COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut

COWLES FOUNDATION DISCUSSION PAPER NO. 113

Some Informal Comments on Models of Decision Processes Under Uncertainty

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May 9, 1961

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1. Games Against Nature

The general formulation of a decision process under uncertainty describes a set of states of Nature S, a set of actions by the player A and a utility function U. Thus the player may be characterized as selecting a row in the matrix illustrated in Figure 1. The triplet (A, S, U) describes a "Game Against Nature."

$$
\begin{array}{c|ccccc}
 & S_1 & S_2 & S_3 & \cdots & S_n \\
\hline
a_1 & U_{11} & U_{12} & U_{13} & \cdots & U_{1n} \\
a_2 & U_{21} & U_{22} & U_{23} & \cdots & U_{2n} \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
a_m & U_{m1} & U_{m2} & U_{m3} & \cdots & U_{m,n}
\end{array}
$$

Figure 1

In this note there are three modifications to this formulation that are considered. They concern:

(1) Knowledge of the number of states of nature, n

(2) Knowledge of the number of actions available, m

(3) Knowledge of the outcome from any pair (a_i, S_j)

*Research undertaken by the Cowles Commission for Research in Economics under Task NR-047-006 with the Office of Naval Research.

2. The States of Nature

Often an individual does not know how many different states Nature has available. Part of his problem is to estimate this.

It appears that under the assumption of "complete ignorance" concerning the behavior of Nature that in some instances the number of states available to Nature will not be of importance.

Luce and Raiffa use two axioms to characterize complete ignorance 1/. The first is that a relabeling of the states of Nature will not change the decision problem. The second axiom indicates that the deletion of any column whose values are identical to those of another column will not change the decision problem. Thus the games:

	S_1	S_2	S_3	S_4
a_1	3	4	4	4
a_2	7	2	2	2

Game 1

	S_1	S_2
a_1	3	4
a_2	7	2

Game 1a

are regarded as the same. A stronger axiom indicates that the deletion of states of Nature which in probability mixtures are equivalent to other states will not change the decision problem. Thus the games:

	S_1	S_2	S_3	S_4
a_1	3	2	1	0
a_2	7	0	1	2

<u>Game 2</u>

	S_1	S_3
a_1	3	1
a_2	7	1

<u>Game 2a</u>

are regarded as the same. $(\frac{1}{2})\ S_2 + \frac{1}{2}\ (S_4) = S_3$.

In particular it does not matter what was the original size of n , the number of states of Nature if there is redundancy present. The axioms and the examples illustrate that if the decision maker has any subjective probability distribution concerning the states of Nature he cannot be regarded as "completely ignorant."

If the decision-maker has no <u>a priori</u> subjective probability concerning the states of Nature should there be any effect to introducing a new column? The individual is often only vaguely aware of some of the states of Nature. For example the "picnicker" may not know that Nature not only can rain or shine, but can snow, hail and possibly do something else such as rain blood or locusts. Consider the following two games:

	S_1 Rain	S_2 Shine
a_1 Stay home	0	- 2
a_2 picnic	- 4	+ 5

<u>Game 3</u>

	S_1 Rain	S_2 Shine	S_3 Rain Blood
a_1	0	- 2	+ 2
a_2	- 4	+ 5	- 10

<u>Game 3a</u>

Suppose the referee informs the player that he may be playing in either game. If he is completely ignorant should this information be any different from that in which the referee merely informs him that he is playing in the first game? Furthermore suppose the referee tells the player that he may be playing in the following two games:

	S_1	S_2
a_1	0	+ 2
a_2	- 4	- 10

Game 4

	S_1	S_2	S_3
a_1	0	+ 2	- 2
a_2	- 4	- 10	+ 5

Game 4a

Should there be any difference between being informed about the first pair of games or about the second? Intuitively I believe yes, however not under "total ignorance." Implicit in the formulation of complete ignorance is the assumption that the player can assign a probability of 1 that one of the n states of Nature initially specified will prevail and that nothing else will. If this assumption is not made, then any additional column can be added to the matrix.

A Bayesian approach can be introduced to handle just the feature of the addition of rows; or it can be a two-stage application. First we limit ourselves to a "completely ignorant LaPlacian man." Consider Games 4 and 4a below. If he were in either game then he would regard himself as completely ignorant, however he assigns equal probabilities to being in either.

	S_1
a_1	10
a_2	0

Game 4

	S_1	S_2
a_1	10	0
a_2	0	20

Game 4a

If the player uses maximin as a decision criterion then in Game 4 he uses a_1 and the value $V = 10$; in Game 4a he uses $(\frac{2}{3}, \frac{1}{3})$ with $V = \frac{20}{3}$.

If he is told that he may be involved with 4 or 4a then given his LaPlacian belief on extra states of Nature; his complete ignorance in the game and his maximin behavior he will evaluate

$$V(a_1) = \frac{1}{2}(10) + \frac{1}{2}\frac{2}{3}(10) = 8\ ^2/_3$$

$$V(a_2) = \frac{1}{2}(0) + \frac{1}{2}\frac{1}{3}(20) = 3\ ^1/_3$$

hence he will select a_1.

A two-stage LaPlacian would have a probability of $\frac{1}{n}$ on the initial n states, then would assign a probability of $\frac{1}{2}$ for the existence of the $n+1^{st}$ strategy. Thus he would view two games with even chance of occurrence, one with probabilities of $\frac{1}{n}$ and the other with $\frac{1}{n+1}$. Applying the LaPlace decision to 4 and 4a we have:

$$V(a_1) = \frac{1}{2}(10) + \frac{1}{2}\,\frac{1}{2}(10) = 7\ ^1/2$$

$$V(a_2) = \frac{1}{2}(0) + \frac{1}{2}\,\frac{1}{2}(20) = 5$$

hence he will select a_1 .

$$\begin{array}{c|ccccc:cc}
 & S_1 & S_2 & S_3 & \cdots & S_n & \cdots & S_? \\
\hline
a_1 & U_{11} & U_{12} & U_{13} & \cdots & U_{1n} & \cdots & U_{1?} \\
a_2 & U_{21} & U_{22} & U_{23} & \cdots & U_{2n} & \cdots & U_{2?} \\
a_3 & & & & & & & \\
\vdots & & & & & & & \\
a_m & U_{m1} & U_{m2} & U_{m3} & \cdots & U_{mn} & & U_{m?}
\end{array}$$

Figure 2

In summary, the assumption of a well defined number of states of Nature is a strong psychological assumption by itself. In particular it implies that the individual divides all possible states of Nature into two categories, relevant or not relevant to the decision process. Thus the completely ignorant decision-maker is not as ignorant as he could be, he has a bound on the different states of Nature. Games such as 3 and 3a are amenable to simple experimentation. It is my hope to have such games played.

The problem of bounding the states of nature appears to be closely interlinked to the general relationship between the decision-maker, his environment and his history. A Bayesian Approach at least can explicitly assign a small probability to events which are either out of the span of attention or are "highly unlikely."

3. Actions of the Decision-Maker

$$
\begin{array}{c|ccccc}
 & S_1 & S_2 & S_3 & \cdots & S_n \\
\hline
a_1 & U_{11} & U_{12} & U_{13} & \cdots & U_{1n} \\
a_2 & U_{21} & U_{22} & U_{23} & & U_{2n} \\
\vdots & \vdots & & & & \vdots \\
a_m & U_{1m} & & & & U_{mn} \\
\hdashline
\vdots & \vdots & & & & \vdots \\
a_? & U_{1?} & & & & U_{?n}
\end{array}
$$

Figure 3

The search for alternatives is often an important feature of decision-making. A natural way to take this into account is in a dynamic model in which techniques of sequential decision-making, learning or search programs are introduced, as has been shown in recent developments [2]. In a static or "one shot" situation there are several simple modifications which can reflect the lack of knowledge of alternatives. The player may be informed that beyond some number m, the action (i.e., the row) a_n $(n > m)$ exists only with a probability p_n. If he selects this action and it does not exist he is penalized an amount c (which may be regarded as the cost of failing to act).

It can be seen immediately that this formalization is equivalent to the ordinary game against Nature if the number of actions is finite. The limitation to a finite number of actions is reasonable in terms of the individual's perceptions.

Another modification which is essentially two stage can be obtained by offering the player a matrix with m rows or giving him the choice of paying a sum c for a matrix with $m' > m$ rows of which the first m are the same as the first matrix.

4. "Holes" in the Payoff Matrix

A player may face a situation as portrayed in Figure 4. He may have specific gaps in his value function. These can be caused either by not even knowing the resulting state, or by knowing the state but not being able to

$$
\begin{array}{c|ccccc}
 & S_1 & S_2 & S_3 & \cdots & S_n \\
\hline
a_1 & ? & U_{12} & U_{13} & & U_{1n} \\
a_2 & U_{21} & ? & U_{23} & & U_{2n} \\
a_3 & U_{31} & ? & U_{33} & & \\
\vdots & \vdots & \vdots & \vdots & & \\
a_m & a_{m1} & U_{m2} & U_{m3} & & U_{mn}
\end{array}
$$

Figure 4

evaluate it. A bushman may not know what will happen if he pours water on an electrical fire. A strategist may know what will happen to New York after a 1,000 megaton explosion but he may not have a value calculation for the worth of 10,000,000 casualties.

Are there any "natural" ways in which blanks in a payoff matrix can be filled? It would be possible to perform gaming experiments with different scripts written about the same abstract game and have players with different training fill in the blanks and then play the game. For example use the same numbers but describe several games in terms of economics, war, a diplomatic situation or give no description, then have them played by businessmen, diplomats, army men and so forth. The behavior both in filling in the blanks and playing the games could then be analyzed.

Leaving experimentation aside, we can observe that methods of handling this type of uncertainty are interlinked with types of play.

For example one rule is to replace any blank by the row average. This will not effect the Hurwicz criterion, nor will any weighted averaging procedure as this decision rule depends only upon the minimum and maximum utility values for any row.

Suppose a maximin criterion is employed. How should the game below be played?

$$\begin{array}{c c} & \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & ? \end{bmatrix} \end{array}$$

It is possible to calculate sensitivity ranges for interpolation criteria with respect to the decision criterion. In the game above for any value of $? \leqslant 0$ there will be a saddlepoint $v = 0$. For any value of $? \geqslant 1$ the game has a value of $\frac{1}{2}$ and the optimum strategy for the player is $(\frac{1}{2}, 0, \frac{1}{2})$. For $0 < ? < 1$ the optimum strategy is $(\frac{?}{1+?}, 0 \frac{1}{1+?})$. Similarly for any other decision criterion the sensitivity of the decision can be examined as a function of the estimation of the unknown entries.

Possibly some of the dissatisfaction of those in the behavioral sciences with the theory of Games has been due to the lack of discussion of the problems involving the building of normative or descriptive theories of n person general sum games and games against Nature. If we assume that the payoff matrix is meant to reflect some particular structure arising from an economic, political situation or another substantive background the subject matter itself may lead to fruitful methods for evaluating blanks or gaps in the information on payoffs, as well as suggesting properties for specialized decision criteria.

FOOTNOTES

1/ Luce, D. and H. Raiffa, Games and Decisions (New York: Wiley, 1957) ch. 13

2/ See the work in Decision Theory as exemplified by A. Wald Statistical Decision Functions (New York: Wiley, 1950) as well as writings on problem solving and artificial intelligence reviewed by M. Minsky "Artificial Intelligence" Proceedings of IRE January 1961.