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On the Existence of a Subinvariant Measure

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ON THE EXISTENCE OF A SUBINVARIANT MEASURE

by*

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Haar measure is invariant under the homeomorphisms induced by the group operation in the measure space. Consider instead the problem of finding a subinvariant measure for a locally compact space with respect to a set $\mathcal{G}$ of homeomorphisms. That is, we look for a measure λ such that $\lambda(GB) \leqq \lambda(B)$ for all $G \in \mathcal{G}$ and Borel sets B. Clearly the existence of such a measure when $\mathcal{G}$ is a group implies that λ is already invariant, so it is natural to consider semigroups $\mathcal{S}$ of homeomorphisms instead. Furthermore, for a monotone set function λ the relation $GB \supset B$ implies $\lambda(GB) \geqq \lambda(B)$, and it is therefore natural to require $GB \not\supset B$ for all $G \in \mathcal{S}$.

In this paper we take the underlying space to be the open unit interval I. The construction of the set function λ given below follows the construction of Haar measure for compact sets as described in [1, Ch. XI].

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The problem of a subinvariant measure on an interval has arisen from an economic problem in the axiomatics of utility [2,3]. The latter problem concerns choice between consumption programs consisting of an infinite sequence of future consumption vectors. The points of I on which $\mathcal{S}$ operates are utility levels of these programs. The elements G of $\mathcal{S}$ represent the effect on utility levels of postponement of programs by a stated number of time units. Each G is labeled by that number and by the "momentary" utility levels associated with the consumption vectors inserted in the gaps created by postponement. The existence of a measure on I subinvariant for $\mathcal{S}$ signifies a certain lack of patience with regard to the time of availability of desirable goods.

Theorem 1.

Let $\mathcal{S}$ be a semi-group of homeomorphisms from I, the open unit interval, to I, having the property that $G\,U \supset U$ never holds for an interval U of I and a $G \in \mathcal{S}$. Assume that for any given open interval U of I an arbitrary point of I can be covered by $G\,U$ for some $G \in \mathcal{S}$. Then there exists a real function λ defined on closed intervals D of I, finitely additive on intervals with disjoint interiors, positive on non-degenerate intervals, monotone, and such that $\lambda(D) \geqq \lambda(GD)$ for all $G \in \mathcal{S}$ and all $D \subset I$.

Proof:

Fix a point p in I and let U be an open interval containing p . If D is a closed subinterval of I let

$$(D:U) = \min \left\{ n \mid D \subset \bigcup_{i=1}^{n} U_i \ , \ U_i = G_i U, \quad G_i \in \mathcal{G} \right\} .$$

It follows from the compactness of D that (D:U) is finite. Define, for A fixed closed and non-degenerate in I ,

$$\lambda_U(D) = (D:U)/(A:U) \quad .$$

Then, if D is non-degenerate, since a cover of D by images of U can be constructed from a cover of D by images of A and a cover of A by images of U ,

(1) $$0 < 1/(A:D) \leq (D:U)/(A:U) = \lambda_U(D) \leq (D:A) \ .$$

Let Φ be the set of functions f defined for closed intervals D in I and such that $0 \leq f(D) \leq (D:A)$. Provide Φ with the topology of convergence on finite sets $\left\{ D_1, D_2, \ldots, D_i \right\}$ [4, p. 92]. Then Φ is compact, by Tychonoff's theorem on the compactness of a product of compact spaces [4, p. 143].

Let $\Lambda(U) = \left\{ \lambda_V \mid U \supset V , p \in V \right\}$. It is straightforward to verify [1, p 255] that the family of all sets $\Lambda(U)$ has the property that any finite subfamily $\{\Lambda(U_1), \ldots, \Lambda(U_n)\}$ has a non-empty intersection. Since Φ is compact there is therefore a function λ in $\bigcap_{p \in U} \overline{\Lambda(U)}$. For D non-degenerate, (1) implies $\lambda(D) > 0$.

To show that λ is finitely additive we use two lemmas.

__Lemma__ 1 . If $\{U_n\}_{n = 1, 2 \ldots}$ is a nested sequence of neighborhoods of p converging to p, then $\lim_{n \to \infty} (A:U_n) = \infty$.

__Proof of Lemma__ 1 . Clearly $(A:U_n)$ is a nondecreasing function of n. Suppose $(A:U_n) \leq N$ for all n for some fixed integer N. Choose $2N$ disjoint open intervals A_i, $i = 1, \ldots, 2N$, in A. Find open intervals U_i' about p such that $G_i A_i = U_i'$ for some $G_i \in \mathcal{G}$. Let $U_o = \bigcap_{i=1}^{2N} U_i'$. Then no A_o, with $G\, U_o = A_o$ for some G, can contain an A_i. Hence N images of U_o under $\mathcal{G}$ could not cover A, which contradicts the premise.

__Lemma__ 2 .

$$- 1/(A:U) + \lambda_U(D) + \lambda_U(E)$$

$$\leq \lambda_U(D \quad E) \leq \lambda_U(D) + \lambda_U(E) ,$$

where U is a neighborhood of p and D and E are closed intervals with disjoint interiors and a common end point.

__Proof of Lemma__ 2 . The first inequality holds because minimal coverings of D and E can be turned into a minimal covering of $D \cup E$, perhaps by removing an interval that covers the common endpoint of D and E .

The second inequality holds because the union of minimal coverings of D and E is a covering of $D \cup E$, perhaps not minimal.

To prove additivity of λ, notice that $\lambda \in \overline{\Lambda(U)}$, for all neighborhoods U of p, implies that there is a nested sequence U_n converging to p, such that $\lambda_n \equiv \lambda_{U_n} \in \Lambda(U_n)$ and λ_n converges to λ in the topology of Φ. For let U'_n be a sequence of neighborhoods of p converging to p. Since $\lambda \in \bigcap_{p \in U} \overline{\Lambda(U)}$, we have $\lambda \in \bigcap_{n=1}^{\infty} \overline{\Lambda(U'_n)}$. Then, for any given finite set $\{D_1, \ldots, D_N\}$, there is for all n a $U''_n \subset U'_n$ such that $| \lambda_{U''_n}(D_k) - \lambda(D_k) | < \frac{1}{n}$, $k = 1, \ldots, N$. Since this sequence converges to p, it contains a nested subsequence U_n converging to p such that λ_n converges to λ.

Now given $\epsilon > 0$ there is an n_ϵ such that $n > n_\epsilon$ implies

$$| \lambda_n (D \cup E) - \lambda(D \cup E) | < \epsilon/3$$

$$| \lambda_n (D) - \lambda(D) | < \epsilon/3$$

$$| \lambda_n (E) - \lambda(E) | < \epsilon/3 .$$

From these inequalities it follows that

$$- \epsilon + \lambda_n(D) + \lambda_n(E) - \lambda_n(D \cup E)$$

$$\leq \lambda(D) + \lambda(E) - \lambda(D \cup E)$$

$$\leq \lambda_n(D) + \lambda_n(E) - \lambda_n(D \cup E) + \epsilon \ .$$

But by Lemma 2 ,

$$1/(A:U_n) \geq \lambda_n(D) + \lambda_n(E) - \lambda_n(D \cup E) \geq 0 \ .$$

Therefore, for all n ,

$$- \epsilon \leq \lambda(D) + \lambda(E) - \lambda(D \quad E) \leq 1/(A:U_n) + \epsilon$$

By Lemma 1, $1/(A:U_n)$ tends to zero. Since ϵ is arbitrary, λ is additive.

To check that $\lambda(G\ D) \leq \lambda(D)$ it is enough to check the same condition for arbitrary λ_U. Now $(GD:U) \leq (D:U)$ because a minimal covering of D by sets G_iU gives rise to a covering, not necessarily minimal, of GD by sets $G\ G_iU$. The desired result follows on division by $(A:U)$.

Monotonicity follows similarly.

<u>Corollary</u>. λ is zero on one-point sets.

Proof: If D is a one-point set, $(D:U) = 1$ for all U . The corollary follows by Lemma 1 .

Theorem 2 . The interval function λ of Theorem 1 is continuous in the sense that, if D_n is a sequence of intervals, containing a fixed point q and converging to it, then $\lim_{n\to\infty} \lambda(D_n) = 0$.

Proof.* There are in I intervals having λ - measure at most ϵ for any $\epsilon > 0$. To see this take an interval having finite positive measure M and partition it into at least $M\epsilon^{-1}$ non-degenerate intervals. One of these must have measure less than ϵ . Let D_ϵ be a non-degenerate interval of measure at most ϵ . Then, for some $G \in \mathcal{S}$, GD_ϵ contains q in its interior and $\lambda(GD_\epsilon) \leq \lambda(D_\epsilon) \leq \epsilon$. For n sufficiently large $D_n \subset G D_\epsilon$ so $\lambda(D_n) \leq \epsilon$.

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