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Spatial Distribution of Industry

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1. Introduction

This paper is concerned with the determination of the optimum spatial distribution of production and its concentration in "centers" (towns or villages) of different population size. The problem is primarily one of economic planning, but it is hoped that the analysis will also contribute to our understanding of the empirical findings about the regularities in the frequency distribution or ranking of centers of different population size.[1]

[1] For a survey of these findings, see e.g., Walter Isard, Location and Space-Economy, New York/London, 1956, pp. 55-60.

This paper does not offer more than a first introduction to the problem. The approach is very tentative, but proves to be of some help to study the fairly simple situations considered in this paper.

As with any other economic problem, it might be useful to state explicitly **which** factors we consider as data of our problem.

First, we have the data present in most economic problems, like consumer preferences, production technology, what are the scarce factors of production, etc.

The data which are specific for problems of spatial economics seem to be the following:

1. The existence of space-consuming activities, like agriculture, which are bound to the spot and which by their nature cannot be concentrated in "centers;"

2. To transport goods gives rise to transportation costs. In the limiting cases of perfectly mobile and immobile goods the transportation costs are zero and infinite, respectively;

3. There exist indivisibilities (fixed costs) in the production of "industries," as distinguished from agriculture, leading to economies of scale.

The three elements mentioned here are both necessary and sufficient to lead to spatial dispersion of industries. Without agriculture or without transportation costs the best location is to concentrate all industries in one center. Without indivisibilities the optimum situation has no centers, but only self-sufficient farmers.

The following sections present a number of simple mathematical models which are based on the three spatial elements.

Section 2 considers a one-dimensional "area." Though no claim is made that any of our models is of direct practical importance, the one-dimensional "area" can be considered as an approximation to a country like Chile, to valleys or economies oriented on the coast. Section 3 extends the model to two-dimensional areas. In both sections we consider the location of plants of one industry which is vertically integrated and produces only consumer goods. Section 4 considers this problem for an economy with several industries which produce consumer and/or intermediate goods.

2. One-Dimensional Area and One Industry

2.1 In this section we consider the problem of how to spread the production units of a given industry along a straight line.

We consider as given an area shaped as a straight line along which an agarian population is spread continously and with a given constant density. Let there be an industry which has production units (plants) each with the following cost function:

(1) $$C = \bar{\gamma} + \gamma v + T$$

where
C = total cost
v = production volume
T = total transportation cost
$\bar{\gamma}$ = total fixed production cost
γ = constant variable production cost per unit of product

In the simplest case the cost functions are identical for all production units and are independent of their location.

Transportation cost can arise either from the distribution of the industrial product to the agricultural area or from collecting agricultural raw materials to the industrial center. In the following, we make the first assumption. We further assume that the transportation costs per unit of product and per unit of distance are a given constant Θ and that the demand density for the product in every point of the line is a given constant Φ . If the maximum distance over which the product has to be delivered on each side of the production center is r distance units, then the total transportation costs T for one plant are:

$$(2) \qquad T = 2 \int_0^r \varphi \Theta r' \, dr' = \varphi \Theta r^2$$

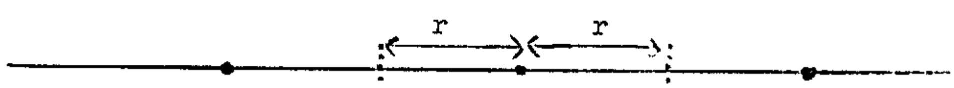


Figure 1

Under the assumptions made the production volume is proportional with the length of the market area over which the production center delivers its product:

$$(3) \qquad v = 2 \varphi r$$

On the basis of the above data we can determine what the optimum distance between production centers is, i.e., the distance at which the average total costs c are a minimum.

The average total costs are equal to:

(4) $$c = \frac{C}{v} = \frac{\bar{\gamma}}{v} + \gamma + \frac{T}{v}$$

Substituting (2) and (3) in (4) and minimizing the resulting expression for variations in r, we find:

(5) $$\tilde{r} = \left(\frac{\bar{\gamma}}{\varphi \, \Theta} \right)^{\frac{1}{2}}$$

where $\tilde{}$ indicates the optimum value of the variable.

In the optimum situation the transportation costs equal:

(6) $$\tilde{T} = \varphi \, \Theta \, \tilde{r}^{\,2} = \bar{\gamma}$$

The solutions for the other variables $\tilde{v}$ and $\tilde{c}$ can easily be derived from equations (3), (4) and (5):

(7) $$\tilde{v} = 2 \, \varphi \, \tilde{r} = 2 \left(\frac{\bar{\gamma} \, \varphi}{\Theta} \right)^{\frac{1}{2}}$$

(8) $$\tilde{c} = \tilde{r} \, t + \gamma = \left(\frac{\Theta \, \bar{\gamma}}{\varphi} \right)^{\frac{1}{2}} + \gamma$$

2.2 In the previous model it was assumed that the demand density φ is the same in every point of the line. This, however, is too strong an assumption. Even with a homogenous demand density in the agricultural area, discontinuity in the demand will arise from the population concentration in the production centers. Assuming that demand is proportional with population size and that the population in the centers is again proportional with the production volume, the previous model can be adapted in the following way:

The equations (1), (2) and (4) remain unchanged. The production volume v now equals:

$$v = 2 \varphi r + \alpha v \quad \text{or}$$

$$v = \frac{2 \varphi r}{1 - \alpha} \tag{3'}$$

In the first expression $2 \varphi r$ indicates the sales to the agricultural area and αv the sales to the production center. The coefficient α is dependent on the demand per worker and the number of workers per unit of product and $0 < \alpha < 1$.

In this case the average costs are a minimum when

$$\tilde{r}' = \left(\frac{\bar{\gamma}}{\varphi \Theta} \right)^{\frac{1}{2}}$$

Comparison with the previous model shows that the optimum distance between two production centers is the same in both cases. However, the optimum production volume increases and the average cost decreases, since $0 < \alpha < 1$:

(7') $$v' = \frac{2}{1-\alpha}\left(\frac{\bar{\gamma}\,\varphi}{\Theta}\right)^{\frac{1}{2}}$$ and

(8') $$\tilde{c}' = (1-\alpha)\left(\frac{\bar{\gamma}\,\Theta}{\varphi}\right)^{\frac{1}{2}} + \gamma$$

2.3 In the previous models, an unbounded area (line) was assumed. In the case of a bounded area, no complications arise as long as the length R of this line is an integer multiple of the optimum length of the market area of one plant, i.e., when $\frac{R}{2\tilde{r}} = n$, where n is an integer indicating the total number of plants in the area.

However, if R is not an integer multiple of $2\tilde{r}$, the market area of all the plants has to be larger or smaller than calculated under the assumption of an unbounded area. Since the cost functions are not symmetrical in r , we have to determine for which value of $\frac{R}{2\tilde{r}}$ the market areas have to be larger (or smaller) than $2\tilde{r}$.

Suppose $\frac{R}{2\tilde{r}} = m$

where $n < m < n + 1$ and n is an integer .

The question then is for which value of m the total production and transportation cost for the whole area R with n centers is equal to these costs with n + 1 centers

$$\text{or} \qquad C_n = C_{n+1} \qquad \text{or}$$

$$n\,(\bar{\gamma} + \gamma\,\frac{R\,\varphi}{n} + r_n^2\,\varphi\,\Theta) = (n+1)\,(\bar{\gamma} + \gamma\,\frac{R\,\varphi}{n+1} + r_{n+1}^2\,\varphi\,\Theta)$$

Substituting $r_n = \frac{R}{2\,n}$ and $r_{n+1} = \frac{R}{2\,(n+1)}$ in (1), we find

$$\frac{R^2}{4\,n}\,\varphi\,\Theta = \bar{\gamma} + \frac{R^2}{4\,(n+1)}\,\varphi\,\Theta$$

(1) or $$n\,(n+1) = \left(\frac{R}{2\,\tilde{r}}\right)^2 = \tilde{m}^2$$

From this it follows that if $\frac{R}{2\,\tilde{r}} < n\,(n+1)$, the total number of centers should equal n ; if $\frac{R}{2\,\tilde{r}} > n\,(n+1)$, the number should be n+1

In the first case the market areas have to be somewhat larger than $2\tilde{r}$, in the second case somewhat smaller. For high values of n · we can approximate our result (1) by the simple rule that we round off $\tilde{m}$ upwards to the nearest integer if the value of $\tilde{m}$ after the point is .5 or more, otherwise we round off downwards.

3. Two-Dimensional Area and One Industry

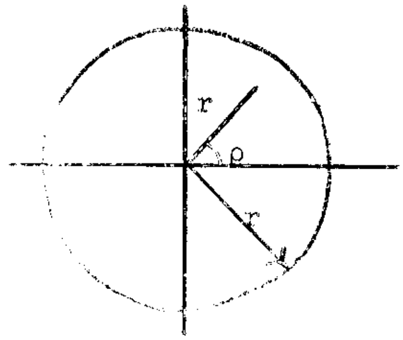
In this section we extend our problem to a two-dimensional market area. Again we consider the most simple situation. We assume an (unbounded) agricultural area. In any point of this area the demand density for the product of a given industry is the same and equal to φ. The cost function of the industry is independent of the location of the plants. The problem of optimum market area for a plant of a given industry is now similar to the problem discussed in Section 2 for a one-dimensional area.

We first determine for a circular market area with the production unit in its center, the optimum size of the area measured by the radius r of the circle. Since demand is given, the optimum radius can be determined by minimizing the average total costs for variations in r:

$$(1) \qquad c = \frac{C}{v} = \frac{\bar{Z}}{v} + \gamma + \frac{T}{v} = \min.$$

Using polar coordinates, the origin coinciding with the center of the circle, every point in the area is defined by its radius r' and the angle ρ (see figure 2)

Figure 2



The demand density in every point is $\varphi\, r'\, dr'\, d\rho$ and the total demand v in a circular area with radius r is:

$$(2) \qquad v = \int_0^{2\Pi} \int_0^{r} \varphi\, r'\, dr'\, d\rho = \Pi\, \varphi\, r^2$$

If the transportation cost per unit of product and per unit of distance is a constant Θ, the costs to transport a quantity $\varphi\, r'\, dr'\, d\rho$ over a distance r' equal $\varphi\, \Theta\, (r')^2\, dr'\, d\rho$ and, therefore, the total transportation costs to supply a circular market area with radius r equal:

$$(3) \qquad T = \int_0^{2\Pi} \int_0^{r} \varphi\, \Theta\, (r')^2\, dr'\, d\rho = \frac{2}{3}\, \Pi\, \varphi\, \Theta\, r^3$$

Substituting (2) and (3) in (1) and minimizing the resulting expression with respect to r, we find:

(4) $$\tilde{r} = \left(\frac{3\,\bar{\gamma}}{\Pi\,\varphi\,\Theta}\right)^{\frac{1}{3}}$$

Since $\left(\frac{3}{\Pi}\right)^{\frac{1}{3}}$ is approximately equal to 1 , we have as an approximation:

(4') $$\tilde{r} = \left(\frac{\bar{\gamma}}{\varphi\,\Theta}\right)^{\frac{1}{3}}$$

In the optimum situation the transportation costs equal:

$$\tilde{T} = \frac{2}{3}\,\Pi\,\varphi\,\Theta\,\tilde{r}^{\,3} = 2\,\bar{\gamma}$$

Comparison with the one-dimensional case show the great similarity in the results for $\tilde{r}$ and $\tilde{T}$.

4. Two-Dimensional Area and Several Industries Producing Consumer and Intermediate Goods

Strictly speaking the models of the previous sections only apply to industries that are vertically integrated and which produce only consumer goods. Further, the models consider the location of just one industry. In this section we present a model which elaborates the previous models on these points.

We consider a closed economy of a more or less circular or square shape with an evenly spread agricultural population. We assume that the level and composition of the GNP by industries for the economy as a whole is given. Gross output of industry h, which produces only one product, is indicated by v^h, where $h = 1, 2 \ldots H$ if H is the total number of industries. In principle, all industries can produce both consumer goods (v^{ho}) and intermediate goods $(v^{hh'})$.

$$(1) \qquad v^h = v^{ho} + \sum^{h'} v^{hh'}$$

We further assume that the plants of a given industry are of the same size and evenly spread over the economy and that they have a more or less circular market area. If A is the given surface of the economy and if n^h indicates the total number of plants of industry h, we have:

$$(2) \qquad n^h = \frac{A}{\Pi \, (r^h)^2}$$

where r^h indicates the radius of the market area of one plant of industry h.

Further we have, since we assumed all plants of a given industry to be of the same size:

$$(3) \qquad v^h = n^h v_i^h \, , \quad \text{where } i = 1, 2 \ldots n^h$$

The cost function of a plant i of industry h is formulated as follows:

$$(4) \qquad C_i^h = \bar{\gamma}^h + \Sigma^{h'} v_i^{h'h} \rho^{h'} + T_i^h$$

We assume that the producer bears the transportation cost on his product and that there is a uniform price for the products of the same industry.

Total transportation cost for a producer in industry h consists of costs of transporting to the consumer (T_i^{ho}) and to other industries. These last costs will be further divided in transportation cost $T_i^{hh_1}$ for deliveries to industries h_1 for which $n^{h_1} > n^h$ and costs $T_i^{hh_2}$ for deliveries to industries h_2 for which $n^{h_2} < n^h$. Thus

$$(5) \qquad T_i^h = T_i^{ho} + \Sigma^{h_1} T_i^{hh_1} + \Sigma^{h_2} T_i^{hh_2}$$

Assuming that we can approximate the transportation cost to evenly spread centers by the transportation cost to a continously spread population, we can estimate the T_i's as follows:

(6) $$T_i^{ho} = \frac{2}{3} \Pi \varphi^{ho} \Theta^h (r^h)^3 \quad \text{where} \quad \varphi^{ho} = \frac{v^{ho}}{A}$$

(7) $$T_i^{hh_1} = \frac{2}{3} \Pi \varphi^{hh_1} \Theta^h (r^h)^3 \quad \text{where} \quad \varphi^{hh_1} = \frac{v^{hh_1}}{A}$$

(8) $$T_i^{hh_2} = \frac{n^{h_2}}{n^h} \cdot \frac{2}{3} \Pi \varphi^{hh_2} \Theta^h (r^{h_2})^3 \quad \text{where} \quad \varphi^{hh_2} = \frac{v^{hh_2}}{A}$$

To determine the optimum number of plants for each industry, we choose to minimize total fixed cost and transportation cost for all industries:

(9) $$\sum^h n^h (\bar{\gamma}^h + T_i^h) = \text{Min.}$$

Substituting equations (6), (7) and (8) in (5) and the resulting expression in (9) and using further equation (2), we find:

$$\sum^h \left\{ \frac{\bar{\gamma}^h A}{\Pi (r^h)^2} + \frac{2}{3} \Pi \varphi^{ho} \Theta^h (r^h)^3 \frac{A}{\Pi (r^h)^2} + \sum^{h_1} \frac{2}{3} \Pi \varphi^{hh_1} \Theta^h (r^h)^3 \cdot \frac{A}{\Pi (r^h)^2} \right.$$

$$\left. + \sum^{h_2} \frac{2}{3} \Pi \varphi^{hh_2} \Theta^h (r^{h_2})^3 \cdot \frac{-A}{\Pi (r^{h_2})^2} \right\} = \text{Min.}$$

Putting $\frac{\partial}{\partial r^h} \overset{h}{\Sigma} \Big\{ \quad \Big\} = 0$, we find

$$r^h = \left\{ \frac{3 \bar{\gamma}^h}{\Pi \Theta^h (\varphi^{ho} + \Sigma^{h_1} \varphi^{hh_1})} \right\}^{\frac{1}{3}}$$

or since approximately $\left(\frac{3}{\Pi}\right)^{\frac{1}{3}} = 1$,

$$(10) \qquad r^h = \left\{ \frac{\bar{\gamma}^h}{(\varphi^{ho} + \Sigma^{h_1} \varphi^{hh_1}) \Theta^h} \right\}^{\frac{1}{3}}$$

This solution is very similar to the solutions found in previous sections. In the special case where for all industries $v^{hh'} = 0$, and therefore $\varphi^{hh_1} = 0$, the solution of Section 3 can be derived from equation (10).

The unsatisfactory feature of the solution (10) is that it is a quasi-solution, because it only gives the values for the r's if we know the ranking of the industries according to the size of their market areas. This size, however, is defined by the r's . The solution, therefore, consists in a trial-and-error procedure, assuming a certain ranking of industries, and calculating the r^h's which have to be consistent with the assumed ranking.

It may be remarked that there is no difficulty in using equation (10) for purposes of empirical testing.