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A Business Game for Teaching and Research Purposes

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(Part 2A, Revised)

Theory and Mathematical Structure of the Game

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1. DEMAND CONDITIONS

1.1. The Functional Form

The listing and description of variables and parameters used together with limits and bounds on their values and the functional forms of demand in this game are discussed and given below:

A compromise has to be made between the complexity and greater richness of the functional forms chosen and the feasibility of analyzing the resulting functions. A detailed discussion of the assumptions made in the selection of the demand conditions is given in sub-section 1.2.

p_i = price charged by Player i

a_i = advertising expenditure by Player i

$$p = \sum_{i=1}^{n} p_i$$

$$a = \sum_{i=1}^{n} (a_i + \epsilon_i)$$

$$\bar{p} = p/n$$

$$\bar{a} = a/n$$

$0 \leq p_i \leq P$

$0 \leq a_i \leq A$

where A and P are some very large upper bounds, merely introduced explicitly for a computer check. They only affect the solutions if they are relatively small.

d_i = market demand for product of the i^{th} firm.

(1) $$d_i = \max [0 , F_i(p,a)]$$

where for the symmetric case $F_i(p, a)$ is defined as:

(2)

$$F_i(p,a) = \frac{1}{n \left(1 + \eta \sqrt{a}\right)} \left[\alpha - \beta \left(p_i + \delta_1 \left(\left| \delta_2 \, \gamma \, (p_i - \bar{p})^{k + \frac{m\delta_3}{a_i}} + \gamma(1 - \delta_2)\,(p_i - \bar{p}) \right| \right) \right) \right]$$

$$\delta_1 = +1 \text{ if } p_i - \bar{p} \geqslant 1$$

$$= -1 \quad p_i - \bar{p} < 1$$

$$\delta_2 = 1 \text{ if } |p_i - \bar{p}| > 1$$

$$= 0 \quad |p_i - \bar{p}| \leq 1$$

$$\left((1 - \Theta) \; \frac{\delta_4 \eta (a_i + \epsilon_i)}{a} + \Theta \right)$$

$$\frac{m}{\varepsilon_i} \delta_3 = \frac{m}{a_i} \quad \text{if} \quad a > 0$$

$$= M \quad \text{if} \quad a = 0$$

$$\left(r^t (\lambda \sin(\omega t + \nu) + \xi) \right)$$

where M is a large positive number

$$\frac{(a_i + \epsilon_i)}{a} \; \delta_4 = \frac{a_i + \epsilon_i}{a} \quad \text{if} \quad \frac{a_i + \epsilon_i}{a} > 0$$

$$= 1 \quad \text{if} \quad \frac{a_i + \epsilon_i}{a} \leq 0$$

The sizes and limits on the parameters are discussed in detail in later sections but a summary is given below:

$1 \leq n$	defined on integers only
$\alpha > 0$	in general very large
$\beta \geq 0$	
$\gamma \geq 0$	
$k + \frac{m}{a} \geq 1$	or sufficiently $k \geq 1$ $m \geq 0$ and $a > 0$
η	any real number, but in general it is likely to be ≥ 0
$0 \leq \Theta \leq 1$	
r	any real number, but in general ≥ 0 and near 1
$\omega \geq 0$	
ν	any real number
ξ	a random variable
$0 \leq \lambda \leq 1$	cycle amplitude parameter
ϵ_i	$i = 1, \ldots, n$ random variables

<u>Note</u>: if $k + \frac{m}{a} = 1$ (especially if $k = 1, m = 0$) a special case will arise when searching for symmetric solutions owing to an extra term

being left after the derivative with respect to p_i is taken and symmetry imposed.

In this case the term

$$\delta_1 \mid \gamma\left((p_i - \bar{p})\ \delta_2\right)^{k + \frac{m}{a}\delta_3} \mid \qquad \text{simplifies to}$$

$$\gamma\ (p_i - \bar{p})$$

$$1 \leq p_i$$

$$0 \leq a_i$$

If the prices and advertising charges are not identical then demand is obtained as follows:

1. Calculate $F_i(p, a)$ for $i = 1, \ldots, n$

2. Replace this series of n numbers by a new series:

$$F_i^{+}(p, a) = \text{Max}\ [0, F_i(p, a)]$$

2 a. Obtain new series

$$F_i^{++}(p, a) = \text{Min}\ [F_i^{+}(p, a), s_i]$$

s_i = inventory of Player i, or amount offered for sale (if this is different)

3. Form the sum

$$R = \sum_{i=1}^{n} F_i^{++}(p, a)$$

4. If $R \leq (\alpha - \beta \bar{p})(1 + \eta \sqrt{a})$

then $d_i = F_i^{++}(p, a)$

5. (1) If $R > (\alpha - \beta \bar{p})(1 + \eta \sqrt{a})$

then $d_i = \frac{(\alpha - \beta \bar{p})(1 + \eta \sqrt{a})}{R} \; F_i^{++}(p, a)$

(2) If the game is played with

$a \equiv 0$

then $d_i = F_i^{++}(p, a)$ starting with the player with the lowest price and so on in sequence until

$$\sum_{S} F_i^{++}(p, a) \geq (\alpha - \beta \bar{p})$$

The last player who is allocated any sales obtains

$$d_i = (\alpha - \beta \bar{p}) - \sum_{S-1} F_i^{++}(p, a)$$

1.2. Discussion of Demand Functions

For ease of discussion, the demand conditions are considered under two special cases. The first is when all firms charge the same price and have the same advertising expenditure. In other words, when market conditions are symmetric. Referring to condition (1) in Section 1.1 we observe that demand must be greater than or equal to zero.

The functional form $F_i(p, a)$ given as condition (2) in Section 1.1. consists of five major parts. The first is primarily concerned with the effect of price and relative price of the market; the second accounts for the overall institutional effect of advertising; the third deals with the competitive or distributional effect of advertising and includes a random component; and the fourth and fifth include the possibility of a trend, a cycle, and a random element in the economy.

When a functional form becomes as lengthy and complex as this one, it is desirable to examine it by considering special cases and simple examples. Referring back to condition (2), we observe that if we ignore trend, cycle, random effects, set advertising equal to zero, and assume that prices are equal for all competitors, all that remains is a simple linear demand equation of the form.

$$q_i = \frac{1}{n} [\alpha - \beta p_i]$$

Leaving conditions as described above except for the restriction on prices, we observe that a term involving the difference between the price charged by a player and the average market price is introduced.

$$(p_i - \bar{p})^k$$

In the above expression k is a parameter which indicates an inherent degree of substitutability between the products. In a slightly more complicated model, it would be desirable to introduce a weighted average rather than the simple arithmetic average used for average market price.

Introducing an advertising effect, the price difference term now becomes

$$(p_i - \bar{p})^{k - \frac{m\delta_3}{a}} .$$

This adds a dampening effect such that as the amount of advertising spent in the market by an individual is increased, the effect of price differentials is damped.

In this first part of the functional form $F_i(p, a)$ three Kronecker δ symbols appear. They are needed to take care of special conditions where the functional form will either be undefined or have an undesirable property. Thus, δ_1 is used to make sure that if an individual charges less than the average market price, he will increase his demand, whereas if he charges more his demand will be decreased, all other things being equal.

δ_2 is introduced to take care of a special case that will exist when the difference between the price charged by the individual and the average market price is less than one in absolute value. For this case a linear approximation of the competitive effect of price is taken.

δ_3 calls our attention to observing that the exponent $\frac{m}{a}$ becomes infinite if a equals zero. In order to keep the demand function well defined, we must assign a value to it in this instance.

The parameter α controls the overall market size. For example, if the structure of this game were meant to represent the automobile market, α would have a value of several million.

The parameter β controls the sensitivity of overall demand to change in average price. Much of the effect of advertising comes about by the multiplicative modification of demand $[\alpha - \beta\, p_i]$.

The parameters γ , k and m are all relevant to the control of the degree of competitive effect of different prices. It can be observed that if m equals zero, the advertising modification is cut out.

The second major term in the function $F_i(p, a)$ which is

$$1 + \eta \sqrt{a}$$

controls the overall or institutional effect of advertising. The parameter η controls the effectiveness of the overall impact on industry demand. If η equals zero, then advertising has no effect whatsoever on the overall demand in this industry. A square root is introduced on the sum of advertising expenditures to produce the effect of diminishing returns.

The third term

$$\left((1 - \Theta) \frac{\delta_4 \eta \ (a_i + \epsilon_i)}{a} + \Theta \right)$$

gives the competitive effect of advertising. The parameter Θ , which ranges between zero and one is introduced so that the percentage of the market which can be "switched" by advertising is controlled. Thus, if $\Theta = 1$, this indicates that there is no competitive component to advertising. If Θ equals zero, a firm may stand to lose its total market if it fails to advertise.

The competitive and random effects are introduced by the term $\frac{\eta(a_i+\epsilon_i)}{a}$. The random effect, ϵ_i , is added directly to the firm's advertising expenditure and has a mean of zero. Apart from the random effect, the term gives a weighting in proportion to the relative advertising expenditures of the firms.

In the program, the a_i are replaced by weighted averages involving several time periods.

The fourth Kronecker δ_4 is introduced to call for a definition of the value $\frac{a_i + \epsilon_i}{a}$ when it is less than or equal to zero.

The fourth term is a simple expression for introducing the effect of a trend controlled by the parameter r and the fifth for the effect of a cycle controlled by the function

$$\lambda \sin(\omega t + \nu) + \xi$$

together with the effect of a random disturbance in the economy given by the random variable ξ , λ gives the amplitude of the cycle as a fraction of what total demand would be in the absence of the cycle. In order that demand will never be negative, $|\lambda| \leq 1$ and the mean of ξ is 1 . For example, if the variance of ξ were zero and $\lambda = .2$, the term $(\lambda \sin(\omega t + \nu) + \xi)$ would vary from $.8$ to 1.2 [because $\sin(\omega t + \nu)$ must range from $+1$ to -1].

1.3. Properties of the Demand Function

As soon as we begin to consider the division of the market when the firms do not employ symmetric strategies, considerable difficulties are immediately faced in computation. The contingent demand functions may take many shapes and will also depend upon the capacity of firms to satisfy the demands on them. It is obvious that the amount

$$Q = [\alpha - \beta \hat{p}] \; [1 + \eta \sqrt{a}]$$

gives us an upper bound on the total sales to be expected in the market. The $\hat{p}_i$ stands for the lowest price charged by any of the players. Essentially this expression is the amount that would be sold by all the players if they were charging the lowest price in the market. This amount cannot be exceeded, yet, the formula for calculating the $F_i(p,a)$ when applied to non-symmetric strategies may result in a series of numbers which sum to more than this quantity. For this reason a "rationing system" must be introduced, and this is what is done in the last part of 1.2 . There are many alternative rationing systems which must be examined. An extensive exploration of them will not be carried out at this time.

Some of the problems in rationing and the effect of capacity limitations are illustrated below in figures 1 and 2 . Figure 1 illustrates the pattern of contingent demand in a duopolistic market where the firms are selling an undifferentiated product. The line DD' represents the function $F_i(p,a)$ in the symmetric case and is equivalent to the DD' curve of Chamberlin. If the firms had a production limitation equal, for example to $1/2\ \alpha$ then if one firm charged the price p_1 the contingent demand function for the other would be given by the two line segments and the point adcd'a' . If on the other hand, there were no capacity limitations and the products being sold were perfect substitutes, then the low priced firm would capture all the market and the contingent demand function, given the information that his competitor is charging a price p_1 , would be the two broken lines and the point adcd'a' as indicated in figure 2.

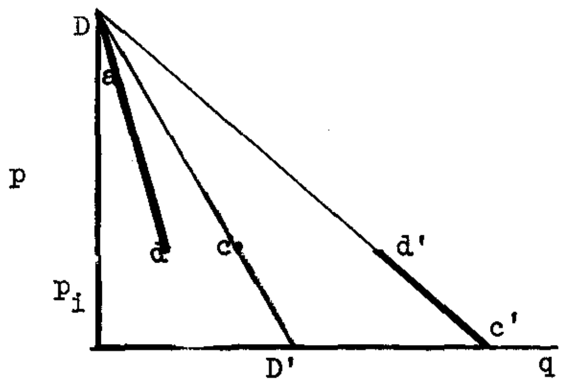


Figure 1

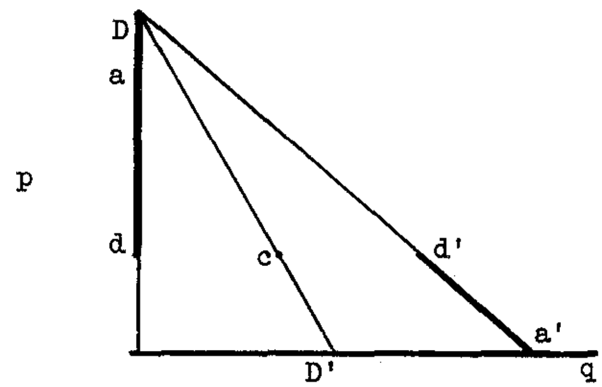


Figure 2

If the goods being traded were substitutes with a varying degree of substitution between them, a contingent demand function of the shape dd' in Figure 3 might be expected. Such a shape would be obtained by introducing a term $\gamma(p_i - \bar{p})^x$ where $x \geq 1$ where $\bar{p}$ is the average price. However this gives a contingent demand of shape dd' as indicated in Figure 4. The zone within aa' is where $| p_i - \bar{p} | < 1$; this causes the change to be relatively insensitive (which may be regarded as undesirable). Figure 3 has a linear approximation for the contingent demand in the zone aa' .

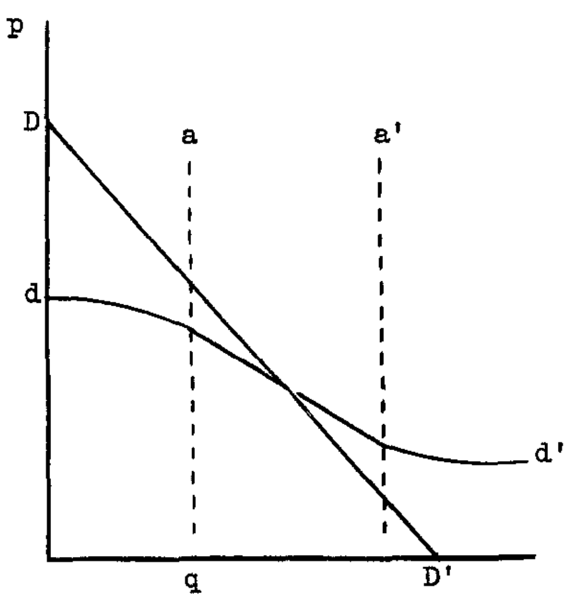


Figure 3

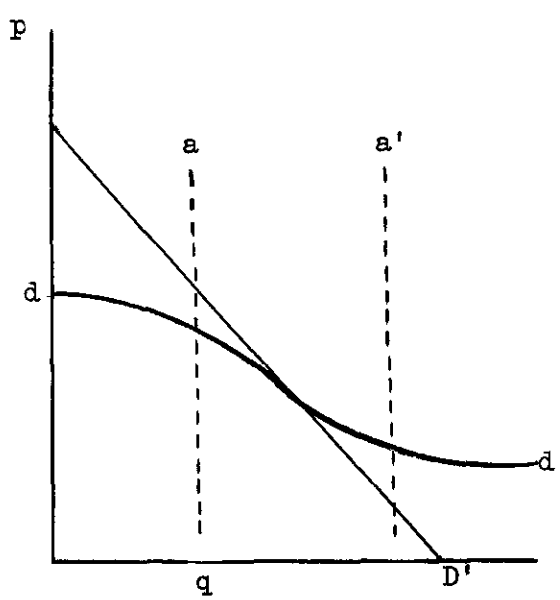


Figure 4

Figures 5 and 6 show two ways for introducing the overall industry effect of advertising. In the model constructed here, the effect as indicated in Figure 6 has been used.

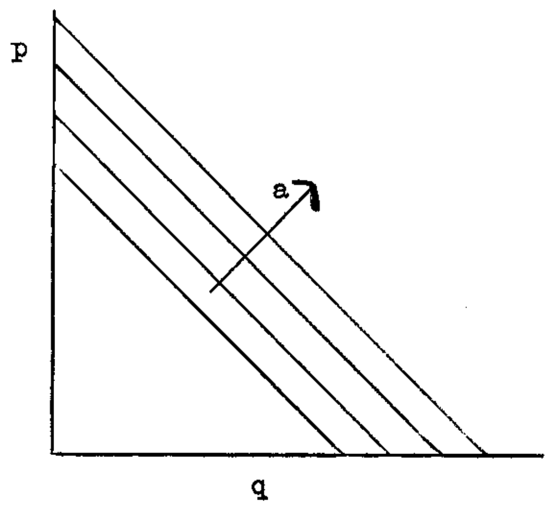


Figure 5

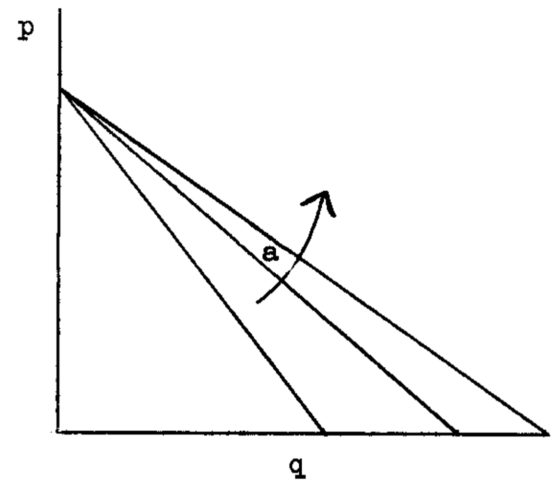


Figure 6

For the parallel shift effect, the terms involving advertising need to be applied only to the constant α in the demand function.

2. COST CONDITIONS

2.1. Production Costs

The average costs of the firms are assumed to be identical and constant. Thus, if the cost of production of q_i items is denoted by $C(q_i)$ then

$$C\ (q_i) = c\ q_i$$

where c is the average cost of production. We place a bound of Q_i on the output of each player. These do not need to be symmetric for the type of solution to be examined. It should be noted that depending upon the size of these bounds on production rates, both the Edgeworth and the Bertrand types of non-cooperative game solutions may be possible.[1]

It can be easily seen that with either linear or constant average costs the revenue function will have quadratic terms and in the symmetric case can be made to have desirable analytical properties depending upon the selection of parameters.

2.2. Advertising

The advertising costs are merely for aggregate advertising, or product differentiation. In actuality, in many markets there are many different mechanisms for advertising, promotion, public relations, etc. These are often used for rebates, inducements to distributors or retailers, and in general affect the market through a series of time lags. The possibility of introducing some of these time lags is dealt with in Section 4.

2.3. The Revenue Function

The revenue function for Player i is given below:

$$\Pi_i = p_i \min \left[s_i , F_i^{++} (p, a) \right] - c\, q_i - I\, s_i - a_i - K$$

I = the unit inventory carrying cost per period. This will be developed further in section 4.

Where the s_i stands for the amount of inventory available to player i . In solving for steady state solutions this particular form will in general be simplified down to the following:

$$\Pi_i = (p_i - c - \frac{1}{2} I)\, F_i\, (p, a) - a_i$$

In the first case, it was necessary to take into account the difference between the demand and inventories as well as the costs of production incurred during that period and the inventory carrying charges. Furthermore, to be formally correct we need to include a constant term K , which represents fixed costs. Referring to section 5 of Part 1 of CFDP 115, this fixed charge appears on the quarterly report in the form of administrative overheads and depreciation.

Furthermore, if the decision is made to play the game with the firms paying dividends, care must be taken to subtract the dividend payments during the computation made to obtain a firm's new asset position after the play of each period. The discussion of assets and ruin is given in section 5. Parenthetically, it may be noted that for most parameter values the asset conditions do not affect the one period equilibrium strategies; although the properties of games of economic survival are considerably different from the simple one period games.2/

3. SYMMETRIC SOLUTIONS

3.1. Preliminary Discussion

Except for an extensive literature on Duopoly there have been few attempts to solve explicit models of markets. As soon as the number of players becomes larger than two, the combinatorics of different solution concepts, different information conditions, etc., become mammoth. Furthermore, when new variables such as advertising are introduced, the very definition of certain solution concepts becomes hazy. Economists have struggled for some time with the problem of the definition of demand under the conditions of oligopoly. Chamberlain's Industry and Individual Demand Curves, Sweezy's "Kinked Oligopoly Demand," the various schemes of Edgeworth, Bertrand, Stigler and others are all manifestations of attempts to deal with the structure of "contingent demand functions."3/

In general, the very logic of market structures will force the economic model builder to define contingent demand functions with kinks, breaks, or at least "bumps" and other non-convexities depending upon a host of conditions. At best, one way in which there is a sporting chance of beginning to lay bare the anatomy of oligopolistic structure is to concern ourselves with symmetric cases. The model constructed for this game is symmetric in demand conditions and costs and can be solved for any number of players under several solution concepts. Although for exploration purposes we limit ourselves to six or fewer players.

In the one period solutions the terms

$$(r^t)\left(\lambda \sin(\omega t + \nu) + \xi\right) = T$$

do not introduce any extra complications. The time path of the one period solutions can be obtained by multiplying the one period solutions by the function T . When logs are considered this is no longer the case.

3.2. Solution Concepts

Most of the oligopoly solution concepts are dynamic or pseudo-dynamic, as such in order to formalize them, multi-period models must be considered. These are discussed in Section 4. Limiting our investigations to one period models, there are three major types of solution which we can examine: (1) Joint maximization, (2) non cooperative equilibrium, and (3) pure competition. These lead to several variants, as is indicated in 3.3.

3.3. Solutions

3.3.1. Joint Maximization

If $k + \frac{m}{a} > 1$ we may drop any consideration of the terms involving $(p_i - \bar{p})$ as both they and their first derivatives will vanish when p_i is set equal to $\bar{p}$. In what follows, $c' = c + \frac{1}{2} I$

$$\Pi = \sum_{i=1}^{n} \Pi_i =$$

$$\sum_{i=1}^{n} \left(\frac{p_i - c'}{n}\right) \left[\alpha - \beta\, p_i \right] (1 + \eta \sqrt{a}) \left((1 - \Theta) \frac{a_i}{a} + \Theta \right) T - a$$

(1) $\frac{\partial \Pi}{\partial p_i} = 0$ for $i = 1, 2, \ldots, n$

(2) $\frac{\partial \Pi}{\partial a_i} = 0$ for $i = 1, 2, \ldots, n$

(1)' $\left(\frac{p_i - c'}{n}\right) (-\beta) + \frac{1}{n} (\alpha = \beta\, p_i) = 0$

or $\boxed{p_i = \frac{\alpha + c'\beta}{2\beta}}$ for $i = 1, 2, \ldots, n$

Set $(p_i - c')(\alpha - \beta\, p_i)\, T = A$

Then $\Pi = \frac{A}{n} (1 + \eta \sqrt{a}) \left(\sum_{i=1}^{n} (1 - \Theta) \frac{a_i}{a} + \Theta \right) - a$

$$= \frac{A}{n} (1 + \eta \sqrt{a}) - a$$

(2) $\frac{\eta}{2\sqrt{a}} = \frac{n}{A}$ or $a = \left(\frac{A\eta}{2n}\right)^2$

$A = \left(\frac{\alpha - c'\beta}{2\,\beta}\right) \left(\frac{\alpha - c'\beta}{2}\right)$ hence

$$\boxed{a = \left(\frac{(\alpha - c'\beta)^2}{8\,\beta\, n}\, \eta \right)^2}$$

3.3.2. Non Cooperative Equilibrium (Price and Advertising)

If advertising is held constant, the remaining price game analysis falls into two cases which can be described respectively as the pure strategy equilibrium, Bertrand or Chamberlin case or the "cycling" or Edgeworth case. The shape of the contingent revenue functions will determine which of the two types of solution exists. Figures 7 and 8 represent the extreme cases arising from the market with each firm selling a perfect substitute or a good identical with that of all others. Figure 1 showed the contingent demand given that there is a production limitation preventing the low priced firm from supplying the total market, thus Figure 7 shows

Figure 7

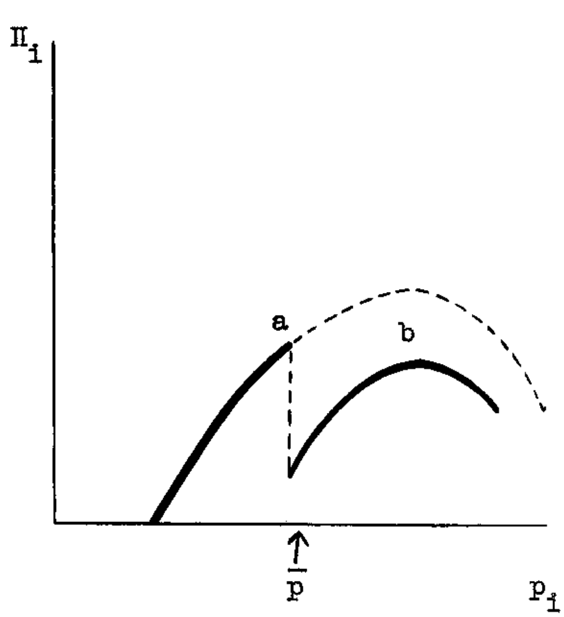
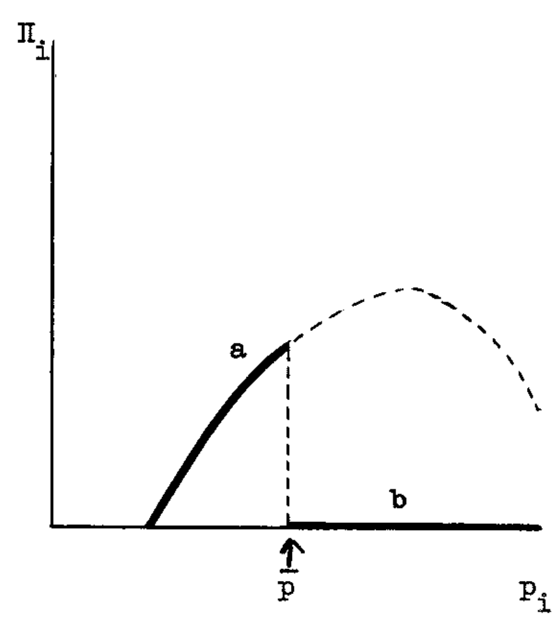


Figure 8



the revenue of the i^{th} firm as it varies its price given that all others are charging $\bar{p}$ and do not supply the whole market. If the secondary maximum "b" is ever greater than the maximum "a" for any contingent demand function, then there will be an incentive for a firm that is barely undercut under some circumstances to raise its price. This gives rise to the Edgeworth cycle.[4/]

The revenue function in Figure 8 arises from the contingent demand as illustrated in Figure 2. If a firm charges a price higher than its competitors it loses all of its market, hence the secondary maximum "b" (in Figure 8) is zero. In this case there will be an equilibrium point at the "purely competitive market" equilibrium.

For undifferentiated products this is the Bertrand solution. With differentiated products a non-cooperative pure strategy equilibrium will be obtained which approaches the pure competition solution as the differentiation is reduced. This is the Chamberlin large group solution.

In this paper we do not deal with general values for k and m; instead for initial experimentation, ease of analysis and discussion the case with $k = 1$ and $m = 0$ is examined exhaustively.

The advertising terms do not appear in the equations below as they can be divided out after the derivatives with respect to p_i are taken.

For $k = 1$ and $m = 0$

$$q_1 = \frac{1}{n}\left(\alpha - \beta(1 + \gamma - \frac{\gamma}{n})\, p_1 + \frac{\gamma}{n}\beta\, (p_2 + p_3 + \dots + p_n)\right)$$
$$\vdots$$

or
$$q = \frac{1}{n}\left(\alpha + \begin{bmatrix} -\beta\,(1+\gamma-\frac{\gamma}{n}), & \frac{\gamma\,\beta}{n}, & \cdots & , \frac{\gamma\,\beta}{n} \\ \frac{\gamma\,\beta}{n}, & -\beta\,(1+\gamma-\frac{\gamma}{n}), & \cdots & , \frac{\gamma\,\beta}{n} \\ \vdots & & & \\ \frac{\gamma\,\beta}{n} & & & -\beta\,(1+\gamma-\frac{\gamma}{n}) \end{bmatrix} p\right)$$

Solving for price:

$$\Pi_i = \frac{1}{n}(p_i - c')\left(\alpha - \beta\,(1+\gamma-\frac{\gamma}{n})\right) p_i + \frac{\gamma\,\beta}{n}\left((p_1 + \ldots + p_{i-1} + p_{i+1} \ldots p_n)\right)$$

$$\frac{\partial \Pi}{\partial p_i} = \frac{1}{n}(p_i - c')\left(-\beta\,(1+\gamma-\frac{\gamma}{n})\right) + \frac{1}{n}\left((\alpha - \beta\,(1+\gamma-\frac{\gamma}{n})\,p_i + (\frac{n-1}{n})\,\beta\,\gamma\,p_i\right) = 0$$

$$\left[-2\,\beta\,A + \gamma\,\beta\,(\frac{n-1}{n})\right] p_i = -(c'\,\beta\,A + \alpha)$$

$$p_i = \frac{\alpha + c'\beta\,(1+\gamma-\frac{\gamma}{n})}{2\beta(1+\gamma-\frac{\gamma}{n}) - \gamma\,\beta(\frac{n-1}{n})}$$

or
$$\boxed{p_i = \frac{n\alpha + c'\beta(n + (n-1)\,\gamma)}{\beta\,(2\,n + (n-1)\,\gamma)}}$$

Where as γ becomes larger the substitutability between the products is increased. We further note that if we wish to analyze the non-symmetric cases in many instances the problem amounts to solving a linear program.

Solving for advertising

$$\Pi_i = \frac{1}{n} A (1 + \eta\sqrt{a}) \left((1 - \Theta) \frac{a_i}{a} + \Theta \right) - a_i$$

where

$$(p_i - c') (\alpha - \beta p_i) T = A$$

$$\frac{\partial \Pi_i}{\partial a_i} = \frac{A}{n} \left[\frac{\eta}{2\sqrt{a}} \left((1 - \Theta) \frac{1}{n} + \Theta \right) + (1 + \eta\sqrt{a}) (1 - \Theta)(\frac{n-1}{na}) \right] - 1 = 0$$

$$\frac{\eta}{2\sqrt{a}} \left((1 - \Theta) \frac{1}{n} + \Theta \right) + (1 + \eta\sqrt{a}) (1 - \Theta) \left(\frac{n-1}{na} \right) = \frac{n}{A}$$

set $\frac{\eta}{2} \left((1 - \Theta) \frac{1}{n} + \Theta) \right) = B$, $(1 - \Theta) \left(\frac{n-1}{n} \right) = C$

$$\frac{n}{A} = A'$$

then $\frac{B}{\sqrt{a}} + (1 + \eta\sqrt{a}) \frac{C}{a} = A'$

set $\sqrt{a} = x$

$$A' x^2 = B x + C (1 + \eta x)$$

or $A' x^2 - (B + \eta C) x - C = 0$

$$a_i = \left(\frac{B + \eta C + \sqrt{(B + \eta C)^2 + 4 A'C}}{2 A'} \right)^2$$

For example for $n = 1$ (a monopolist)

$$a_i = \left(\frac{\frac{\eta}{2} + \sqrt{\left(\frac{\eta}{2}\right)^2}}{2\,A} \right)^2 = \left(\frac{\eta}{2\,A}\right)^2$$

$$= \left(\frac{\eta\,(\alpha - c'\,\beta)^2}{8\,\beta} \right)^2$$

3.3.3. Purely Competitive Market Equilibrium

3.3.3.1 Perfect Substitutes: No Advertising

Leaving out of consideration trend cycle and the random element, the demand conditions become

$$q_i = \frac{1}{n}(\alpha - \beta\,p_i) \text{ or}$$

$$p_i = \frac{\alpha - n\,q_i}{\beta}$$ or equivalently where the q_i

are the independent variables and symbolize the amounts offered for sale by the players, in general

$$p_i = \frac{\alpha - \sum_{i=1}^{n} q_i}{\beta}$$

Under pure competition market price is regarded as unchanged by individual action thus if

$$\Pi_i = p_i\,q_i - c'q_i$$

the competitive conditions yield

$$\frac{\partial \Pi_i}{\partial q_i} = p_i - c' = 0 \quad \text{or}$$

$$\boxed{p_i = c'}$$

which gives $\Pi_i = (p_i - c')\, q_i = 0$ for any q_i . In particular the outputs for which this condition holds will be

$$q_i = \frac{\alpha - \beta c'}{n}$$

and

$$\Pi_i = 0$$

3.3.3.2 Perfect Substitutes: "Industry Advertising"

The demand conditions

$$q_i = \frac{1}{n} (\alpha - \beta p_i) (1 + \eta \sqrt{a})$$

reflect the cooperative component of advertising. If the firms were to disregard their individual effect on the market via advertising then the purely competitive solution would be:

$$a_i = 0$$

3.3.3.3 Product Differentiation: The Competitive Solution

If product differentiation is allowed for, then a competitive solution may be obtained by solving a general equilibrium system for all prices as parameters. This will not be developed further at this time.

3.3.4. Quantity - Variation Non-Cooperative Solutions

3.3.4.1 Perfect Substitutes: "Industry Advertising"

The demand conditions become

$$p = \frac{\alpha - \dfrac{\sum_{i=1}^{n} q_i}{(1 + \eta\sqrt{a})}}{\beta}$$

$$\Pi_i = p\, q_i - c'\, q_i - a_i$$

Equilibrium is given by

$$\frac{\partial \Pi_i}{\partial q_i} = 0 \quad \text{and}$$

$$\frac{\partial \Pi_i}{\partial a_i} = 0$$

$$(1) \quad \left(\frac{-1}{\beta\,(1 + \eta\sqrt{a})}\right) q_i + \left(\frac{\alpha}{\beta} - \frac{n\, q_i}{\beta(1 + \eta\sqrt{a})} - c'\right) = 0$$

$$\frac{-q_i}{1 + \eta\sqrt{a}} + \left(\alpha - \frac{n\, q_i}{1 + \eta\sqrt{a}}\right) - c'\beta = 0$$

$$\boxed{q_i = \frac{1 + \eta\sqrt{a}}{(n+1)}\,(\alpha - c'\beta)}$$

$$(2) \quad \frac{\partial \Pi_i}{\partial a_i} = \frac{\partial}{\partial a_i}\left\{(p - c')\, q_i - a_i\right\} = q_i \frac{\partial p}{\partial a_i} + p\,\frac{\partial p_i}{\partial a_i} - 1 = 0$$

substituting we obtain

$$p = \frac{1}{\beta}\left(\alpha - \frac{n}{n+1}\,(\alpha - c'\,\beta)\right) \text{ which is independent}$$

of a_i hence:

$$(2)' \quad \frac{1}{\beta}\left(\alpha - \frac{n}{n+1}\,(\alpha - c'\beta)\right)\left(\frac{\eta\,(\alpha - c'\beta}{2\,(\eta+1)\sqrt{a}}\right) = 1$$

$$\boxed{a_i = \left(\frac{\eta}{2}\right)^2 \left(\frac{\alpha - c'\beta}{(n+1)^2}\right)^2 \left(\frac{\alpha + n\,c'\beta}{\beta}\right)^2 \frac{1}{n}}$$

3.3.5 Fixed Costs

In this model the pure competition solution is not stable if there are fixed costs present. All firms must run at a loss. Unless there are capacity limitations the eventual ruin of some of the firms does not change the loss position of the remaining firms.

If capacity limitations are considered then in the symmetric case

$$p_i = \text{Max}\left[c'\,,\ (\alpha - \sum_{i=1}^{n} \hat{q}_i)/\beta\right] \text{ where } \hat{q}_i$$

is the production limit of the i^{th} firm. The full development of the effect of fixed costs requires an investigation of games of economic survival. This is not done at this time.

It is of further interest to note that both experience with business games and industrial folk-lore indicate a far greater importance attached to fixed costs than that indicated by economic theory.

3.3.6 Threats

It is conceivable that the goal of a firm is to maximize the losses of its competitors. In the two person case such behavior converts a non-zero sum game into a zero sum game on the difference matrix $\Pi_1 - \Pi_2$. Furthermore if the game is symmetric the solution to this coincides with the efficient point or purely competitive solution. Thus care must be taken in interpreting results of plays if they indicate a purely competitive solution.[4]

For games with more than two players a complex of degrees of co-operation must be considered as is indicated by the parameter values below.

We assume each player wishes to maximize $\bar{\Pi}_i$ where:

$$\bar{\Pi}_1 = \Pi_1 + \Theta_{1,2}\Pi_2 + \Theta_{1,3}\Pi_3 + \cdots + \Theta_{1,n}\Pi_n$$

$$\bar{\Pi}_2 = \Theta_{2,1}\Pi_1 + \Pi_2 + \Theta_{2,3}\Pi_3 + \cdots + \Theta_{2,n}\Pi_n$$

$$\vdots$$

$$\bar{\Pi}_n = \Theta_{n,1}\Pi_1 + \Theta_{n,2}\Pi_2 + \Theta_{n,3}\Pi_3 + \cdots + \Pi_n$$

Where the $\Theta_{j,k}$ are real numbers.[5] For example if they are all zero this gives the non-cooperative equilibrium conditions. If they are all $+1$ this gives joint maximization and if they are all -1 this gives a game in which each man is interested in trying to do better than his competitors.

Further development requires a consideration of games of economic survival.[6]

FOOTNOTES

1. Shubik, M., Strategy and Market Structure, John Wiley and Sons, Inc., New York, 1959, Ch. 5.

2. Shubik, M., pages 206-267.

3. Shubik, M., *Op. Cit.*, pages 143-150.

4. Shubik, M., *Opp.*, Ch. 4.

5. The idea of introducing the parameter $\Theta_{j,k}$ was put forward by Dr. George Feeney in his own work in this area.

6. Shubik, M., Ch. 10.