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On a Theorem of Scarf

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On a Theorem of Scarf[1]

by

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In [3], Herbert Scarf has given a remarkable solution for a classical problem of economics. In this note, I wish to suggest a simplification of his proof, and a slight weakening of his assumptions.

Let Ω denote the non-negative orthant of the commodity space R^{ℓ}. The economy is made up of N infinite sequences of consumers. For each $j = 1, \ldots, N$, all the consumers of the j^{th} sequence have the same resources I_j in the interior of Ω, and the same preference preordering $\underset{j}{\precsim}$ on Ω satisfying

(1) $\{x \in \Omega \mid x \underset{j}{\succsim} x'\}$ and $\{x \in \Omega \mid x \underset{j}{\precsim} x'\}$ are closed for every x' in Ω,

(2) for every x in Ω, there is x' in Ω such $x' \underset{j}{\succ} x$,

(3) $x' \underset{j}{\succ} x$ implies $t\,x' + (1 - t)\,x \underset{j}{\succ} x$ for every t such that $0 < t < 1$,

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(4) $x \underset{j}{\succ} x'$ for some x' implies that x is interior to Ω .

An allocation is an N - tuple of infinite sequences $\left((x_1^i), \ldots, (x_N^i)\right)$ of points of Ω , where x_j^i is the consumption of the i^{th} consumer in the j^{th} sequence, such that

$$(5) \quad \lim_{n \to \infty} \left(\sum_{i=1}^{n} \sum_{j=1}^{N} x_j^i - n \sum_{j=1}^{N} I_j \right) = 0 .$$

A finite coalition S of consumers blocks an allocation $\left((x_1^i), \ldots, (x_N^i)\right)$ if, for every consumer (i,j) in S , there is a consumption y_j^i in Ω such that $\sum_{(i,j) \in S} y_j^i = \sum_{(i,j) \in S} I_j^i$, and $y_j^i \underset{j}{\succsim} x_j^i$ for every (i,j) in S , while $y_j^i \underset{j}{\succ} x_j^i$ for at least one (i,j) in S .[2] The core of the economy is the set of allocations that no finite coalition blocks.

An allocation $\left((x_1^i), \ldots, (x_N^i)\right)$ and a price system p form an equilibrium of the economy if, for every (i,j) , the consumption x_j^i is a greatest element of the set $\left\{ x \in \Omega \mid p \cdot x \leqq p \cdot I_j \right\}$ for $\underset{j}{\precsim}$.

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[2] It is convenient, here, to identify the resources of consumer (i,j) by I_j^i , although I_j^i is a constant with respect to i . Given the assumptions made on preferences, our definition of a blocking coalition is easily seen to be equivalent to H. Scarf's.

<u>Theorem</u>: <u>Given an allocation $\left((x_1^i), \ldots, (x_N^i)\right)$ in the core, there is a price system p with which it forms an equilibrium.</u>

<u>Proof</u>: By (1), there is a continuous utility function u_j on Ω for every j ([1], p. 56). We denote $u_j(x_j^i)$ by v_j^i. Two cases have to be distinguished:

(a) <u>for every j, $\text{Inf}_i\ v_j^i = \underline{\lim}_i\ v_j^i$.</u>

We introduce the notation

$$C_j^i = \left\{x \in \Omega \mid u_j(x) > v_j^i\right\}, \quad \Gamma_j^i = C_j^i - \left\{I_j\right\};$$

$$C_j = \left\{x \in \Omega \mid u_j(x) > \text{Inf}_i\ v_j^i\right\}, \quad \Gamma_j = C_j - \left\{I_j\right\}.$$

All these sets are non-empty, by (2), and convex, by (3) and (1) ([1], p. 60). They also have non-empty interiors, for every C_j^i does. Indeed, let x be a point in C_j^i, i.e., such that $x \underset{j}{\succ} x_j^i$. By (1), x has a neighborhood in Ω all of whose elements $\underset{j}{\succ} x_j^i$. But, in that neighborhood, there are points interior to Ω. Any one of them is interior to C_j^i.

The basic property of the sets Γ_j is

(6) $\quad 0$ is not interior to the convex hull of $\bigcup_{j=1}^{N} \Gamma_j$.

To establish this, we denote the interior of a set S by $\mathrm{Int}\ S$, its convex hull by $H(S)$, and its closure by $\bar{S}$, and we first prove that

(7) $$\mathrm{Int}\ H(\bigcup_j \Gamma_j) \subset H(\bigcup_j \mathrm{Int}\ \Gamma_j) .$$

$$\mathrm{Int}\ H(\bigcup_j \Gamma_j) \subset \mathrm{Int}\ H(\bigcup_j \overline{\mathrm{Int}\ \Gamma_j}) \subset$$

$$\mathrm{Int}\ H(\overline{\bigcup_j \mathrm{Int}\ \Gamma_j}) \subset \mathrm{Int}\ \overline{H(\bigcup_j \mathrm{Int}\ \Gamma_j)} =$$

$$\mathrm{Int}\ H(\bigcup_j \mathrm{Int}\ \Gamma_j) .$$

Assume now that (6) does not hold. According to (7), there are, for each j , a joint y'_j in $\mathrm{Int}\ \Gamma_j$, and a non-negative real number α_j , with $\sum_{j=1}^{N} \alpha_j = 1$, such that

$$\sum_j \alpha_j\ y'_j = 0 .$$

Thus, one can find, for each j , a point y_j in Γ_j , and a non-negative <u>rational</u> number r_j , with $\sum_{j=1}^{N} r_j = 1$, such that

$$\sum_j r_j y_j = 0 .$$

Multiplying by a common denominator of the r_j , we obtain

$$\sum_j k_j y_j = 0$$

for an N - tuple (k_j) of non-negative integers, not all zero. Since $y_j \in \Gamma_j$, one has $u_j (y_j + I_j) > \text{Inf}_i \ v_j^i$. Therefore, according to (a), we can select, in the j^{th} sequence, k_j consumers whose v_j^i are less than $u_j(y_j + I_j)$. This means that y_j belongs to the set Γ_j^i of each one of these k_j consumers. Consequently, 0 belongs to the sum of the sets Γ_j^i of the $k_1 + \dots + k_N$ consumers we have selected. And the coalition of these consumers would block the given allocation.

Having established (6), we apply Minkowski's theorem to the situation it describes, and we obtain a hyperplane through 0 , with normal p , bounding for $\bigcup_{j=1}^{N} \Gamma_j$, hence for every Γ_j . We write this as

$p \cdot \Gamma_j \geqq 0$, or $p \cdot C_j \geqq p \cdot I_j$. However, C_j^i is contained in C_j for every (i,j). In addition, by (3), every x such that $x \underset{j}{\succ} x_j^i$ is adherent to C_j^i . Therefore

(8) for every (i,j), $x \underset{j}{\succsim} x^i_j$ implies $p \cdot x \geqq p \cdot I_j$.

In particular, $p \cdot x^i_j \geqq p \cdot I_j$ for every (i,j) . If any of these inequalities were strict, the inner product of p and the vector in the parenthesis of (5) would not tend to zero when $n \to \infty$. Hence

$$p \cdot x^i_j = p \cdot I_j \quad \text{for every } (i,j) \ .$$

Finally, since I_j is interior to Ω , it follows readily from (8) ([1], p. 69), that x^i_j is a greatest element of the set

$$\left\{ x \in \Omega \mid p \cdot x \leqq p \cdot I_j \right\} \quad \text{for } \underset{j}{\precsim}$$

<u>(b) for some j' , $\operatorname{Inf}_i \; v^i_{j'} < \underline{\lim}_i \; v^i_{j'}$</u> .

We will show that this case cannot occur. Notice first that, according to (5),

$$\lim_{n \to \infty} \left(\sum_{j=1}^{N} x^n_j - \sum_{j=1}^{N} I_j \right) = 0 \ .$$

Therefore the sequence of N - tuples (x^n_j) is bounded, and we can extract a subsequence converging to the N - tuple (x^o_j) . Clearly

(9) $$\sum_{j=1}^{N} x_j^o = \sum_{j=1}^{N} I_j \quad .$$

Moreover

$$u_j(x_j^o) \geqq \text{Inf}_i \; v_j^i \quad \text{for every } j \; , \text{ and } \; u_{j'}(x_{j'}^o) > \text{Inf}_i \; v_{j'}^i \quad .$$

The last inequality, which follows from (b), implies $x_{j'}^o \succ_{j'} x_{j'}^i$ for some i , hence, by (4),

$$x_{j'}^o \text{ is interior to } \Omega \quad .$$

Let $s(x,r)$ denote the open sphere with center x and radius $r > 0$. We can choose r small enough for $s(x_{j'}^o, r)$ to be contained in Ω , and for the utility of every consumption in $s(x_{j'}^o, r)$ to be greater than $\text{Inf}_i \; v_{j'}^i$. By (2) and (3), there is, for every $j \neq j'$, a consumption x_j^* in $s(x_j^o, \frac{r}{N})$ such that

$$u_j(x_j^*) > u_j(x_j^o) \quad (j \neq j') \quad .$$

We define $x_{j'}^*$ as equal to $\sum_{j=1}^{N} x_j^o - \sum_{j \neq j'} x_j^*$.

Thus $| x_{j'}^* - x_{j'}^o | < r$. Consequently $x_{j'}^*$ is in Ω and

$$u_{j'}(x^*_{j'}) > \mathrm{Inf}_i \; v^i_{j'} \quad .$$

Also, by (9),

$$\sum_{j=1}^{N} x^*_j = \sum_{j=1}^{N} I_j \quad .$$

To conclude, select for each j, a consumer (i,j) such that $x^i_j \prec_j x^*_j$. The coalition of these N consumers blocks the given allocation.

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The theorem can be generalized without modification of the proof. For instance, the common consumption set X_j of the consumers of the j^{th} sequence may be any closed, convex set with a non-empty interior (instead of being Ω), provided that the asymptotic cone of $X = \sum_{j=1}^{N} X_j$ satisfies $AX \cap (-AX) = \{ 0 \}$ (to insure that the sequence (x^n_j), at the beginning of (b), is bounded). Assumptions (1), (2), (3), and (4) are made on the preferences $\prec_j$ on X_j. Then, given an allocation in the core, there is a price system with which it forms a quasi-equilibrium (a definition of this concept, and a discussion of its relation to the concept of equilibrium will be found in [2]).

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References

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