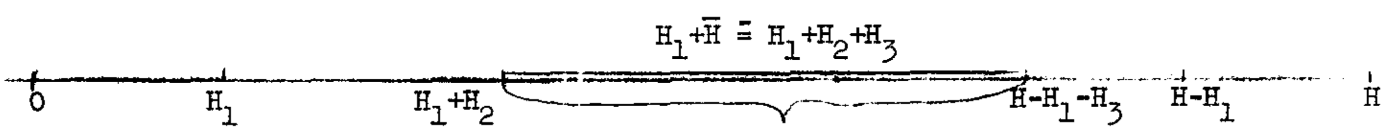
and by the same argument conclude that

$d(y^{H_1+H_2+t'}, w^*) < \delta$ $t'=1, \ldots, H - 2H_1 - H_2 - 2H_3$, holds.

Thus $d(y^t, w^*) < \delta$ for $H_1 + H_2 \leqq t \leqq H_1 - H_1 - H_3$ holds.

Let $H' = H_1 + H_2$, $H'' = H_1 + H_3$.

The convergence period is indicated in Figure 1.



(b) The proof in the case of efficient paths of infinite duration is similar. All three steps are the same. Then we consider[38] in turn finite subsequences $\left\{y^t\right\}_{t=H_1+t'}^{H_1+H_2+H_3+t'}$, for $t' = 1, 2, \ldots$, ad inf., of $\left\{y^t\right\}_{t=0}^{\infty}$ and we finally show that $d(y^t, w^*) < \delta$ for $H_1 + H_2 \leqq t$.||.

38. We emphasize that the procedure followed is possible since the approximations established in Lemmas 5 and 6 are uniform; see footnotes 35 and 37. Because of this we do not have to attempt to make $\bar{H}$ larger and larger. As a matter of fact, if we do this, it is possible that for $\bar{H} \to +\infty$, $\epsilon \to 0$, and thus $H_1 \to +\infty$. Namely, if we wish to associate an arbitrarily large part of $\left\{y^t\right\}_{t=0}^{\infty}$ with a sequence $\left\{w^t\right\}$ (of the same length) satisfying (12), then we may have to wait an arbitrarily large number of periods before this is possible.

7. SOME REMARKS ON THE UNIQUENESS OF EFFICIENT PATHS

7.1. We will make some observations on the uniqueness properties of efficient paths of finite or infinite duration. When we examine paths of finite duration, then of course we consider u-maximal paths with the preference function u specified. In the sequel we assume that the preference function u is also quasi-concave on the nonnegative orthant of R^n_H, i.e., $u(y'^H) \geqq u(y^H)$ implies that $u(\alpha\, y'^H + (1-\alpha)\, y^H) \geqq u(y^H)$, for $1 \geqq \alpha \geqq 0$.

It is easily seen that u-maximal paths are uniquely determined by their initial commodity bundle, if the economy's production set is a strictly convex cone. For then $T^H(y^o)$ is a strictly convex cone for all $H > 0$.

7.2. Furuya and Inada [2] were the first to consider more closely the problem of the uniqueness of efficient paths of infinite duration starting from a common initial position. They established that, in an economy whose production set is a strictly convex cone, such paths are uniquely determined by the initial commodity bundle they start from.

7.3. The situation however in an economy with decentralized production is more complex. The following lemmas are straightforward but nevertheless important:

<u>Lemma 7</u>: Let (T_1)-(T_4) be satisfied. If $(-y^t, y^{t+1})$, $(-y'^t, y'^{t+1})$ are on different facets[39] of T , then there exists $(-\bar{y}^t , \bar{y}^{t+1}) \in T$ with $\bar{y}^t < \alpha y^t + (1-\alpha) y'^t$, and $\bar{y}^{t+1} > \alpha y^{t+1} + (1-\alpha) y'^{t+1}$, $1 > \alpha > 0$.

Proof: Let $\tilde{y}^t = \alpha y^t + (1-\alpha) y'^t$, $\tilde{y}^{t+1} = \alpha y^{t+1} + (1-\alpha) y'^{t+1}$, $1 > \alpha > 0$ $(-\tilde{y}^t , \tilde{y}^{t+1}) \in T$. If $(-\bar{y}^t, \bar{y}^{t+1}) > (-\tilde{y}^t , \tilde{y}^{t+1})$ implies that $(-\bar{y}^t, \bar{y}^{t+1}) \notin T$, then $(-\tilde{y}^t , \tilde{y}^{t+1}) \in$ boundary T . But through each point in boundary T passes a supporting hyperplane of T . By (T_2) and (T_4) the normal to this hyperplane is $(p^t, p^{t+1}) \geq \Theta$. Thus $-p^t \bar{y}^t + p^{t+1} \bar{y}^{t+1} = 0$. Moreover, $-p^t y^t + p^{t+1} y^{t+1} \leqq 0$, $-p^t y'^t + p^{t+1} y'^{t+1} \leqq 0$. Hence, $(-y^t , y^{t+1})$, $(-y'^t , y'^{t+1}) \in F(p^t , p^{t+1})$. This contradicts the hypothesis. || .

39. Two closed facets of T (or of $T(y^o)$) are called different if neither one contains the other. Alternatively, two facets F and F' of T are different, if and only if for any (p^t, p^{t+1}) , (p'^t , p'^{t+1}) , elements of the cone of normals to T on F and F' , respectively, $(p^t, p^{t+1}) \neq \lambda (p'^t, p'^{t+1})$, $\lambda > 0$.

<u>Lemma 8:</u> Let (T_1)-(T_4) be satisfied. If $\{y^t\}_{t=0}^{H}$, $\{y'^t\}_{t=0}^{H'}$ with $y^o = y'^o \geq \Theta$ are any two efficient paths such that y^τ and y'^τ are on the same facet of $T^\tau(y^o)$, then y^t and y'^t are on the same facet of $T^t(y^o)$, $t = 1, \ldots, \tau$.

Proof: Suppose that y^n and y'^n are on different facets of $T^n(y^o)$ for any $n = 1, \ldots, \tau - 1$. Then, as in the previous lemma, there exists $\bar{y}^n \in T^n(y^o)$ and $\bar{y}^n > \alpha\, y^n + (1-\alpha)\, y'^n$, $1 > \alpha > 0$. Also, by (T_4), there exists $(-\bar{y}^n, \bar{y}^{n+1}) \in T$ such that $\bar{y}^{n+1} > \alpha\, y^{n+1} + (1-\alpha)\, y'^{n+1}$, and similarly for $t = n + 2, \ldots, \tau$. Thus the path $\{\tilde{y}^t\}_{t=0}^{\tau}$, with $\tilde{y}^t = \alpha\, y^t + (1-\alpha)\, y'^t$ for $t < n$ and $\tilde{y}^t = \bar{y}^t$ for $n \leqq t \leqq \tau$, is feasible and such that $\tilde{y}^\tau > \alpha\, y^\tau + (1-\alpha)\, y'^\tau$. This shows that y^τ and y'^τ cannot be on the same facet of $T^\tau(y^o)$. Contradiction.|| .

7.4. The above lemma shows that any two u-maximal paths of the same duration, which start from the same initial commodity bundle, are on the same facet of $T^t(y^o)$, $t = 1, \ldots, H$. For in this case a path $\{\tilde{y}^t\}_{t=0}^{H}$, with $\tilde{y}^H > \alpha\, y^H + (1-\alpha)\, y'^H$, $1 > \alpha > 0$, is feasible. Then $u(\tilde{y}^H) > u(\alpha\, y^H + (1-\alpha)\, y'^H) \geqq u(y^H)$, and this contradicts the hypothesis that $\{y^t\}_{t=0}^{H}$ is a u-maximal path.

7.5. Similarly, efficient paths of infinite duration, which start from the same initial commodity bundle, are on the same facet of $T^t(y^o)$, $t \geqq 1$.

<u>Lemma 9</u>:[40] Let (T_1)-(T_9) be satisfied. If $\{y^t\}_{t=0}^{\infty}$, $\{y'^t\}_{t=0}^{\infty}$ with $y^o = y'^o \geq \theta$ are any two efficient paths, then y^t and y'^t are on the same facet of $T^t(y^o)$, for all $t \geqq 1$.

Proof: If y^τ and y'^τ are on different facets of $T^\tau(y^o)$ for any $\tau \geqq 1$, then similarly to Lemma 8 above, we can find a feasible path $\{\tilde{y}^t\}_{t=0}^{\infty}$ such that $\tilde{y}^t = \alpha\, y^t + (1-\alpha)\, y'^t$ for $0 \leqq t < \tau$ and $\tilde{y}^t > \alpha\, y^t + (1-\alpha)\, y'^t$ for $\tau \leqq t$. If $\tilde{y}^\tau = \alpha\, y^\tau + (1-\alpha)\, y'^\tau + y$, where $y \geqq \gamma\, y^* > 0$ and (y^*) is the von Neumann ray, then $\tilde{y}^t \geqq \alpha\, y^t + (1-\alpha)\, y'^t + \rho^{*t-\tau}\, \gamma\, y^*$ for $\tau \leqq t$.
Since both $\{y^t\}_{t=0}^{\infty}$ and $\{y'^t\}_{t=0}^{\infty}$ converge uniformly to (y^*), the sequences y^o, $\frac{1}{\rho^*} y^1, \ldots, \frac{1}{\rho^{*t}} y^t, \ldots$, and y^o, $\frac{1}{\rho^*} y'^1, \ldots, \frac{1}{\rho^{*t}} y'^t, \ldots$ have the same limit, $\beta\, y^*$.
Let $\bar{\epsilon} = \{\epsilon, \ldots, \epsilon\} \in R^n$. For every $\epsilon > 0$ there exists $H > 0$ such that for $t \geqq H$

$$\beta\, y^* - \bar{\epsilon} \leqq \frac{1}{\rho^{*t}}\, y^t \leqq \beta\, y^* + \bar{\epsilon}$$

$$\beta\, y^* - \bar{\epsilon} \leqq \frac{1}{\rho^{*t}}\, y'^t \leqq \beta\, y^* + \bar{\epsilon} \qquad \text{hold.}$$

Since $\frac{1}{\rho^{*t}}\, \tilde{y}^t \geqq \alpha \frac{1}{\rho^{*t}}\, y^t + (1-\alpha) \frac{1}{\rho^{*t}}\, y'^t + \frac{1}{\rho^{*\tau}}\, \gamma\, y^*$,
$\beta\, y^* - \bar{\epsilon} + \frac{1}{\rho^{*t}}\, \gamma\, y^* \leqq \frac{1}{\rho^{*t}}\, \tilde{y}^t$. ϵ can be selected so small that

$$\frac{1}{\rho^{*\tau}}\, \gamma\, y^* \geqq 2\, \bar{\epsilon}. \quad \text{Then} \quad \frac{1}{\rho^{*t}}\, \tilde{y}^t \geqq \beta\, y^* + \bar{\epsilon} \geqq \frac{1}{\rho^{*t}}\, y^t,$$

which contradicts the hypothesis of $\{y^t\}_{t=0}^{\infty}$ being efficient.|| .

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40. The proof follows that in [2]. We note that here we use all the assumptions needed to insure the convergence of any efficient path of infinite duration to the von Neumann ray.

7.6. What are the implications of the above results for our model with decentralized production? It is easily seen that in our production system, where each industry's production set T_i is a strictly convex cone, any efficient facet of T is generated by a unique production process from each participating industry. Thus if we examine u-maximal paths (of the same duration), or efficient paths of infinite duration, starting from a common initial commodity bundle, we find that in every period within the horizon each industry uses at most one production process only. This is true despite the possibility that the efficient price path p^t, associated with these output paths, may not be unique. Although more than one u-maximal (or efficient of infinite duration) path may start from $y^o \geq \theta$, the individual production processes used by the participating industries in any period are unique. Differences -- if any -- in the output paths correspond to differences in the proportions of participation of the various industries only.

7.7. Let $\{y^t\}_{t=0}^{H}$ be a u-maximal path (or an efficient path of infinite duration), with $y^o \geq \theta$. We have:

$$y^o = A_o u^o , \quad y' = B_o u^o = A_1 u' , \ldots , y^t = B_{t-1} u^{t-1} = A_t u^t , \ldots$$

where $u^t \in R^n_t$, $u^t \geq \theta$, and $A_t = [a^1_t , \ldots , a^n_t]$, $B_t = [b^1_t , \ldots , b^n_t]$ are nxn matrices.[41] A_t and B_t are unique for every t in the horizon.

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41. The columns of A_t , B_t for which u^t has zero components need not be specified.

A strong sufficient condition for such a path $\{y^t\}_{t=0}^{H}$ to be unique is that the set $\{a_t^i\}$, of the inputs components of the processes used at t ,[42] is linearly independent in R_t^n , for all t in the horizon.[43]

7.8. Let us further consider the case where $\{a_t^i\}$ is not necessarily linearly independent in R_t^n . We can still insure -- under a rather weak assumption -- the uniqueness of a u-maximal path, if the preference function u attains its maximum at a unique point of $\bar{\bar{T}}^H(y^o)$.[44]

We assume that the null-space of B_t is properly contained in that of A_t , for all t in the horizon.[45] Let $\{y^t\}_{t=0}^{H}$ and $\{y'^t\}_{t=0}^{H}$ be two paths with $y^o = y'^o$, and $y^t = A_t u^t$, $y'^t = A_t u'^t$, $t \in [1, + 1]$. If $y^\tau \neq y'^\tau$ for some $\tau \in [1, H-1]$, then we can show that $y^t \neq y'^t$ for all $t \in [\tau, H]$. Thus with a strictly concave preference function $\{y^t\}_{t=0}^{H}$ is uniquely determined, given y^o .

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42. I.e., those for which u_i^t is positive.

43. Of course, even if this condition is not satisfied uniqueness of the path $\{y^t\}_{t=0}^{H}$ is insured if $\mathcal{N}(B_t) \supset \mathcal{N}(A_t)$ for all t in the horizon; $\mathcal{N}(A)$ denotes the null-space of the linear operator A .

44. E.g. if u is strictly concave.

45. This assumption holds if e.g. $\{b_t^i\}$ is linearly independent.

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