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The Accumulation of Risky Capital: A Discrete-Time Sequential Utility Analysis

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THE ACCUMULATION OF RISKY CAPITAL:

A DISCRETE-TIME SEQUENTIAL UTILITY ANALYSIS

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This paper investigates the nature of the optimal lifetime consumption strategy of a household whose wealth holding opportunities are confined to a single, risky asset. Consumption, nonwealth income and capital growth are treated here as periodic. A continuous-time formulation of the same problem was presented in a previous Cowles Foundation Discussion Paper [12] .

The problem described belongs mainly to the theory of personal saving. Models of saving behavior have thus far been entirely deterministic [4, 7, 8, 11, 13, 14].** In fact the saver is always exposed to the possibility of a

** An exception is an unpublished Cowles Foundation Discussion Paper by Martin Beckmann [2] . Although that paper deals with wage rather than capital uncertainty, the technique employed is the same one used here and I have benefited from reading it.

capital loss. So it seems relevant to ask which results of that theory carry over and which have to be qualified upon admitting capital risk into the theory. The effect of variation in the riskiness and expected return of capital upon

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the optimum consumption rate is also of interest. The model developed below to meet these questions resembles Ransey's [13] more than contemporary models [7, 11] so that it is largely Ramseyan results which are modified and extended.

The plan of the paper is this. In the following sections the individual's utility function and the stochastic capital growth process are specified and discussed. Subsequently, the "structure" of the optimal consumption policy, that is, the way in which consumption depends upon the individual's age and capital, is established. Little else appears deducible without further restrictions upon the utility function.

Thereafter we focus our attention on individuals having certain monomial utility functions. These special cases serve to show possibilities. Among these is the possibility that the classical phenomenon of "hump saving" [8, 13] need not occur, quite apart from reasons of time preference, if capital is risky. Instead a low-capital "trap" region is possible in which it is optimal to decumulate capital, no matter how distant the planning horizon. These utility functions have the intriguing property that they make consumption linear homogeneous in capital and permanent nonwealth income and linear in each of these variables. Finally the directions of the risk and return effects upon consumption are investigated.

1. The behavior of capital.

In this model all wealth is held in the form of a single asset which we may call capital. The asset is homogeneous in that the same probability distribution governs the rate of return on each unit of the asset. Moreover, each unit of the asset experiences ex post the same rate of return.*

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* Alternatively capital might have been envisioned more like identical female rabbits. In any short time period, some units of the asset would multiply while others not. This type of stochastic growth was investigated in [12].

The individual's consumption opportunities occur at discrete, equally spaced points in time. These points divide the lifetime of the consumer into N periods. The state of the system at the beginning of each period, $n = 1, 2, \ldots, N$, is described by the variable x_n, the amount of capital then on hand. At this time the individual chooses to consume some amount c_n of this capital.

The unconsumed capital is left to grow at a rate which is not then known. In addition to the capital growth, the individual receives an amount y, of nonwealth income at the end of the period. This income is the same each period. Consequently the amount of capital available for consumption in the next period is given by the difference equation

$$(1.1) \qquad x_{n+1} = \beta_n (x_n - c_n) + y , \qquad x_1 = k$$

where $\beta_n - 1$ is the rate of return earned on capital in the nth period.

We will assume that the random variables β_n are independent and drawn from the same probability distribution. There are m possible rates of return, $0 \leq \beta_i$, $i = 1, 2, \ldots, m$. The probability of the ith rate of return will be denoted p_i (the same from period to period). In addition we will assume that $\bar{\beta} = \sum_1^m p_i\beta_i > 1$ so that the consumer expects capital to be productive. However, $\sum_1^m p_i(\beta_i - \bar{\beta})^2 > 0$ so the realized return may differ from the expected one.

2. The utility function.

This model postulates a consumer who obeys the axioms of von Neumann-Morgenstern utility theory [1, 5]. His consumption strategy (or policy) can therefore be viewed as that which maximizes the expected value of utility, which is determined up to an increasing linear transformation.

Second, we suppose that the lifetime utility associated with any consumption history is a continuously differentiable function of the amount consumed at the beginning of each period.

The lifetime utility function is assumed to be of the independent and additive form

$$(2.1) \qquad U = \sum_{i=1}^{N} \alpha^{n-1} \; u(c_n) \; , \qquad 0 < \alpha \leq 1$$

The implications of this functional form are several. Preferences for the consumption "chances" or distributions of any period are invariant to the consumption levels befalling the individual in other periods (separability). Preferences among consumption subhistories in the future are independent of the age of the individual (stationarity). Preference for a consumption strategy is independent of or unaffected by any serial correlation in the random consumption sequence associated with that strategy (independence). However the necessary and sufficient conditions for independence of utilities when choice takes place under uncertainty have yet to be uncovered.*

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* The independence of utilities when choice takes place in an environment of certainty has been axiomatized by Debreu [6]. The meaning of additivity with a variable utility discount factor and an infinite number of periods has also been investigated by Koopmans [9].

The same axioms which yield the Neumann-Morgenstern utility indicators also imply that $U(c_1, \ldots, c_N)$ is bounded from above and below.* Consequently

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* A proof of boundedness may be found in [1] and [5]. The proof uses the "continuity axiom" and a generalization of the St. Petersburg game, the idea for which Arrow [1] credits to K. Menger.

$u(c_n)$ is also a bounded function. Let $\overline{u}$ and $\underline{u}$ denote the upper and lower bounds of $u(c_n)$, respectively.

Finally we postulate that the individual strictly prefers more consumption to less (monotonicity) and that he is strictly averse to risk (concavity). The latter means that for every pair of consumption histories $(c_1, \ldots, c_N)$ and $(c_1^o, \ldots, c_N^o)$ to which he is not indifferent, he will strictly prefer the certainty of the compromise history $\Theta c + (1 - \Theta)c^o$ to the mixed prospect offering him the history c with probability Θ and the history c^{o} with probability $1 - \Theta$, $0 < \Theta < 1$. It follows trivially that $u(c_n)$ is a strictly increasing and strictly concave function.

3. Derivation of the functional equations.

We seek the consumption strategy (or equivalently, policy) -- denoted by the sequence of functions $\{c_n(x)\}$ for $x \geq 0, n = 1, 2, \ldots, N$ -- which maximizes

$$(3.1) \qquad J_N(c) = \underset{\beta}{\exp} \left[\sum_{n=1}^{N} \alpha^{n-1} u(x_n, c_n, \beta_n) \right]$$

subject to the relation (1.1). Notice that the optimal c_n, $n = 1, \ldots, N$, will be a stochastic rather than predetermined function of n .

To treat this variational problem we turn to the technique of dynamic programming [3] . Observing that the maximum expected value of lifetime utility depends only upon the number of stages in the process and the initial capital, k , we define the function

$$(3.2) \qquad w_N(k) = \max J_N(c)$$

where the maximum is taken over all admissible policies. The function defined may be interpreted as the utility of wealth function of the optimizing consumer having N periods of life remaining.

Next we reduce the problem with N decision variables to a sequence of N problems, each involving only one policy variable, the decision which must be taken at the current moment. The argument starts with the observation that with the elapse of each period the individual is confronted with another multistage decision problem which differs only in having one less stage and, in general, a different initial capital. By the "principle of optimality" [3], if the individual's consumption strategy is optimal for the original N stage process then that part of the strategy relating to the last $N-1$ stages must also constitute a complete optimal strategy with respect to the new $N-1$ stage process. This principle, equation (1.1), the additive utility function (3.1) and the definition (3.2) combine to yield the following sequence of equations in the unknown utility of wealth functions

$$(3.3) \qquad w_N(x) = \max_{0 \leq c \leq x} \left[u(c) + \alpha \sum_{i=1}^{m} p_i \, w_{N-1}(\beta_i(x-c) + y) \right]$$

for $N \geq 2$, and

$$(3.4) \qquad w_1(x) = \max_{0 \leq c \leq x} u(c)$$

which defines the utility of wealth in the single stage process. Without a subscript, the symbol c shall always denote the value of consumption in the first period of the (not necessarily original) multistage process. Similarly x shall denote capital at the start of whatever process is being considered.

4. Properties of the optimal consumption policy.

We show first that the optimal consumption strategy is unique upon the assumptions made above. In other words, the optimum consumption c_n is a unique function of x_n for every n. To do this we have only to show that $w_N(x)$ is strictly concave for all N and $x \geq 0$.

That each function in the sequence $\{w_N(x)\}$ is strictly concave can be shown inductively as follows. By (3.4) and the postulate that $u(c)$ is strictly increasing and strictly concave we obtain

$$(4.1) \qquad c(x) = x , \qquad x \geq 0$$

and

$$(4.2) \qquad w_1(x) = u(x) , \qquad x \geq 0$$

as the solutions to the single-stage process. Since $u(x)$ is strictly concave, $w_1(x)$ must also be strictly concave.

Suppose now that $w_N(x)$ is strictly concave. Then the function

$G_{N+1}(x, c) = u(c) + \alpha \Sigma p_i w_N(\beta_i(x-c) + y)$ is a strictly concave function of x and c for $x,c \geq 0$.*

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* Strict concavity of $f(x,y)$ in x and y means $f(\lambda x_1 + (1-\lambda)x_2 , \lambda y_1 + (1-\lambda)y_2) > \lambda f(x_1,y_1) + (1-\lambda) f(x_2,y2)$ for $0< \lambda < 1$.

Then $w_{N+1}(x)$ which is given by

$$(4.3) \qquad w_{N+1}(x) = \max_{0 \leq c \leq x} G_{N+1}(x, c) , \quad N \geq 1$$

is also strictly concave in x for $x \leq 0$ as the following lemma [3, pp. 21-22] demonstrates.

For $0 < \lambda < 1$,

$$(4.4) \qquad w(\lambda x_1 + (1 - \lambda) x_2) = \max_{0 \leq c \leq \lambda x_1 + (1-\lambda) x_2} G(\lambda x_1 + (1-\lambda)x_2, c)$$

Replace c by the quantity $c = \lambda c_1 + (1 - \lambda)c_2$ where $0 \leq c_1 \leq x$ and $0 \leq c_2 \leq x_2$. Then

$$(4.5) \qquad w(\lambda x_1 + (1-\lambda)x_2) \qquad \max_{\substack{0 \leq c_1 \leq x_1 \\ 0 \leq c_2 \leq x_2}} G(\lambda x_1 + (1-\lambda)x_2, \lambda c_1 + (1-\lambda)c_2)$$

Since $G(x,c)$ is strictly concave in x and c,

$$(4.6) \qquad G(\lambda x_1 + (1-\lambda) x_2, \lambda c_1 + (1-\lambda)c_2) > \lambda G(x_1, c_1) + (1-\lambda) G(x_2, c_2)$$

whence

$$(4.7) \qquad w(\lambda x_1 + (1-\lambda)\, x_2) > \max_{\substack{0 \le c_1 \le x_1 \\ 0 \le c_2 \le x_2}} [\lambda\, G(x_1, c_1) + (1-\lambda)\, G\,(x_2, c_2)$$

$$> \lambda \max_{0 \le c_1 \le x_1} G(x_1, c_1) + (1-\lambda) \max_{0 \le c_2 \le x_2} G(x_2, c_2)$$

$$> \lambda\, w(x_1) + (1-\lambda)\, w\,(x_2)$$

On the application of the lemma to equation (4.3) it follows that $w_{N+1}(x)$ is strictly concave provided that $w_N(x)$ is strictly concave. Thus the strict concavity of $w_1(x)$ and the result just obtained imply that each function in the sequence $\{w_N(x)\}$ is a strictly concave function.

By the definition of $G_N(x, c)$ it follows that every function in the sequence $\{G_N(x, c)\}$ is also a strictly concave function of x and c. Therefore each "consumption function" in the sequence $\{c_n(x_n)\}$, $n = 1, 2, \ldots, N$, determines c_n as a unique function of x_n.

We wish to show next that consumption is an increasing function of capital and age. The latter result depends upon the further assumption made now that $\alpha \bar{\beta} > 1$. It is clear that this inequality is also a necessary condition for positive accumulation of capital.

It is worth pointing out again that the sequence of consumption functions $c_n(x_n)$, $n = 1, 2, \ldots, N$, which comprise the strategy for the N stage decision process can be viewed as the sequence of initial, first-period consumption functions corresponding to a sequence of decision processes of one stage, two stage, etc., up to N stages in duration.

Equations (7) and (8) contain the solutions for the single-stage process. In the two stage process,

$$(4.8) \qquad w_2(x) = \max_{0 \leq c \leq x} \left[u(c) + \alpha \Sigma\, p_i w_1(\beta_i(x - c) + y) \right].$$

Consider the function $D_2(c) = u'(c) = \alpha \Sigma\, p_i\, p_i\, w_1{}' \ (\beta_i(x-c) + y)$.

If $x = 0$, $D_2(0) = u'(0) - \alpha \overline{\beta}\, u'(y)$. By (4.2) it follows that $D_2(0) \gtreqless 0$

according as $\alpha \overline{\beta} \gtreqless \dfrac{u'(0)}{u'(y)}$. Since $y > 0$, $\dfrac{u'(0)}{u'(y)} > 1$

so that either case is possible. Let us consider first the case in which $\alpha \overline{\beta} > \dfrac{u'(0)}{u'(y)}$.

Then $D_2(0) < 0$ and the maximum in (14) occurs at $c = 0$ for small x. Since $w_1(x)$ is strictly concave and bounded, as x is increased a value of x , denoted $\overline{x}_2$, is eventually reached where $D_2(0) = 0$. This value is given by

$$(4.9) \qquad u'(0) - \alpha \Sigma\, p_i\, \beta_i\, w_1' \,(\beta_i\, x + y) \;=\; 0$$

For $x > \bar{x}_2$, $D_2(0) > 0$ so that, for $x \geq \bar{x}_2$, $c = c_2(x)$ gives the optimal consumption in the first period of a two-stage process,* where $c_2(x)$ is the unique solution of

$$(4.10) \qquad u'(c) - \alpha \Sigma p_i B_i w_1' (B_i(x - c) + y) = 0$$

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* Note then that $c = c_2(x)$ and $c_2 = c_2(x_z)$ have entirely different meanings.

Hence

$$(4.11) \qquad c = \begin{cases} 0, & 0 \leq x \leq \bar{x}_2 \\ c_2(x), & x \geq \bar{x}_2 \end{cases}$$

where $\quad c_2(x) = 0$ at $x = \bar{x}_2$.

Since $w_1'(x) > 0$ by (4.2), $c_2'(x) > 0$. Thus $c_2(x) > 0$ for all $x > \bar{x}_2$. Since $u'(c)$ is monotone and decreasing, $u'(x) < u'(0)$ for $x > 0$ so that $c_2(x) < x$, $x > \bar{x}_2$.

We show now that $w_2'(x) > w_1'(x)$. By (4.8)

$$(4.12) \qquad w_2'(x) = \begin{cases} \alpha \Sigma p_i \beta_i w_1' (\beta_i x + y), & 0 \leq x \leq \bar{x}_2, \\ \alpha \Sigma p_i \beta_i w_1' (\beta_i(x - c) + y), & x \geq x_2 \end{cases}$$

There are two intervals to be considered. If $x < \bar{x}_2$, $D_2(0) < 0$ whence, by (4.9), $u'(x) < \Sigma \; p_i \; \beta_i \; w_1'(p_i x + y)$ <u>a fortiori</u>. Since $w_1(x) = u'(x)$ and $w_2'(x) = \Sigma \; p_i \; B_i \; w_1'(\beta_i x + y)$ in this range of x , $w_1'(x) < w_2'(x)$. If $x \geq \bar{x}_2$ by (4.10) and (4.12), $w_2'(x) = u'(c)$ while $w_1'(x) = u'(x)$ by virtue of (4.2). But $c = c_2(x) < x$ whence $w_1'(x) < w_2'(x)$ also for $x \geq \bar{x}_2$ and therefore for all $x \geq 0$.

The final step in the inductive proof follows. Assume that $w_{N-1}'(x) < w_N'(x)$ and assume that $D_N(0) = u'(0) - \alpha \, \Sigma \; p_i \; \beta_i \; w_{N-1}'(y) < 0$.

For the $N + 1$ stage process

$$(4.13) \quad w_{N+1}(x) = \max_{0 \leq c \leq x} \; [u(c) + \alpha \, \Sigma \, P_i w_N(\beta_i(x - c) + y)]$$

so that $D_{N+1}(c) = u'(c) - \alpha \, \Sigma \; p_i \; \beta_I \; w_N'(\beta_i(x - c) + y)$ is the critical function.

Since $w_{N-1}(x) < w_N'(x)$ and $D_N(0) < 0$ at $x = 0$ it follows that $D_{N+1}(0) < 0$ at $x = 0$. Thus $c = 0$ for small x .

As x is increased a value, $\bar{x}_{N+1}$, is reached where

$$(4.14) \quad u'(0) - \alpha \, \Sigma \; p_i \; \beta_i \; w_N' \; (\beta_i \; x + y) \; = \; 0$$

which determines $\bar{x}_{N+1}$. Similarly $\bar{x}_N$ is determined by

$$(4.15) \qquad u'(0) - \alpha \Sigma p_i \beta_i w'_{N-1} (\beta_i x + y) = 0$$

and is also positive by virtue of the assumption $D_N(0) < 0$ at $x = 0$.

Since $w'_{N-1}(x) < w'_N(x)$, $\bar{x}_N < \bar{x}_{N+1}$.

For $x \geq \bar{x}_{N+1}$, c is given by the unique solution, denoted $c_{N+1}(x)$, of the equation

$$(4.16) \qquad u'(c) - \alpha \Sigma p_i \beta_i w'_N(\beta_i(x - c) + y) = 0$$

Similarly, the optimal c in the first period of the N stage process is the solution, $c_N(x)$, of

$$(4.17) \qquad u'(c) - \alpha \Sigma p_i \beta_i w'_{N-1} (\beta_i(x - c) + y) = 0$$

from all $x \geq \bar{x}_N$. Since $w'_{N-1}(x) < w'_N(x)$, $c_N(x) > c_{N+1}(x)$ for $x \geq \bar{x}_{N+1}$.
Differentiating (4.13) gives

$$(4.18) \qquad w'_{N+1}(x) = \begin{cases} \alpha \Sigma p_i \beta_i w'_N(\beta_i x + y) , & 0 \leq x \leq \bar{x}_{N+1} , \\ \alpha \Sigma p_i \beta_i w'_N(p_i(x - c) + y) , & x \geq \bar{x}_{N+1} \end{cases}$$

while

$$(4.19) \qquad w_N'(x) = \begin{cases} \alpha \Sigma p_i \beta_i w_{N-1}'(\beta_i x + y) , & 0 \leq x \leq \bar{x}_N \\ \alpha \Sigma p_i \beta_i w_{N-1}' (\beta_i(x - c) + y) , & x \geq \bar{x}_N \end{cases}$$

There are three intervals to be considered. In the first interval $0 \leq x \leq \bar{x}_N$, $w_{N+1}(x) = \alpha \Sigma p_i \beta_i w_N' (\beta_i x + y)$ and $w_N'(x) = \alpha \Sigma p_i \beta_i w_{N-1}' (\beta_i x + y)$. Since $w_{N-1}'(x) < w_N'(x)$, $w_N'(x) < w_{N+1}'(x)$ in this interval.

In the second interval, $x_N \leq x \leq x_{N+1}$, $w_{N+1}'(x) = \alpha \Sigma p_i \beta_i w_N'(p_i x + y)$ and $w_N'(x) = u'(c)$ where $c = c_N(x)$. Now $c_N(x) \geq 0$ for $x \geq x_N$ with equality holding only at $x = x_N$. Since $u'(c)$ is monotone decreasing, $w_N'(x) \leq u'(0)$ for $x \geq x_N$ with equality only at $x = x_N$. But $\alpha \Sigma p_i \beta_i w_N'(\beta_i x + y) \geq u'(0)$ with equality holding only at $x = \bar{x}_{N+1}$. Therefore $w_N'(x) < w_{N+1}'(x)$ in this interval.

In the last interval, $x \geq \bar{x}_{N+1}$, we have $w_{N+1}'(x) = u'(c)$ where $c = c_{N+1}(x)$ and $w_N'(x) = u'(c)$ where $c = c_N(x)$. Since $c_N(x) > c_{N+1}(x)$ for all x , $w_N'(x) < w_{N+1}'(x)$, $x \geq \bar{x}_{N+1}$. This concludes the proof.

We have shown inductively that if $\alpha \bar{\beta} > \frac{u'(0)}{u'(y)}$ then initial consumption is the following function of initial capital

$$(4.20) \qquad c = \begin{cases} 0\ , & 0 \leq x \leq \bar{x}_N \\ c_N(x)\ , & x \geq \bar{x}_N \end{cases}$$

where $c_N(x) = 0$ at $x = \bar{x}_N$ and $c_N'(x) > 0$.

Furthermore,

$$(4.21) \qquad \begin{aligned} & w_1'(x) < w_2'(x) < \ldots . < w_N'(x) < \ldots \\ & c_1(x) > c_2(x) > \ldots . > c_N(x) > \ldots \\ & 0 < \bar{x}\ , < \bar{x}_2 < \ldots . < \bar{x}_N < \ldots . \end{aligned}$$

Consider now the other case in which $\alpha\,\bar{\beta} \leq \frac{u'(0)}{u'(y)}$.

Then $D_2\,(0) \geq 0$ and the maximum in (4.8) occurs at $c = x$ for small x . Since $u(c)$ is strictly concave and bounded, as x is increased a value of x is reached for which $D_2(x) = 0$. This value, $\hat{x}_2$, is determined by the equation

$$(4.22) \qquad u'(x) - \alpha\,\bar{\beta}\ w_1'(y) = 0$$

Since $w_1'(x) = u'(x)$ and $1 < \alpha\,\bar{\beta} \leq \frac{u'(0)}{u'(y)}$, $0 \leq \hat{x}_2 < y$.

For $x \geq \hat{x}_2$, $c = c_2(x)$ where $c_2(x)$ is the unique solution of (4.10).

Hence,

$$(4.23) \qquad c = \begin{cases} x , & 0 \leq x \leq \hat{x}_2 , \\ c_2(x) , & x \geq \hat{x}_2 \end{cases}$$

where $c_2(x) = x$ at $x = \hat{x}_2$. Since $c_2'(x) > 0$, $c_2(x) > \hat{x}_2$ for all $x > \hat{x}_2$. Since $u'(c)$ is monotone decreasing, $u'(x) < u'(\hat{x}_2)$, $x > \hat{x}_2$, so that, by (4.10) , $c_2(x) < x$ for $x > \hat{x}_2$.

Further,

$$(4.24) \qquad w_2'(x) = \begin{cases} u'(x) , & 0 \leq x \leq \hat{x}_2 , \\ \alpha \Sigma p_i \beta_i w_1' (\beta_i(x - c) + y) , & x \geq \hat{x}_2 \end{cases}$$

with $w_2'(x)$ continuous at $x = \hat{x}_2$.

In the interval $0 \leq x \leq \hat{x}_2$, $w_1'(x) = w_2'(x) = u'(x)$ by (4.2) and (4.24). For $x \geq \hat{x}_2$, $w_2'(x) = u'(c_2)$ by virtue of (4.10) and (4.24) which $w_1'(x) = u'(x)$. Since $c_2(x) < x$ for $x \geq \hat{x}_2$ and $u'(c)$ is monotone decreasing, $w_1'(x) < w_2'(x)$ in this interval.

Completion of the induction is left to the reader. The remaining part of the proof starts by assuming that $D_N(0) \geq 0$ so that $\hat{x}_N \geq 0$ with

$w'_{N-1}(x) = w'_N(x)$ for $0 \leq x \leq \hat{x}_N$ and $w'_{N-1}(x) < w'_N(x)$, $x > \hat{x}_N$. By now familiar reasoning one can show that (1) $D_{N+1}(0) < D_N(0)$ at $x = 0$ and that if $D_{N+1}(0) \geq 0$, (2a) the maximum occurs at $c = x$ for $0 \leq x \leq \hat{x}_{N+1}$ when $0 \leq \hat{x}_{N+1} < \hat{x}_N$ and at $c = c_{N+1}(x)$ for $x \geq \hat{x}_{N+1}$, and (3a) $w'_N(x) = w'_{N+1}(x)$ in the interval $0 \leq x \leq \hat{x}_{N+1}$ and $w'_N(x) < w'_{N+1}(x)$ for $x > \hat{x}_{N+1}$; if $D_{N+1}(0) < 0$, we are back in the first case in which (2b) the maximum occurs at $c = 0$, $0 \leq x \leq \bar{x}_{N+1}$ when $\bar{x}_{N+1} > 0$ and at $c = c_{N+1}(x)$ for $x \geq \bar{x}_{N+1}$, and (3b) $w'_N(x) < w'_{N+1}(x)$ for all $x \geq 0$. In either case, (4) $c_N(x) > c_{N+1}(x)$ as shown above.

The following results have been obtained

a. $w'_1(x) \leq w'_2(x) \leq \ldots \leq w'_N(x) \leq \ldots$

b. $c_2(x) > c_3(x) > \ldots$

c. $c_N(x) < x$ for all $x \geq y$ and every $N \geq 2$

d. $c'_N(x) > 0$ for every $N \geq 2$

e. For every $N \geq 2$, either a value $\hat{x}_N \geq 0$ is defined by the equation $c_N(x) = x$ in which case

$$\hat{x}_2 > \ldots > \hat{x}_N \geq 0$$

or a value $\bar{x}_N \geq 0$ is defined by the equation $c_N(x) = 0$ in which case

$0 \leq \bar{x}_N < \bar{x}_{N+1} < , \ldots$

f. If $\alpha \bar{\beta} > \frac{u'(0)}{u'(y)}$ then $\bar{x}_1 > 0$. If $\alpha \bar{\beta} < \frac{u'(0)}{u'(y)}$

then $\hat{x}_1 > 0$. Otherwise $\hat{x}_1 = \bar{x}_1 = 0$.

g. The optimal consumption policy makes initial consumption the following function of initial capital:

$$c = \begin{cases} 0 \,, & 0 \leq x \leq \bar{x}_N \text{ if } \bar{x}_N \geq 0 \text{ is defined} \\ c_N(x) \,, & x \geq \bar{x}_N \text{ if } \bar{x}_N \geq 0 \text{ is defined} \\ x \,, & 0 \leq x \leq \hat{x}_N \text{ if } \hat{x}_N \geq 0 \text{ is defined} \\ c_N(x) \,, & x \geq \hat{x}_N \text{ if } \hat{x}_N \geq 0 \text{ is defined} \end{cases}$$

The figure below illustrates one possibility.

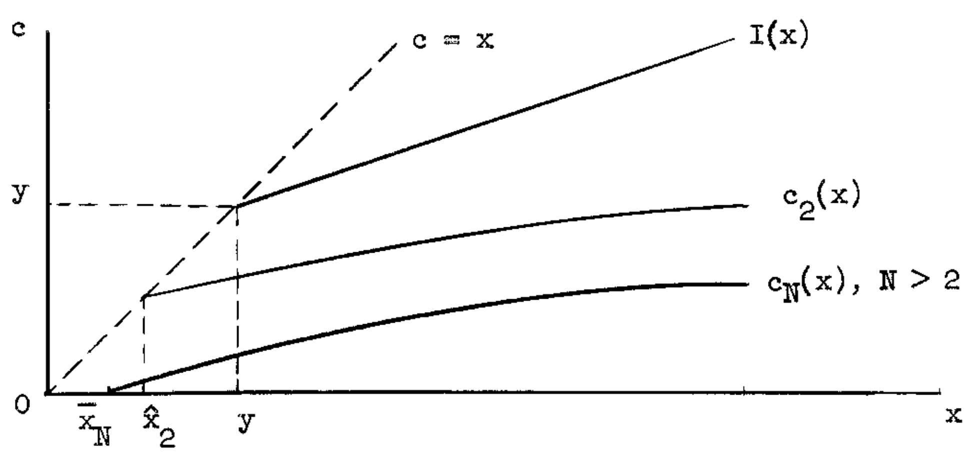

Figure 1

5. Conditions for expected accumulation.

The preceding theorems confirm our expectations about the qualitative behavior of optimal consumption. They do not go far enough to permit inferences about the behavior of capital as a function of age and initial capital. One might ask if the model generates "hump saving" [8] , so important in the theory of aggregate capital formation. This question reduces to: Can one find a value of N sufficiently large that, for $x \geq y$, the individual's consumption will cause the expected value of his subsequent capital to exceed the value of his present capital?

Let us define "expected income," $I(x)$, to be the amount of consumption such that the expected value of capital in the next period equals present capital. Now $\exp x_{n+1} = y + \bar{\beta}(x_n - c_n)$. Expected stationarity, $\exp x_{n+1} = x_n$, implies $c_n = \frac{y}{\bar{\beta}} + \frac{\bar{\beta} - 1}{\bar{\beta}} \equiv I(x)$. Expected income is displayed as a function of capital in Figure 1 . Our question is then whether, in the limit, as N approaches infinity, $c_N(x) < I(x)$ for all $x \geq y$.

The answer is clear cut when capital is riskless. Then $\beta_i = \beta$ for all i and we obtain the following recurrence relation in the limiting utility of wealth function:

$$(5.1) \qquad w(x) = \max_c \{u(c) + \alpha w(\beta(x - c) + y)\}$$

The maximum is an interior one for $x \geq y$ so that $c(x)$ defined by

$$(5.2) \qquad u'(c) - \alpha\,\beta\, w'(\beta(x - c) + y) = 0$$

determines c as a function of x.

Differentiating totally with respect to x gives

$$(5.3) \qquad w'(x) = \alpha\,\beta\, w'(\beta(x-c) + y) + c'(x)\,[u'(c) - \alpha\,\beta\, w'(\beta(x-c) = y)]$$

$$= \alpha\,\beta\, W'(\beta(x-c) + y)\,, \quad \text{by } (5.2).$$

Since $w'(x)$ is monotone decreasing, (5.3) implies that $x_{n+1} > x_n$ if and only $\alpha\,\beta > 1$. Therefore, denoting the limiting consumption function by $c(x)$, $c(x) < I(x)$ for all $x \geq y$.

This simple result fails to extend to risky capital. When $\beta_i \neq \bar{\beta}$ for some i, (5.3) becomes

$$(5.4) \qquad w'(x) = \alpha\, \Sigma\, p_i\, \beta_i\, w'(\beta_i(x - c) + y)$$

From (5.4) no general conclusions concerning the conditions for expected capital growth can be drawn. Of course capital cannot be expected to grow very long unless $\bar{\beta} > 1$. But $\alpha\,\bar{\beta} > 1$ is insufficient to guarantee expected capital growth.

It is clear that the critical value which $\alpha\,\bar{\beta}$ must exceed if capital growth is to be expected will depend upon the distribution of β_i and the shape of the marginal utility function $w'(x)$. The only practical procedure here is to investigate the implications for capital growth of particular classes of utility functions.

6. Consequences of selected monomial utility functions.

In this section we investigate the implications of certain types of monomial utility functions for the consumption function and for the expected path of capital.

We consider first the function

$$(6.1) \qquad u(c_n) = \bar{u} - \lambda\, c_n^{-\gamma} \ , \quad \bar{u} \ , \lambda > 0 \ , \ \gamma > 1$$

The function (6.1) fails to have the boundedness property assumed to this point and thus it contradicts the "continuity axiom" mentioned in section 2 . Whatever the merits of that axiom, the function has received sufficient study in the context of deterministic models [4, 13, 14] to deserve our attention here.

Solving successively for the sequence of unknown functions $\{w_n(x)\}$, $N = 1 \ , \ 2, \ \ldots,$ yields

$$(6.2) \qquad w_N(x) = \bar{u}\,(1 + \alpha + \ldots + \alpha^{N-1}) - \lambda(\alpha\, b^{-\gamma})^{N-1}\left(1+(\alpha\, b^{-\gamma})^{\frac{-1}{\gamma+1}}+\ldots+(\alpha\, b^{-\gamma})^{\frac{-(N-1)}{\gamma+1}}\right)^{\gamma+1} [x + (b^{-1} + b^{-2} + \ldots + b^{-(N-1)})\ y]^{-\gamma}$$

and

(6.3)
$$c_N(x) = \frac{(\alpha\, b^{-\gamma})^{\frac{-(N-1)}{\gamma+1}}}{1+(\alpha\, b^{-\gamma})^{\frac{-1}{\gamma+1}} + \ldots + (\alpha\, b^{-\gamma})^{\frac{-(N-1)}{\gamma+1}}}\,[x + (1 + b + \ldots + {}^{N-2})y]$$

where $$b = (\Sigma\, p_i\, \beta_i^{-\gamma})^{\frac{-1}{\gamma}}$$

If the reader applies (6.3) to $c_{N+1}(x)$ and uses (6.2) he will obtain an expression for $w_{N+1}(x)$ having the same form as (6.2). Note also that if $\alpha = \beta_i \doteq 1$ for all i, formula (6.3) calls for consuming a fraction $\frac{1}{N}$ of the individual's net worth, $x + (N-1)\, y$.

Provided that $\alpha\, b^{-\gamma} < 1$ (for which $\alpha < 1$, $\beta > 1$, $\gamma > 0$ is sufficient in the certainty case), the expressions in (6.2) and (6.3) converge as N approaches infinity giving, in the limit, the solutions to the "infinite stage" process

(6.4)
$$w(x) = \frac{\bar{u}}{1-\alpha} - \lambda \left[\frac{(\alpha\, b^{-\gamma})^{\frac{-1}{\gamma+1}}}{(\alpha\, b^{-\gamma})^{\frac{-1}{\gamma+1}} - 1}\right]^{\gamma+1} \left(x + \frac{y}{b-1} \quad {}^{-\gamma}\right)$$

and

(6.5)
$$c(x) = \left(1 - (\alpha\, b^{-\gamma})^{\frac{1}{\gamma+1}}\right)\left(x + \frac{y}{b-1}\right)$$

This limiting consumption function is useful as an approximation to $c_N(x)$ for large N .

A number of properties of the consumption functions (6.3) and (6.5) can be observed immediately. First, the consumption function is linear homogeneous in capital and nonwealth income. Of two households, both having identical utility functions like (6.1), if one household enjoys twice the capital and nonwealth income of the other, it will also consume twice as much.

Another observation is that the consumption function is linear in capital and nonwealth income. It follows that $c(x) < I(x)$ for all $x \geq y$ if and only if $c(y) < y$ and $c'(x) \leq I'(x)$.

$c(y) < y$ if and only if

$$\left(1 - (\alpha\, b^{-\gamma})^{\frac{1}{\gamma+1}}\right) \quad \frac{b}{b-1} \quad < \quad 1$$

or
$$\left(b - (\alpha\, b)^{\frac{1}{\gamma+1}}\right) < \; b - 1$$

or
$$\alpha\, b > 1 \;.$$

In the limiting case of riskless capital, $b = \beta$. The condition $\alpha\,\beta < 1$ was found earlier to be necessary and sufficient for the growth of riskless capital. But $b < \bar{\beta}$ when capital is risky, that is, when $\beta_i \neq \bar{\beta}$ for some i . This can be seen from a diagram showing $\beta_i^{-\gamma}$ as a function of β_i . Since $\beta^{-\gamma}$ is a convex function of β , $\Sigma\, p_i\, \beta_i^{\;-\gamma} > \bar{\beta}^{-\gamma}$ whence

$b = (\Sigma\, p_i\, \beta_i^{-\gamma})^{\frac{-1}{\gamma}} < \bar{\beta}$. Therefore $\alpha\, \bar{\beta} > 1$ does not guarantee that $\alpha\, b > 1$.*

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* The condition $\alpha\, b^{-\gamma} < 1$ implies only $\alpha\, b > \alpha^{\frac{\gamma}{\gamma+1}}$ which is less than unity.

A sufficient condition is that $\alpha\, \beta_i \geq 1$ for all i .**

** This result may appear to conflict with the results of section 4 that, for every N , $c'_N(x) < 1$ and that if the function $c_N(x)$ intersects the $c = x$ line it is at a value of $c < y$. The difference in results lies in the failure of (6.1) to satisfy the restrictions imposed on the utility function in section 4, in particular the boundedness of $u(c)$.

The condition that $c'(x) < I'(x)$ is

$$1 - (\alpha\, b^{-\gamma})^{\frac{1}{\gamma+1}} < \frac{\bar{\beta} - 1}{\bar{\beta}}$$

It was found above that $b < \bar{\beta}$ so that $\alpha\, b > 1$ is sufficient though unnecessary to satisfy this inequality also. Hence, if $c(y) < y$, then $c'(x) < I'(x)$ but not conversely. Therefore $\alpha\, b > 1$ is necessary and sufficient if the limiting consumption function implied by (6.1) is to produce expected growth of capital at all initial values of capital, that is, if $c(x) < I(x)$ for all $x \geq y$.

Recalling the assumption $\alpha\, b^{-\gamma} < 1$, one easily obtains the following properties of the limiting consumption function in (6.5). First, $0 < c'(x) < 1$.

$c(y) > 0$ if and only if $b > 1$. The Keynesian marginal propensity to consume, defined in this model by $\frac{d\ c(x)}{d\ y}$, where $x = y$ + nonnegative constant, is equal to $c(y)/y$. Hence it is strictly positive if and only if $b > 1$ and between zero and one if and only if $\alpha b > 1$.

If $b > 1$, therefore, the effect of an addition dollar of sure, permanent income on consumption will exceed the effect of an addition dollar of initial capital by a factor $b/(b - 1)$, greater than one.

If capital is riskless, this factor equals the capitalization factor, $\beta/(\beta - 1)$, used to convert a sure, permanent income stream into net worth or present value. This reflects the fact that, provided there is no constraint on borrowing or that such a constraint is not binding, only total net worth or total income matters in determining consumption under conditions of certainty. Consumption in that case is independent of the composition of net worth between current capital and future nonwealth income.

If capital is risky, however, consumption cannot be expressed simply as a function of total "expected income," $I(x)$. Instead, the consumption effect of a dollar increase in expected income arising from an increase in sure, nonwealth income exceeds the corresponding effect of an increase in current capital which is sufficient to raise expected income by a dollar.

The first effect is the marginal propensity to consume, $c'(x)\,\frac{b}{b-1}$.

Writing $x = \frac{\bar{\beta}}{\bar{\beta}-1}\left(I(x) - \frac{y}{\beta}\right)$, we see that the second effect is

$c'(x)\,\frac{\bar{\beta}}{\bar{\beta}-1}$. Recalling that $b < \bar{\beta}$, whence $\frac{\bar{\beta}}{\bar{\beta}-1} < \frac{b}{b-1}$, it is seen

that the former effect is the stronger. This suggests the hypothesis that households whose incomes are comparatively uncertain will be comparatively thrifty.

The last questions of interest relate to the effects upon consumption of variation in the riskiness and expected return from capital. These are thorny questions, for their answers depend frequently upon what other parameters are held constant. The treatment here is deliberately incomplete.

In the riskless case, $b = \beta$ and from (6.5) it follows that an increase in the rate of return, $\beta - 1$, <u>increases</u> consumption at every initial capital, provided $y = 0$. Therefore an increase in the expected return on capital must have the same effect if we mean by that a uniform shift in the probability distribution of β_1 which leaves all its moments the same except the mean, $\bar{\beta}$.

When capital is risky, $b < \beta$ so that, by (6.5) the presence of risk can be said in a structural or global sense to cause a decrease of consumption at every capital, again if $y = 0$.

A second kind of risk effect, one arising from a change in the degree of risk somehow measured, can also be distinguished.

A probability distribution which offers a simple measure of risk is the uniform or rectangular distribution. This is a two-parameter distribution with mean $\bar{\beta}$ and range $2h$. The variance is $\frac{h^2}{3}$ so that h is the measure of risk.

By the definition of b, $b^{-\gamma} = \int_{\bar{\beta}-h}^{\bar{\beta}+h} \beta^{-\gamma} \left(\frac{1}{2h}\right) d\beta$.

An increase in risk will reduce consumption at every capital if $\frac{d\, b^{-\gamma}}{dh} > 0$.

Evaluating the integral we find

$$b^{-\gamma} = \frac{1}{(1-\gamma)2h} \left[(\bar{\beta} + h)^{1-\gamma} - (\bar{\beta} - h)^{1-\gamma} \right] .$$

Differentiating with respect to h yields

$$\frac{d\, b^{-\gamma}}{d\, h} = \frac{1}{2(1-\gamma)h^2} \left[(\bar{\beta} - h)^{-\gamma} (\bar{\beta} - \gamma h) - (\bar{\beta} + h)^{-\gamma} (\bar{\beta} + \gamma h) \right]$$

Assuming $\gamma > 1$, $\frac{d\, b^{-\gamma}}{d\, h} > 0$ if and only if

$$\frac{\bar{\beta} - \gamma h}{\bar{\beta} + \gamma h} < \left(\frac{\bar{\beta} - h}{\bar{\beta} + h} \right)^{\gamma} .$$

Values of β equal to zero are excluded, else b is not defined. Consequently $h < \bar{\beta}$ and the righthand side of the critical inequality must be positive. Hence the inequality will be satisfied if $\gamma \geq \frac{\bar{\beta}}{h}$. Otherwise it may not be satisfied. ($\gamma = 2$ happens to work whatever the value of $\frac{\bar{\beta}}{h}$) . Therefore no general statement can be made on the marginal effect of variations in the degree of risk.

Thus far the effects of risk and return have been examined on the assumption that $y = 0$. But the individual's consumption depends also upon his nonwealth income and the valuation of this stream will depend upon the riskiness and expected return on capital. Thus an increase in the expected return on capital will reduce the subjective present value, $\frac{y}{b-1}$, of future income in terms capital and current consumable goods. This revaluation acts to offset the effect of an increase in return when there is no nonwealth income to be revalued. As a consequence, no general conclusion can be drawn concerning the effect of variations in the expected return. However, if the improvement in capital productivity is accompanied by an equal improvement in labor productivity, so that y increases, then the earlier results for $y = 0$ will apply.*

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* The effect of a change in b upon the marginal propensity to consume out of nonwealth income is unambiguous. $\frac{d}{db}\left\{\frac{b}{b-1}\left[1 - (\alpha\, b^{-\gamma})^{\frac{-1}{\gamma+1}}\right]\right\} \geq 0$ iff $(\alpha\, b^{-\gamma})^{\frac{-1}{\gamma+1}} \leq \frac{1 + b\,\alpha}{1 + \gamma}$ which is satisfied for all $b > 1$, $\gamma > 0$ and $\alpha \leq 1$.

To see that the implications of the utility function (6.1) for the effects of variations in risk and return are not general, one has only to modify the utility function thus.

$$(6.6) \qquad u(c_N) = \lambda\, c^{\gamma} \; , \quad \lambda > 0 \; , \quad 0 < \gamma < 1$$

All the equations (6.2) - (6.5) continue to hold with the difference that λ and γ , then are replaced by $-\lambda$ and $-\gamma$, respectively. Hence the limiting consumption function is

$$(6.7) \qquad c(x) = [1 - (\alpha\, b^{\gamma})^{\frac{1}{1-\gamma}}] \;\; (x + \frac{y}{b-1})$$

where $b^{\gamma} = \Sigma\, p_i\, \beta_i^{\gamma}$

An increase in $\bar{\beta}$, other moments of the distribution unchanged, will increase b , hence decrease $c(x)$ for all x even if $y > 0$. The re-capitalization of nonwealth income cooperates with the substititon effect.

Once again the structural effect of risk is easy to ascertain. Since β^{γ} is a concave function of β , $\Sigma\, p_i\, \beta_i^{\gamma} < \bar{\beta}^{\gamma}$ whence $b = (\Sigma p_i \beta_i^{\gamma})^{\frac{1}{\gamma}} < \bar{\beta}$.

Therefore, the presence of risk makes consumption greater than it would be in the limiting riskless case.* These conclusions are opposite those obtained

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* The effect of a change in b upon the marginal propensity to consume out of nonwealth income is also unambiguous in the case of (6.6).

$$\frac{d}{db}\left\{\frac{b}{b-1}\left[1-(\alpha\, b^{\gamma})^{\frac{1}{1-\gamma}}\right]\right\} \leq 0 \quad \text{iff}$$

$(\alpha\, b^{\gamma})^{\frac{-1}{1-\gamma}} \geq \frac{1-b\gamma}{1-\gamma}$ which is satisfied by all values of $b \geq 1$ and $0 < \gamma < 1$.

---

for the utility function in (6.1) with the qualification that the latter are complicated by the capitalization effect.

The effect of a marginal increase in risk depends again upon its effect upon b . Turning again to the uniform distribution we find that the "natural" result $\frac{d\, b^{\gamma}}{d\, h} < 0$ (meaning that global and marginal risk effects have like signs) depends upon the problematical condition $\frac{\bar{\beta}-\gamma h}{\bar{\beta}+\gamma h} > \left(\frac{\bar{\beta}-h}{\bar{\beta}+h}\right)^{\gamma}$.

We examine finally a utility function which frequently holds surprises, the logarithmic function in (6.8) . It should be pointed out

$$(6.8) \qquad u(c_N) = \ell_N\, c_N$$

that this function, like the others studied here, lacks the boundedness property invoked in section 4.

It appears to be impossible to solve for $c_N(x)$ explicitly in terms of x and y except in the case $y = 0$. Then we easily find

$$(6.9) \qquad w_N(x) = (1 + \alpha + \ldots + \alpha^{N-1})\, \ell_N\, x + v\,(\theta, \alpha, N)$$

where $v(\theta, \alpha, N)$ depends only upon the parameters, denoted by θ, of the probability distribution of β_i, α and N and not upon x.

Also

$$(6.10) \qquad c_N(x) = \frac{x}{1 + \alpha + \ldots + \alpha^{N-1}}$$

When the utility function is logarithmic, the optimum consumption rate is independent both of the expected return and riskiness of capital. Consumption is linear homogeneous in capital. As N is increased, the consumption function flattens asymptotically until, in the limit,

$$(6.11) \qquad c(x) = (1 - \alpha)\, x$$

A limiting function exists only if $\alpha < 1$.*

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* For certain utility functions the existence of a limiting solution does not require $\alpha < 1$. Ramsey [13] argued that boundedness was sufficient but a condition on the elasticity or rate of approach to the upper bound is also necessary, at least in models not containing risk. Samuelson and Solow [15] assume that the upper utility bound is attained at a finite consumption rate which appears to be overly strong.

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