COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station

New Haven, Connecticut

COWLES FOUNDATION DISCUSSION PAPER NO. 110 REVISED

An Analysis of the New View of Investment

Edmund S. Phelps

March 27, 1961

Revised July 10, 1961

AN ANALYSIS OF THE NEW VIEW OF INVESTMENT*

Edmund S. Phelps

July 10, 1961

In 1956 appeared the first in a series of papers [1, 2, 5, 6, 7, 9, 11] disputing the traditional thesis that capital deepening is the major source of productivity gains and conjecturing that we owe our economic growth to our progressive technology.

Thesis and antithesis were synthesized by 1960. Investment is now married to Technology. In the new view, investment is prized as the principal carrier of technological progress [8, 12, 16] .

No effort is made here to criticize what we shall call the "new view" of the role of investment. Nor is the need for accelerated investment, public and private, questioned. This paper is concerned only with the logic of certain conclusions which the new view has shown a tendency to inspire. In what sense does its new role make investment more important? Does the new view of investment present any new reasons, any new incentives -- if added ones were needed -- for faster capital accumulation? The analysis will be confined to investment-thrift policies described by a fixed saving ratio. The results of the inquiry are summarized at the conclusion of the paper.

---

* The author is grateful to Arthur Okun for uncovering several mistakes in an earlier draft.

Early Work

The empirical work cited above spans a great variety of analytical methods and historical materials. One of the best known papers is that by Professor Solow [11] . A number of other investigators followed the same "neoclassical" approach.

This approach postulates aggregate output, Q_t , to be a continously differentiable function of capital, K_t , employment, N_t , and "time" (standing for the state of technology). Solow assumed that technical progress was "neutral" so that output became a separable function of time, as follows.

$$(1) \qquad Q_t = A(t)\, F(K_t, N_t)$$

Such a production function implies that technical progress is organizational in the sense that its effect on productivity does not require any change in the inputs.

It follows that the growth rate of output is equal to the rate of technical progress plus a weighted average of the growth rates of the inputs. These weights are the elasticities of output with respect to capital and to labor. Assuming constant returns to scale, the weights add to one and we obtain

$$(2) \qquad \frac{\dot{Q}_t}{Q_t} = \frac{\dot{A}_t}{A_t} + a_t \frac{\dot{K}_t}{K_t} + (1-a_t) \frac{\dot{N}_t}{N_t}$$

where a_t is the capital elasticity of output, that is $\frac{F_K(K_t, N_t)K_t}{Q_t}$.

There are two unknowns in equation (2), the rate of technical progress and the capital elasticity. Solow, and later Massell [7], relied on an "outside" estimate of the capital elasticity and proceeded to focus on the rate of technical progress. Solow took capital's relative share of national income in year $\underline{t}$ as a measure of a_t and Massell, who assumed a_t was constant over time, used the average share going to capital. It is not known how close such approximations are. The practice presumes pure competition as well as constant returns to scale.

Once the capital elasticity was determined in this manner, investment pessimism was bound to follow. This pessimism found its darkest expression in the following conclusion from U. S. time series: Less than one-third of the growth rate of output per worker in the last quarter century could be credited to the increase in capital per worker which occurred.*

---

* From equation (2) it is easy to derive the proportion of the growth rate of output per worker which is attributable to capital deepening. It is

$$\frac{a_t(k_t - n_t)}{q_t - n_t} = \frac{a_t(k_t - n_t)}{r_t + a_t(k_t - n_t)}$$

where k_t, n_t, q_t and r_t denote the (relative) growth rates of capital, labor, output and technology respectively, at time $\underline{t}$. If there is no capital deepening, meaning $k_t = n_t$, then the proportion is equal to zero. If there is no technical progress, the proportion is equal to one.

The Solow-Massell result is easy to explain. In the U. S. time series they employed, capital and output grew at approximately the same rate. But if k_t equals q_t then the proportion equals a_t. Their factor share data put a_t at less than one-third.

Of course it does not follow from this conclusion that capital deepening is ineffectual. It might mean only that in recent history little capital deepening has taken place. For policy purposes, the effectiveness of additional investment is of greater interest. On this score too, however, the approach outlined above produces some gloomy results.

Consider the effect of doubling the (net) investment-income ratio from .09 to .18 . If the capital-output ratio is about 3 then this increase in the saving ratio would in a year increase the capital stock by about 3 percent (beyond what it would have increased otherwise). Now capital's share in (net) national income is less than one third. Therefore, according to equation (2), the three percent increase in the capital stock would increase (net) output by less than one percent (and it would increase output even less if the capital-output ratio rose).* Professor Solow has remarked of such a

---

* H. Stein and E. Denison's remarkably pessimistic paper [3] for the President's Commission on National Goals is based on calculations of this kind.

---

calculation: "This seems like a meager reward for what is after all a revolution in the speed of accumulation of capital" [12].

The New View

At a time when the reputation of investment seemed at low ebb there appeared the first signs of a new tide. The Economic Commission for Europe [14] argued in 1959 that European population growth had stimulated productivity by necessitating a high rate of gross investment -- thus bringing about a younger and more modern capital stock. On the same grounds, PEP's diagnosis of the British economy [8] discounted Britain's comparatively high investment per worker because her population growth was small.

By 1961 this new view of investment had reached some high places. The President's Economic Message to Congress [15] in January, 1961 states:

> "Expansion and modernization of the Nation's productive plant is essential to accelerate economic growth and to improve the international competitive position of American industry. Embodying modern research and technology in new facilities will advance productivity, reduce costs, and market new products."

Expansion and modernization are put on equal footing and the latter is stressed. A statement by the Council of Economic Advisers before the Joint Economic Committee in March 1961 [16, p.338] amplifies this view:

> "One of the reasons for the recent slowdown in the rate of growth of productivity and output is a corresponding slowdown in the rate at which the stock of capital has been renewed and modernized... As has been confirmed by more recent research, the great importance of capital investment lies in its interaction with improved skills and technological progress. New ideas lie fallow without the modern equipment to give them life. From this point of view the function of capital formation is as much in modernizing the equipment of the industrial worker as in simply adding to it. The relation runs both ways: investment gives effect to technical progress and technical progress stimulates and justifies investment."

The dismal spell cast by the early researchers has been broken.

The fundamental theory on which the new view of investment now rests is due to Professor Solow. In a 1960 piece [12] which is already a classic, he pointed out that a production function like (1) makes old and new capital share alike in technological progress while expressing the belief that "many if not most innovations need to be embodied in new kinds of durable equipment before they can be made effective." He then constructed a model which accords to investment this new role.

Solow says of this model: "[It] redresses the balance somewhat and attributes greater importance to capital investment. The reason is, of course, that capital formation is a vehicle for carrying technical change into effect" [12, p. 97].

We shall now sketch this model and then inquire into the increased importance which the new view attributes to investment.

Unlike the earlier model of production, which permitted much more generality, Solow assumes that the index of technology, $B(t)$, advances exponentially at the constant relative rate $\underline{r}$. Every capital good embodies the latest technology at the moment of its construction but it does not participate in subsequent technical progress. "Capital" thus becomes a continuum of heterogeneous vintages.

It is assumed that the output $Q_v(t)$ produced at time $\underline{t}$ by equipment $K_v(t)$ of vintage $\underline{v}$ is given by the Cobb-Douglas function:

$$Q_v(t) = B_o e^{rv} K_v(t)^a \; N_v(t)^{1-a} \tag{3}$$

Since technical progress is neutral, the elasticity parameter $\underline{a}$ is the same for capital of all vintages.

If we are to estimate capital of vintage v still existing at time t from a gross investment time series, we must make an assumption about depreciation. Solow assumes a constant "force of mortality," δ, to which capital is continuously exposed. Hence,

$$K_v(t) = K_v(v)\ e^{-\delta(t-v)} \equiv I(v)\ e^{-\delta(t-v)} \tag{4}$$

where $I(v)$ is gross investment -- unconsumed output -- at time v. This makes the average life of a capital good about $1/\delta$ years.

The last step is to determine the distribution of the labor force over the vintages of capital. This he does by introducing the condition that the marginal productivity of labor be everywhere equal. Then aggregate output -- the sum of the homogeneous outputs of the various vintages of capital -- is given by

$$Q_t = B_o\ e^{-a\delta t}\ N_t^{1-a}\ J_t^a \tag{5}$$

$$\text{where} \qquad J_t = \int_{-\infty}^{t} e^{(\delta + \frac{r}{a})v}\ I(v)dv$$

Since the new model uses the Cobb-Douglas assumption, let us specialize (1) in the same way to facilitiate comparisons. If all technology is organizational, then

$$Q_t = A_o\ e^{rt}\ K_t^a\ N_t^{1-a} \tag{6}$$

or equivalently

(7) $$Q_t = A_o N_t^{1-a} \left[\int_{-\infty}^{t} e^{\frac{r}{a} t} K_v(t)\, dv \right]^a$$

We can bring (5) into the same form but with a crucial difference:

(8) $$Q_t = B_o N_t^{1-a} \left[\int_{-\infty}^{t} e^{\frac{r}{a} v} K_v(t)\, dv \right]^a$$

The basis for the optimism engendered by (8) can be illustrated by the following example.

Suppose that existing machines are of just two vintages, v_1 (old) and v_2 (new), and that there are an equal number of machines of the two vintages.

According to (7) a two per cent increase in the number of machines of the current vintage, v_2, will bring about a one per cent increase in the value of the bracketed expression in (7); we are weighting a two per cent and a zero increase equally.

Consider the case in equation (8). The bracketed expression is the weighted sum of the machines of the two vintages with the weight for the contemporary machines, namely e^{rv_2}, being greater. Consequently a two per cent increase in the number of machines of current vintage will produce a proportionate increase in the value of the bracketed expression in (8) in excess of one per cent. Hence, current investment increases output per man partly through decreasing the average age of the capital stock.

What if we lengthen our view and ask what happens as the program of capital accumulation continues? Pretty soon we will be confronted by a new situation; large investments today will present us with a large number of old machines tomorrow. To achieve that one percent increase in the value of the bracketed expression tomorrow, a greater absolute increase in the number of new machines will then be required. Of course the weights accorded the old machines in the bracket in (8) are small and are smaller the older the vintage. This contrasts with (7) where all vintages get the same weight. But the essential point is that investment must grow in order to maintain a constant average age of capital. Hence, the future consequences of a permanent change in investment policy are not so clear as the immediate consequences. The remainder of this paper is devoted to a study of the longer run effects of different investment policies as predicted by the "old view" and the "new view" of the relation between investment and technology. First we shall "model" a simple type of investment policy from the old and new point of view.

Constructing Comparable Models

We shall confine our analysis to investment policies which make (gross) investment a fixed proportion of (gross) output.

The choice of an investment policy in this case reduces to selecting the investment-output ratio $\underline{s}$. Hence

$$(9) \qquad I(t) = s\,Q(t)$$

The second critical restriction is the assumption that the labor force grows exponential at the constant relative rate <u>n</u> :

$$(10) \qquad N_t = N_0\, e^{nt}$$

Let us first construct the growth model corresponding to the <u>old</u> view of investment and technology. We make the exponential depreciation assumption which corresponds to (4):

$$(11) \qquad K_t = \int_{-\infty}^{t} e^{-\delta(t-v)}\, I(v)\, dv$$

As a consequence, $\dot{K}_t = I(t) - \delta K_t$.

In order to make the "old" model of investment and growth comparable with the new model formulated by Solow we shall make the same Cobb-Douglas assumption; that is, we shall work with the production function in equation (6) . Differentiating Q_t in (6) with respect to time yields

$$(12) \qquad \frac{dQ}{dt} = rQ + (1-a)\, Ae^{rt}\, N^{-a}\, K^{a}\, \frac{dN}{dt} + aAe^{rt}\, N^{1-a}\, K^{a-1}\, (I(t) - \delta K)$$

where we have used the relation $\dot{K}_t = I(t) - \delta K_t$ by virtue of (11) .

Using (9), (10) and (11) (to express K^{a-1} in terms of Q and N) we obtain the fundamental differential equation corresponding to the "old view" of investment:

$$(13) \qquad \frac{dQ}{dt} = c_1 Q + c_2 Q^{c_3} e^{c_4 t}$$

where

$$c_1 = r + (1 - a)n - a\,\delta$$

$$c_2 = a\, s\, A^{\frac{1}{a}}\, N_o^{\frac{1-a}{a}}$$

$$c_3 = \frac{2a - 1}{a}$$

$$c_4 = \frac{r + (1 - a)n}{a}$$

According to the new view of investment, production takes place according to equation (5). Differentiating that equation with respect to time yields

$$(14) \quad \frac{dQ}{dt} = - a\,\delta\, Q + (1-a)Be^{-a\delta t}\, N^{-a}\, J^{a} \frac{dN}{dt} + aBe^{-a\delta t}\, N^{1-a}\, J^{a-1}\, e^{(\delta+\frac{r}{a})t}\, I(t)$$

where we have used the relation $\dot{J} = e^{(\delta + \frac{r}{a})t} I(t)$.

Using (9), (10) and (5) (to write J^{a-1} in terms of Q and N) we obtain the fundamental differential equation corresponding to the new view of investment:

$$\frac{dQ}{dt} = c_1' Q + c_2' Q^{c_3'} e^{c_4' t} \tag{15}$$

where
$$c_1' = (1 - a)n - a\,\delta$$

$$c_2' = a\, s\, B^{\frac{1}{a}} N_o^{\frac{1-a}{a}}$$

$$c_3' = \frac{2a - 1}{a}$$

$$c_4' = \frac{r + (1 - a)n}{a}$$

These two growth models -- which differ only in respect to the embodiment or non-embodiment of technical change in capital goods -- exhibit differential equations having the same form. There exists an explicit solution for the growth path $Q(t)$ resulting from such a differential equation [4]. Consequently we are ready to compare the behavior of the two models.

Comparing the Long Run

These models have a nice property [10, 13, 4] . Starting from the initial position, the path of growth will be asymptotic to a balanced-growth, "golden-age" equilibrium growth path along which path production, consumption, investment and the capital stock (of all ages) all grow exponentially at the same rate.

The limiting or asymptotic solution to equation (13) or (15) is

$$\bar{Q}(t) = \bar{Q}_0 \, e^{\frac{c_4}{1 - c_3} t} \tag{16}$$

In the limit, growth is exponential at the relative rate $\frac{c_4}{1-c_3}$.

On the balanced-growth equilibrium path, the "initial point" is arbitrary; but the height of the path depends upon the existing labor force at whatever initial point is selected <u>and</u> depends upon the investment ratio. This <u>equilibrium</u> value of Q at time zero, $\bar{Q}_0$, is to be distinguished carefully from the actual value of output, Q_0 , at time zero; the two will be equal only if the initial capital-output ratio happens to equal that ratio which the chosen investment ratio will ultimately bring about.

$\bar{Q}_0$ is given by

(17) $$\bar{Q}_0 = \left[\frac{(1 - c_3)\, c_2}{c_4 - (1 - c_3)\, c_1} \right]^{\frac{1}{1 - c_3}}$$

We are at last in a position to examine the consequences of the new view for this limiting or golden age mode of growth -- especially those consequences relating to the importance of the investment ratio.

Since $c_3 = c_3'$ and $c_4 = c_4'$ it is plain that the limiting growth rates under the two systems are identical. It is a "natural" growth rate in the usual sense that it is independent of the investment ratio.' The growth rate is:

(18) $$g = \frac{r + (1 - a)n}{1 - a}$$

The fact that the limiting growth rate of the old-style Cobb-Douglas model is independent of the investment ratio is well known. It is not surprising that the limiting solution of the new Cobb-Douglas model exhibits the same property. Associated with any exponential mode of growth is a certain unchanging age distribution of capital. Capital which is $(t - v)$ years old will grow at the rate $\underline{g}$ like most everything else; the proportion of capital which is $(t - v)$ years old or less is constant over time. The fact that capitals of different vintages get different technical weights is immaterial in the determination of the exponential equilibrium growth rate.

In what way, then, does the new view attribute more importance to investment? While the limiting growth rate in both models is independent of the investment ratio, the height of the equilibrium growth path will depend upon that ratio. We might well ask therefore if the new view imputes to the equilibrium growth path - in short, $\bar{Q}_0$ - a greater sensitivity to the investment ratio than is implied by the old view. This is a conjecture concerning the elasticity of $\bar{Q}_0$ with respect to $\underline{s}$.

Equations (13) and (17) yield the equilibrium output rate corresponding to an investment ratio $\underline{s}$ at some arbitrary zero point in time according to the old model (6) :

$$(19) \qquad \bar{Q}_0 = s^{\frac{a}{1-a}} \left[\frac{(1-a)\, A_o^{\frac{1}{a}}\, N_o^{\frac{1-a}{a}}}{r + (1-a)\,(n+\delta)} \right]^{\frac{a}{1-a}}$$

Equations (15) and (17) yield the equilibrium output rate associated with an investment ratio $\underline{s}$ at the arbitrary zero point according to the new model based on (5) :

$$(20) \qquad \bar{Q}_0' = s^{\frac{a}{1-a}} \left[\frac{(1-a)\, B_o^{\frac{1}{a}}\, N_o^{\frac{1-a}{a}}}{\frac{r}{a} + (1-a)\,(n+\delta)} \right]^{\frac{a}{1-a}}$$

Equations (19) and (20) disprove the conjecture. The elasticity of $\bar{Q}_0$ with respect to $\underline{s}$ is $\frac{a}{1-a}$ in both equations. Whether one takes the new view or the old, it follows that, in the long run, a one percent increase in the investment ratio will yield asymptotically a rate of output which is $\frac{a}{1-a}$ percent in excess of what asymptotically it would otherwise have been (i.e., had the original investment ratio prevailed).

This result seems at first appearances to be in flat contradiction to the observations of the previous section. The explanation of the puzzle lies in the behavior of the average age -- or more precisely, the age distribution -- of capital. The trivial but easily overlooked fact is that, in exponential growth, the age distribution of capital depends upon the rate of growth and the rate of depreciation and upon nothing else. Since both rates are, in the long run, independent of the investment ratio, a once-for-all change in that ratio can have no persistent long-run influence on the age distribution of capital. Consequently, in the long run, any increase in thrift must rely for its effectiveness upon the prosaic mechanism of capital deepening -- of an equiproportionate deepening of capital of every age.

This is easily proved. Consider a point far in the future of the kind of economy discussed here in which the effects of any aberrant investment policies of the distant past are no longer felt; the economy has been growing smoothly at the rate $\underline{g}$, along the growth path corresponding to the chosen fixed investment ratio, for quite some time.

The time path of gross investment traced out by such an economy is the upper exponential curve shown in Figure 1 . We are looking backward from a point in time at which, with no loss in generality, $t = 0$.

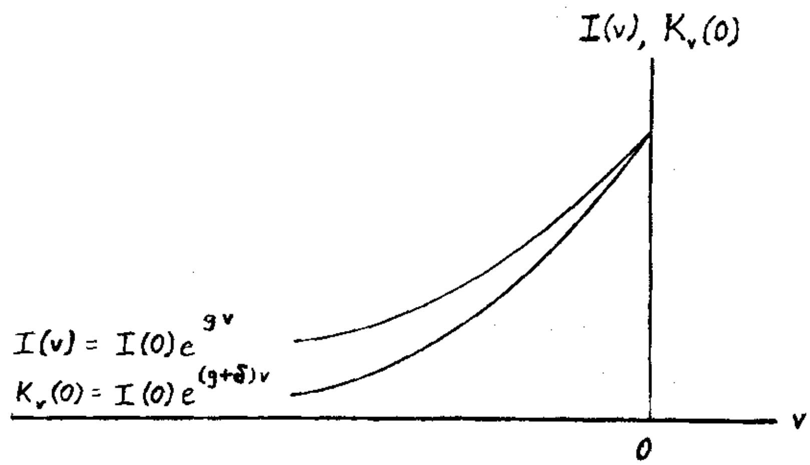

Figure 1

In order to obtain the amount of capital of vintage v still in use at $t = 0$, $K_v(0)$, we have to multiply $I(v)$ by $e^{\delta v}$. This gives the lower curve.

The lower curve is an exponential curve but not the curve of statistical theory with unit area under the curve. To obtain the mean age and the other moments of the age distribution of capital, it is necessary to normalize the curve so that its area will equal one. This requires dividing $K_v(0)$ by $I(0)/(g+\delta)$ for all v .* The normalized age-distribution curve is therefore

* $I(0)/(g+\delta)$ is the total area under the $Kv(0)$ curve, by the familiar "capitalization" formula.

(21) $$f(v) = (g + \delta)\, e^{(g + \delta)v}$$

The mean age of capital, $-\bar{v}$, in this case is

(22) $$-\bar{v} = \int_{-\infty}^{0} (g + \delta)\, e^{(g + \delta)v} (-v)\, dv$$

$$= \left. \frac{-v(g + \delta)e^{(g+\delta)v}}{g + \delta} \right]_{-\infty}^{0} + \int_{-\infty}^{0} \frac{(g + \delta)e^{(g+\delta)v}}{(g + \delta)}\, dv$$

$$= 0 + \left. \frac{e^{(g + \delta)v}}{g + \delta} \right]_{-\infty}^{0}$$

$$= \frac{1}{g + \delta}$$

Other moments of the distribution can be derived in the same manner. But (21) shows simply that the asymptotic age distribution will depend only upon g and δ , whatever the investment ratio.

Has the new view of investment no significance for the selection of the fixed investment ratio? This depends upon the nature of the investment policy, of the decision rule employed.

Suppose that we wished to achieve a particular equilibrium growth path. Which model - new or old - predicts the smaller investment ratio which is required asymptotically to achieve it? Or suppose we are concerned with the absolute increase in output (or consumption), rather than the relative increase, which results from a one per cent increase in the investment ratio. Which model predicts the greater absolute effect of a given increase in the investment ratio? These two questions are the same. They both ask whether the coefficient T' of the investment ratio term in equation (20) is greater than the corresponding coefficient T in equation (19). If the answer is yes, then it would seem reasonable to assert that the new view makes investment policy more important than does the old view. However it could not be said, in this event, that the new view provides additional incentives to increase the long-run investment. (Such incentives would then have to be based upon the transient behavior of the system "before the asymptote is reached.")

Looking at (19) and (20) we can see that $T' > T$ if and only if

$$(23) \qquad \frac{B}{A} > \frac{\frac{r}{a} + (1-a)(n+\delta)}{r + (1-a)(n+\delta)} \equiv \psi$$

Now $\psi > 1$ since $r > 0$ and $0 < a < 1$. Can we make any a priori deductions about $\frac{B}{A}$?

First of all, note that at any point in time, $\underline{t}$, Q_t , N_t and $K_v(t)$ are data. Anyone subscribing to the "old view" must estimate A_o in such a way that his production function, equation (7), fits the facts. Similarly, anyone adhering to the "new view" will estimate B_o such that his production function, equation (8), fits those same facts. If we are to make a meaningful comparison of the two models, we must estimate the parameters of each of them in such a way that both models are admissible theories of the same economy. Equations (7) and (8) must "predict" the same current rate of output, given current resources. It is worth adding that the other parameters, a , r , n and δ , must be assigned the identical values in the two models.

From this consideration and (7) and (8) we find

$$(24) \qquad \frac{B_o}{A_o} = \frac{\int_{-\infty}^{t} e^{\frac{r}{a} t} K_v(t)\, dv}{\int_{-\infty}^{t} e^{\frac{r}{a} v} K_v(t)\, dv} \geq 1$$

It is apparent from (24) that $\frac{B_o}{A_o} > 1$ -- unless all capital is brand new. B_o must exceed A_o by an amount which is necessary to "compensate" for the drag on productivity which the new view attributes to old capital. A new-view man might say that an old-view man's estimate of A_o was really an estimate of the <u>average</u> level of technology, whereas B_o was the latest or best-practise level of technology, which was embodied only in the newest capital goods.

The above deduction is the only a priori statement that can be made about B_o/A_o. It is insufficient to satisfy condition (24). We have to consider the current age distribution of capital to determine the exact ratio of the two estimates of the level (as distinct from the rate of advance) of technology.

Let us consider an age distribution whose only claim to our attention is its simplicity. Suppose that gross investment has been growing steadily at the rate h for a long time. Let the present time constitute the zero point, for convenience; then $K_v(0) = 0$ for all $v > 0$ since future capital vintages have not yet been built. From the growth rate assumption, the depreciation assumption in (4) and (11), and equation (24) one obtains

$$(25) \qquad \frac{B_o}{A_o} = \frac{\int_{-\infty}^{0} e^{\delta v} I(v)dv}{\int_{-\infty}^{0} e^{\frac{r}{a} v} e^{\delta v} I(v)dv}$$

$$= \frac{\int_{-\infty}^{0} I(0)e^{(\delta+h)v}\, dv}{\int_{-\infty}^{0} I(0)e^{(\frac{r}{a}+\delta+h)v}\, dv}$$

$$= \frac{\frac{r}{a} + \delta + h}{\delta + h} .$$

Let us write $h = g + u$ where u may be positive or negative and g is defined in (18). Then, from (25)

$$(26) \qquad \frac{B_o}{A_o} = \frac{\frac{r}{a} + (1 - a)(n + \delta + u)}{r + (1 - a)(n + \delta + u)}$$

If $u = 0$ then, by (26), $\frac{B_o}{A_o} = \psi$ so that $T = T'$. In other words, if the distribution of capital by vintage happened to be described by the exponential curve $I(0)e^{(\delta+g)v}$, then the two models would yield the same coefficient in (19) and (20).*

---

* This result could have been anticipated. Suppose the economy had been traveling along the exponential balanced-growth path corresponding to the prevailing investment ratio. Both models would have to predict the ruling growth path; A_o would have to be estimated such that $\bar{Q}_o = Q_o$ and B_o would have to be estimated such that $\bar{Q}'_o = Q_o$. Hence $T = T'$ in this very special case.

If $u < 0$, then the previous growth rate is smaller than the natural rate. $\frac{B_o}{A_o} > \psi$ by virtue of (26) and the assumptions on $\underline{r}$ and $\underline{a}$. In this case, $T' > T$.

Correspondingly, if $u > 0$, then $T' < T$.

These cases are symmetrical so that it is necessary to consider only the former in any detail.

This result can be explained quite simply. If growth has been slow, then the average age of capital exceeds the mean age which would be produced by a policy of proportional investment. The mean age at $t = 0$ is $\frac{1}{h+\delta}$ while equilibrium balanced growth must produce asymptotically a mean age of capital equal to $\frac{1}{g+\delta}$, as we have seen. Therefore, once we gear the economy to a fixed investment ratio - no matter what that ratio is! - the average age of capital will decline in the limit. According to the old view, this shift in the age distribution is of no significance; but the new view implies that such a change in the age structure will make a separate contribution to growth by raising the average level of technology embodied in the capital structure.

The exponential age distribution may have been worth considering for purposes of insight. But any empirical age distribution would be bound to show many irregularities. From (25) it is possible to derive the general condition on the time path of past gross investment such that $\frac{B_o}{A_o} > \psi$. This is

$$(27) \qquad \int_{-\infty}^{0} e^{\delta v} I(v) \left[1 - \psi e^{\frac{r}{a}v}\right] dv \quad > 0$$

Let $\psi e^{\frac{r}{a} v_\sigma} = 1$ define v_o. Then we can say that gross investment in the interval $-v_o \leq t \leq 0$ receives negative weight in (27); gross investment prior to $-v_o$ receives positive weight. It is still true to say, therefore, that $\frac{B_o}{A_o} > \psi$ is more likely to be satisfied the greater the average age of capital.

If the preceding analysis is correct, there is nothing *a priori* in the new view of investment which makes thrift more important or desirable so far as its ultimate, asymptotic consequences are concerned. However, we need not blind ourselves to the data. Is there evidence that capital in this country has grown so very old that, in terms of the model here, $\frac{B_o}{A_o} > \psi$?

Only in the case of certain simple distributions like the exponential does the average age of capital give us all the information we need about the age distribution in (27). But the mean age is a useful statistic.

The Council of Economic Advisers [16] has recently exhibited estimates of the mean age of equipment in the U. S. in 1959 and 1952-55 . In this interval the mean age of equipment declined from 8.5 years to 9.0 years. However, this is still short of the ripe old age of postwar equipment, 10.6 years in 1945. In 1948, 12.5% of plant and equipment was 5 years old or less; in 1959-60, it was only 9.4% .

This evidence was presented to show that a deceleration of investment will increase the average age of capital. An acceleration of investment can be expected to reduce the mean age, at least for a while. The analysis here suggests that, in the long run, the mean age of capital will depend, for any given investment ratio, only upon the rate of depreciation and the limiting rate of growth. To show that the present mean age of capital is in excess of the equilibrium mean age would require further analysis. It is interesting that if $g = .04$ and $\delta = .06$ then $\frac{1}{g + \delta} = 10$ years. In that case, capital could be expected ultimately to get older, not younger, if a fixed investment ratio were established and if the present model were taken as descriptive of the economy.

Comparing the Adjustment Process

We have been concerned until now with limiting or asymptotic behavior. The economy of these models only approaches (never reaches) its limiting path. Even to get close to that path may take considerable time. It is worthwhile therefore to contrast our two models in respect to the speed with which the economy adjusts to a change in the limiting path (brought about, say, by a change in the saving ratio). This will involve the full solution of each model. We are indebted here to the paper by Dernburg and Quirk [4] .

The complete solution of the fundamental differential equation in (13) is

$$Q(t) = \left[\left(Q_0^{\,1-c_3} - \bar{Q}_0^{\,1-c_3} \right) e^{c_1(1-c_3)t} + \bar{Q}_0 \, e^{c_4 t} \right]^{\frac{1}{1-c_3}} \tag{28}$$

where $\bar{Q}_0$ is given in equation (17). As we are well aware by now, the differential equation (15) corresponding to the "new model," whence its solution, has the same form.

Equation (28) can be rewritten in the form

$$(28a) \qquad Q(t) = \bar{Q}(t) \left\{ 1 + \left[\left(\frac{Q_0}{\bar{Q}_0} \right)^{1-c_3} - 1 \right] e^{[c_1(1-c_3)-c_4]\,t} \right\}^{\frac{1}{1-c_3}}$$

From equations (28) we can find the condition on which output (from any set of initial conditions) will approach, in some sense, the equilibrium (exponential) growth path, $\bar{Q}_t$. Clearly, $\frac{Q_t}{\bar{Q}_t} \rightarrow 1$ as $t \rightarrow \infty$

if and only if $c_1(1-c_3) - c_4 < 0$, in which case the model is said to exhibit <u>absolute stability</u>.* Both our models satisfy this condition.

---

*If only the limiting growth rate (and not also the limiting path) is independent of initial conditions then the model possesses only "relative stability."

We are comparing here the adjustment processes of the two models, not their limiting paths. It will facilitate this comparison if we assume that the two models happen to predict the same limiting path (given the same saving ratio). In that case, two measure of the relative speed of adjustment in the two models suggest themselves. The first is the ratio of output (at time $\underline{t}$) predicted by the new model to output predicted by the old model. Equations (13),

(15) and (28a) yield this ratio in the following form.

$$(29) \qquad \frac{Q(t)'}{Q(t)} = \left\{ \frac{1 + X_0\, e^{-\left[\frac{r}{a} + (1-a)(n+\delta)\right]t}}{1 + X_0\, e^{-\left[r + (1-a)(n+\delta)\right]t}} \right\}^{\frac{a}{1-a}}$$

where

$$X_0 = \left(\frac{Q_0}{\bar{Q}_0} \right)^{\frac{1-\alpha}{\alpha}} - 1$$

Except at $t = 0$ and at $t = \infty$, the time path of this ratio will depend upon the sign of X_0. We shall suppose that X_0 is negative; if X_0 is positive, similar conclusions hold mutatis mutandis. This is precisely the effect of an increase in the saving ratio which is designed to accelerate (for a time) the rate of growth and increase the capital intensity of the economy; it shifts the "equilibrium" path of output above the path of actual output so that $Q_0 - \bar{Q}_0 < 0$ and X_0 is negative.

Figure 2 shows the time path of the ratio. The new model forecasts a faster ascent to the new equilibrium path than does the old model in the sense that, at every future point of time, the new model predicts a higher output than the old. This is because the equilibrium age distribution of the capital stock (like the limiting growth path) is never reached in finite time but only in the limit. These results are easily derived.

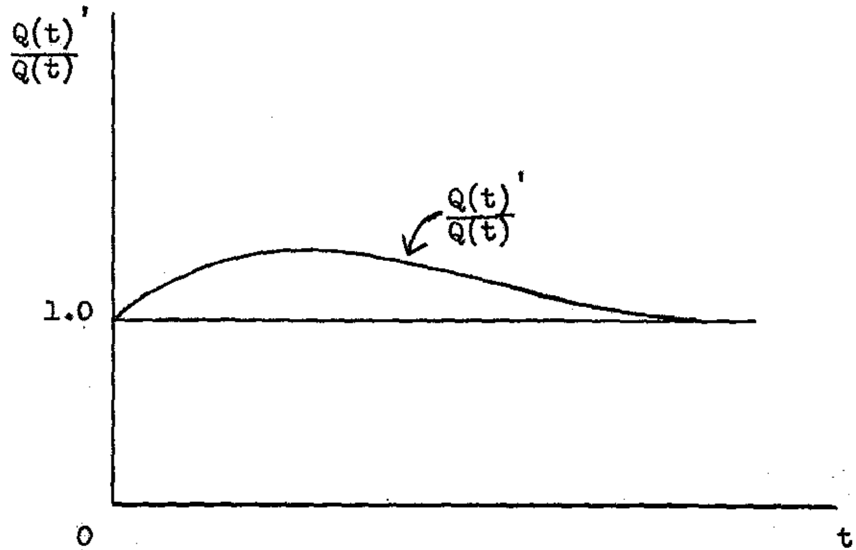

Figure 2

$$\text{Evaluating (29) at } t = 0 \text{, we find } \frac{Q(0)'}{Q(0)} = \frac{Q_0}{Q_0} = 1 \text{ .}$$

Both models must predict the initial (or actual) output. For $t > 0$, $\frac{Q(t)'}{Q(t)} > 1$ since X_0 is negative and $r < \frac{r}{a}$. Next we observe that, as $t \to \infty$, $\frac{Q(t)'}{Q(t)} \to 1$. This follows immediately from the absolute stability of the models and our assumption that they have the same equilibrium paths. The condition for stability of the old model is $r + (1-a)(n+\delta) > 0$. The condition for stability of the new model is $\frac{r}{a} + (1-a)(n+\delta) > 0$. These conditions are taken for granted throughout the paper.*

---

*The results of this paragraph imply that the ratio reaches a maximum at some finite $t > 0$. It can be shown that there is only one such stationary value.

When measured by the ratio of the predicted output rates, the relative optimism of the new model is seen eventually to vanish. For many and perhaps most purposes, the ratio may be the preferable measure. However it should be noted that the alternative measure, the difference between the predicted outputs, can lead to somewhat different and rather surprising results.

Using again equations (13), (15) and (28a), we can express this difference as follows:

$$(30) \quad Q(t)' - Q(t) = \tilde{Q}(t) \left\{ \left[1 + X_0 e^{-[\frac{r}{a} + (1-a)(n+\delta)]t} \right]^{\frac{a}{1-a}} - \left[1 + X_0 e^{-[r+(1-a)(n+\delta)]t} \right]^{\frac{a}{1-a}} \right\}$$

We know that the difference is positive for all $t > 0$ and that the expression inside the braces approaches zero as $\underline{t}$ goes to infinity. However $\tilde{Q}(t)$ will approach infinity at the same time. Therefore the difference cannot be presumed to approach zero without further analysis.

Application of L'hopital's rule yields

$$\text{(31)} \qquad \lim_{t \to \infty} \left(Q(t)' - Q(t) \right) =$$

$$\lim_{t \to \infty} \frac{1}{g} \cdot \frac{a}{(1-a)} \left\{ \left[r + (1-a)(n+\delta) \right] (- X_0) \; \bar{Q}_0 \, e^{[g - r - (1-a)(n+\delta)]t} \right.$$

$$\left. - \left[\frac{r}{a} + (1-a)(n+\delta) \right] (- X_0) \; \bar{Q}_0 \, e^{[g - \frac{r}{a} - (1-a)(n+\delta)]t} \right\}$$

where $\underline{g}$ is the growth rate of $\bar{Q}_t$ and is given in (18).

The algebra of the matter is this: Inside the braces are two terms, both positive, the second being substracted from the first. The exponent in the first term is larger (algebraically) than the exponent in the second. Therefore, if the first exponent is positive, the first term will approach infinity as $t \to \infty$ and it will approach infinity faster than the second term so that the entire expression (the difference) will approach infinity. If, on the other hand, the first exponent is negative then so is the second so that both terms, whence also the difference, will equal zero in the limit. Therefore, the difference approaches zero or infinity according as the first exponent is negative or positive.

This is rather interesting for on the sign of this same exponent depends (in the old model) whether $Q(t) - \bar{Q}(t)$ approaches minus infinity or zero. For proof of this the reader is referred to the paper by

Dernburg and Quirk [4] in which the conditions for this "convergence" (as distinct from stability which entails only convergence of the ratio to unity) is analyzed thoroughly. Also, the sign of the second exponent determines (in the new model) whether $Q(t)' - \bar{Q}(t)$ approaches minus infinity or zero.

The following conclusions can be drawn. If the old model is "convergent" then (since the first exponent is the larger) so is the new model. But if both $Q(t)$ and $Q(t)'$ converge to $\bar{Q}(t)$ then they also converge to each other; that is, $Q(t) - Q(t) \longrightarrow 0$. In short, the predicted outputs of the two models converge if and only if each model is convergent.

How likely is convergence in the old model (which is more prone to divergence than the new model)? Convergence requires only that the first exponent be negative. Using (18) we easily find

$$g - r - (1-a)(n+\delta) < 0 \iff g\left(\frac{a}{1-a}\right) < \delta \tag{32}$$

It is clear that this condition is satisfied by the usual Cobb-Douglas estimates from United States time series.* It is sufficient that

---

* This finding extends the conclusions drawn by Dernburg and Quirk [4] . In omitting depreciation from their model they were led to the belief that, while per capita output probably converges to its equilibrium path, total output could not converge to the equilibrium path. Once depreciation is introduced, however, even total output is likely to be convergent. The faster capital wears out, the smaller the influence of initial conditions upon the ultimate progress of the economy.

$a \leq .5$, $g \leq .05$ and $\delta \geq .05$. Therefore it seems likely that the predicted outputs of the new and old models, when fitted to U. S. data, will converge (to each other) in the limit. In passing, we note that if predicted total outputs fail to diverge then per capita outputs as predicted by the two models will also converge since per capita output grows more slowly than total output.

Conclusions

We have constructed two growth models which differ only in the underlying assumptions about the embodiment of new technology in old capital. An investigation and comparison of the solutions of these models revealed that

(1) the models display the same limiting growth rate

(2) the elasticity of the limiting exponential growth path with respect to the investment ratio is the same in both models

(3) in the limit, the age distribution of capital depends only on the rates of growth and depreciation

(4) the height of the limiting exponential growth path produced by a given investment ratio may be predicted differently by the two models

(5) the difference will depend upon the difference between the initial age distribution of capital and the limiting equilibrium distribution; if capital is initially very old, then the new view will predict a higher equilibrium growth

path corresponding to any fixed investment ratio than will the old view; this is because the change in the age distribution which occurs (in the limit) is favorable to productivity in the new view

(6) the approach to the limiting path is uniformly faster in the new model; consequently the new model forecasts a greater response to an increase in the investment ratio than will the old model

(7) but the comparative optimism of the new model vanishes as the equilibrium path is approached; if the equilibrium paths (corresponding to the previaling investment ratio) are the same in the two models, then the ratio of the predicted outputs will approach one; more generally, the ratio will approach the ratio of the equilibrium paths; this is a consequence of the models' _stability_ which means here that the ratio of predicted to equilibrium output (in each model) approaches one

(8) while both models are stable (barring rapid population contraction) they may or may not exhibit _convergence_ which means that the difference between predicted and equilibrium output approaches zero; the old model is more prone to diverge and both models are divergent if the rate of capital depreciation is sufficiently small; but, unless the depreciation rate is smaller than the growth rate, total output (and per capita output _a fortiori_) can be presumed to be convergent

(9) if the models have the same limiting paths and are convergent, then of course their predicted outputs will converge to each other as well as to their common limiting path; but if the old model is divergent, then the outputs as forecast by the two models will diverge with the new model more optimistic even in the limit, provided it does not predict a lower equilibrium path

REFERENCES

[1] Abramovitz, M. "Resource and Output Trends in the U.S. Since 1870," American Economic Review, 46 (1956).

[2] Aukrust, O., "Investment and Economic Growth," Productivity Measurement Review, 16 (1959), 35-53.

[3] Denison, E.F. and H. Stein, "High Employment and Growth in the American Economy," in Goals for Americans, (Prentice-Hall, 1960), pp. 163-192.

[4] Dernburg, T. and J. Quirk, "Per Capita Output and Technological Progress," Institute for Quantitative Research in Economics and Management, Purdue University, (1960).

[5] Fabricant, S., Basic Facts on Productivity Change, New York, National Bureau of Economic Research, 1959.

[6] Kendrick, J., "Productivity Trends: Capital and Labor," Review of Economics and Statistics, 39 (1956).

[7] Massell, B., "Capital Formation and Technological Change in United States Manufacturing," Review of Economics and Statistics, 42, (1960).

[8] PEP (Political and Economic Planning), Growth in the British Economy, London, Allen and Unwin, 1960.

[9] Schultz, T.W., "Reflections on Agricultural Production, Output and Supply," Journal of Farm Economics, 38 (1956).

[10] Solow, R., "A Contribution to the Theory of Economic Growth," Quarterly Journal of Economics, 70 (1956).

[11] ________, "Technical Change and the Aggregate Production Function," Review of Economics and Statistics, 39 (1957).

[12] ________, "Investment and Technical Progress" in Mathematical Methods in the Social Sciences, 1959. Stanford, Stanford University Press, 1960.

[13] Swan, T., "Economic Growth and Capital Accumulation," Economic Record, 32, (1956).

[14] U. N. Economic Commission for Europe, Economic Survey of Europe in 1958, (Chapter II), Geneva, 1959.

[15] Message From the President of the United States relative to a Program to Restore Momentum to the American Economy, New York Times, February 2, 1961.

[16] U. S. Congress, "The American Economy in 1961: Problems and Policies," Statement of the Council of Economic Advisers, March 1961, in Economic Report of the President; Hearings before the Joint Economic Committee, 1961.