COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut

COWLES FOUNDATION DISCUSSION PAPER NO. 107

Extended Edgeworth Bargaining Games and Competitive Equilibrium

Martin Shubik

January 12, 1961

Extended Edgeworth Bargaining Games and Competitive Equilibrium*

Martin Shubik

1. INTRODUCTION

There has been little, if any, utilization of the von Neumann and Morgenstern concept of solution to an n-person non-constant sum game to economic problems. In this paper an attempt will be made to link up this solution concept with welfare problems and further to give an interpretation relating it to the properties of equilibrium and stability.

The description of an n-person game is given in terms of its characteristic function. As the models dealt with below are primarily based upon extensions of Edgeworth's two-person bargaining model this is used to provide the first example of a characteristic function.

Suppose that two bargainers have initial fortunes consisting respectively of (a,0) and (0,b), where the first number of the pair is the amount of the first two commodities being traded which is possessed by the bargainer. The second number is the amount of the second commodity possessed by the bargainer. Suppose that both bargainers possess a preference system which can be represented by

---

*Research undertaken by the Cowles Commission for Research in Economics under Task NR-047-006 with the Office of Naval Research.

a family of convex curves for each of them. Let them be $\psi_1(x,y)$ and $\psi_2(x,y)$. Suppose that the bargainers were able to compare utilities. The characteristic function below describes this game:

$$v(\Theta) = 0$$

$$v(\{1\}) = \psi_1(a,0)$$

$$v(\{2\}) = \psi_2(o,b)$$

$$v(\{1,2\}) = \max_{x_1,y_1} [\psi_1(a-x_1,y_1) + \psi_2(x_1, b - y_1)]$$

The $v(\Theta)$ merely states that the value of a coalition with no members is zero. The next two values given are the amounts that each player can obtain for himself by individual action. Each bargainer can guarantee for himself a value no lower than that obtained by refusing to trade. The last value given is that for the largest amount that the bargainers can obtain together. On the assumption that utilities are comparable this will in general be a uniquely defined amount. Edgeworth calls this the"utilitarian point". If trading is to be worth while, the sum of the utilities to be obtained by trade should be larger than the sum of the utilities obtained without trade. Thus in the bargaining situation:

$$v(\{1,2\}) > v(\{1\}) + v(\{2\}) .$$

The characteristic function is a super-additive set function. It is defined, i.e., it takes on a value for every possible set of players in a game. In other words, it gives a value for every coalition. For example, if the game has four players the characteristic function will have 2^4 or 16 values; one for the no-member coalition, four one man coalitions, six coalitions of two, four coalitions of three and one coalition of four.

In a game that is worth playing there must be at least two sets of players who can improve their gain by cooperating. Thus in an <u>essential game</u> there will be at least two coalitions without members in common, say M and N such that $v(M \cup N) > v(M) + v(N)$. It is furthermore assumed that no coalition will ever obtain less than can be obtained by smaller coalitions consisting of the same individuals. Hence always for any two sets of players M and N with no member in common, $v(M \cup N) \geq v(M) + v(N)$. The characteristic function is super-additive as the amount to be obtained by a coalition is always at least as much as the sum of the amounts obtained by the same players acting in smaller coalitions.

The characteristic function provides a useful description for complementarity and substitutability relations among indivisible objects, hence the sets can equally well be considered to be groupings of specialized machines as well as coalitions of bargainers.

The amounts that can be obtained by coalitions are given in the values of the characteristic function. In general we will want to know the amount that each individual player receives after the bargaining has taken place and side-payments have been agreed upon. Let x_i be the amount obtained by the i^{th} player. We denote by x the n-tuple of numbers $(x_1, x_2, \ldots, x_n)$ which indicates the amount obtained by each player according to some scheme of division. The first condition we impose upon these numbers is that for all i

$$(1) \qquad v(\{i\}) \leq x_i$$

This states that under any scheme of division no player will obtain less than he can obtain by acting unilaterally.

A second condition to be imposed is that of Pareto Optimality (or even more strongly joint maximality). Let us denote by I the set consisting of all of the players. In this instance they number n. $v(I)$ stands for the amount that the players can obtain from the game by acting as one large coalition. This is the most that they can obtain under any circumstances, thus it is not possible for the sum of the payments to all players to exceed this amount. If the players behave optimally, it should equal this amount.

(2) $$\sum_{i \text{ in } I} x_i = v(I) .$$

An n-tuple which meets the above conditions is referred to as an <u>imputation</u> from a game with the characteristic function v . The von Neumann and Morgenstern solution concept is addressed to the selection of certain sets of imputations as solutions to an n-person game.

The selection of the sets of imputations in a solution depends upon the relationship of <u>domination</u>. An imputation $x = (x_1, x_2, \ldots, x_n)$ dominates an imputation $y = (y_1, y_2, \ldots, y_n)$ with respect to a coalition of players S if (on the assumption that S contains some players)

(3) $$v(S) \geq \sum_{i \text{ in } S} x_i$$

and

(4) $$x_i > y_i \quad \text{for every } i \text{ belonging to } S .$$

In words, for x to dominate y, the set of players S must be in a position to obtain at least the amount they are suggesting for themselves in x. Furthermore, everyone in the coalition S strictly prefers x to y . The set of players S is called an <u>effective set</u>.

It is possible for an imputation x to dominate an imputation y and vice versa (with respect to different effective sets). Furthermore, the relation is not transitive. x may dominate y, which dominates z, which in turn dominates x. If we consider voting agreements, political power and horse-trading arrangements over appropriations, these possibilities are not strange. The stability of an imputation depends upon the presence of a group of individuals with something better to offer rather than upon an atomistic reaction to some price parameters.

A <u>solution</u> to a game in the sense of von Neumann and Morgenstern, consists of a set of imputations B such that:

If x and y are imputations contained in B, then neither dominates the other.

If there exists an imputation z which is not contained in B then there is at least one imputation belonging to B which dominates z.

The meaning of this solution concept can be demonstrated with reference to the Edgeworth bilateral monopoly model. Before this is done, we can view the important distinction between the Edgeworth treatment and the von Neumann and Morgenstern treatment in terms of games with and without side payments. Figure 1 illustrates a different representation of the familiar <u>contract curve</u> shown in Figure 2.

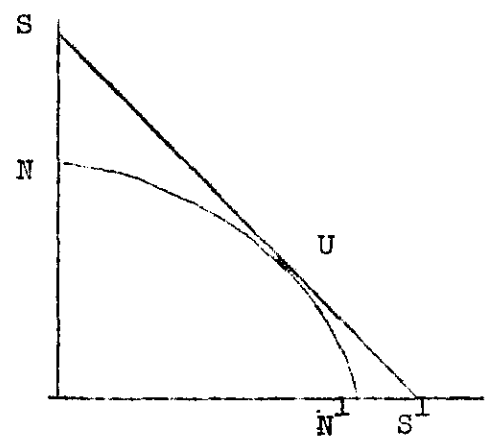

Figure 1

If the utilities of the players were comparable it would then be meaningful to find a point where they were jointly maximized. This is denoted by U in this diagram. Furthermore, if there were a supply of "utiles" or a "money" commodity for which the players had a linear utility, then they could jointly maximize regardless of the asymmetry involved in doing so, and then settle up by means of side-payments along the 45° line denoted by SS' . This is what happens in the von Neumann and Morgenstern theory. Thus their model of the Edgeworth two-person, two-commodity trade is in fact a two-person three-commodity trade model with the third good being utiles.

If side-payments are forbidden, then the outcome must be restricted to the curve NN' or the contract curve (which is part of the Pareto optimal surface). This is the "best" that the players can do if they are unable to settle up outside of the game. An immediate analogy can be made in terms of cartel behavior. The curve NN' represents the best that a pair of firms can obtain if they are not in a position to make cash side-payments after having adjusted their outputs to equalize marginal costs in both plants (in this instance the assumption that there is a third good in the form of money which can be treated as transferable utility appears to be fairly reasonable).

We can modify the von Neumann and Morgenstern assumptions to rule out side-payments in which case the best that a coalition of the two players can do is to obtain some point on the contract curve. When there are only two players it is obvious that by forbidding side-payments and modifying the von Neumann and Morgenstern solutions accordingly, then it is also not necessary to assume the comparability of utilities.

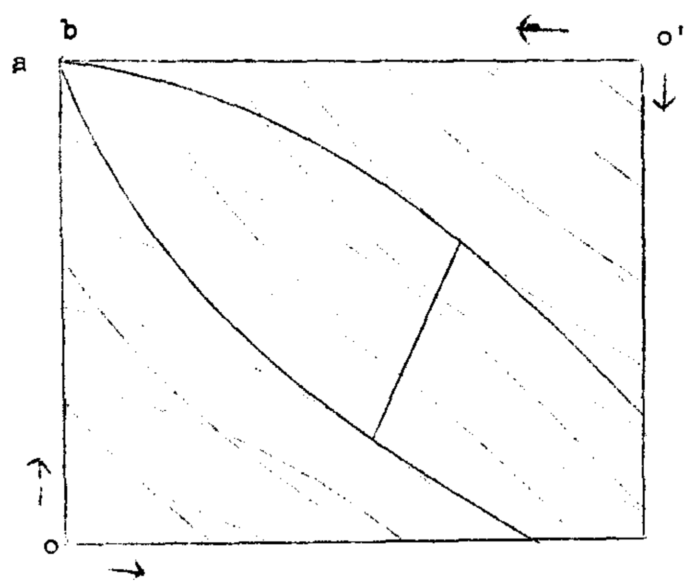

Figure 2

The solution to the bargain in the sense of Edgeworth (or von Neumann and Morgenstern without side-payments) is the set of imputations on the contract curve such that for any imputation (x_1, x_2), $x_1 \geq \psi_1(a,0)$ and $x_2 \geq \psi_2(0,b)$. This is proved by showing that any imputation on this curve does not dominate any other imputation on the curve. Consider two imputations (x_1, x_2) and (y_1, y_2). If $x_1 > y_1$ then $x_2 < y_2$, hence the two person coalition is not an effective set. Furthermore, for any of these imputations excluding the endpoints of the contract curve, condition (3) is violated. For the endpoints if condition (3) is not violated condition (4) will be. This implies that no single player forms an effective set for any of the imputations on the contract curve.

Using a completely similar argument, but assuming side-payments with comparable utilities it can be shown that the von Neumann and Morgenstern solution is the "extended contract curve" SS' of Figure 1, as has been previously noted.

2. SOME GENERAL PROBLEMS

In dealing with models of bargaining and welfare there are four problems, which though related, are distinct. They involve the implications of the assumptions concerning:

(1) The measurability of the utility of an individual

(2) The comparability of utility

(3) The transferability of utility (i.e., side-payments)

(4) The importance of the role of coalitions of players.

The fourth assumption is utterly independent of the others. Although it fast becomes mathematically complex, it has been shown that even on the assumption of ordinal non-comparable utility scales it is possible to define a solution concept broadly analagous to that of von Neumann and Morgenstern taking into account the role of coalitions [1]. Thus even rejecting the first three assumptions, there is still a need and a possibility for a theory of coalitions.

The price-parameter or competitive equilibrium theory is a theory which assumes no measurability, comparability or transferability of utility and furthermore assumes no role for coalitions. The fact that at equilibrium the market is cleared implies no side-payments.

It is remarkable that under relatively general conditions [2] a competitive equilibrium point exists on the Pareto optimal surface. Beyond the observation that it is on this surface, there is no particular ethical content which distinguishes this point from any other on the surface. In fact it may not even be unique. There is, however, a strong <u>administrative argument</u> for the competitive equilibrium. It is simple to administer in the sense that it is utterly decentralized, there is no bargaining or discussion between any individuals. However instead of being an institution-free concept of economic solution, it should be more correctly viewed as one which calls for a very special institutional set-up in which man as an economic animal takes great care to purify himself from contact, collusion or collaboration with his fellows.

The recent concern for the stability of competitive equilibria [3] illustrates the degree of specialty of the assumptions even further. Differential or difference equations must be introduced which depend in some manner upon the excess supply and demand functions [4]. The selection of the specific forms is more or less "dealer's choice". In his series of counter-examples Scarf has shown [5] that even under some of the "most natural" choices the equilibria may be unstable. Intuitively this appears to be a disturbingly undesired result. Yet even the most casual empiricism seems to indicate that whatever stability there is in an economy tends to owe its strength to enforcement by groups or coalitions. For example, the instability of an agricultural market appears to be aggravated by the lack of coalitions.

Any concept of stability which, in my opinion, appears to be natural to an economic system should be related to the coalition structures potentially able to disturb it. The stability of an imputation should be inversely related to the number of <u>effective</u> <u>sets</u> of players that exist for that imputation. This imposes many more conditions than are imposed by the conditions under which the stability of a price-parameter equilibrium has been examined.

The von Neumann and Morgenstern solution (as it stands, or modified to rule out side-payments) to an n-person game is a socio-economic concept. The economic bounds are reflected in the utility functions and assets of the participants. However the stability of the imputation of wealth is examined in terms of the sociological conditions reflected by the power of groups.

The remainder of this paper is devoted to illustrating the properties of the von Neumann and Morgenstern solution and associating them with the concept of equilibrium by examining an extension of the Edgeworth bargaining model. The proofs and initial discussion are given in terms of games with comparable transferable utilities; however it will be seen that at least for the relatively simple games under discussion the results generalize immediately to the case where no side-payments are permitted.

3. EDGEWORTH MARKET GAMES

Edgeworth examined bargains with more than two players and concluded that if the players became numerous on both sides of the market (under certain conditions) the bargaining would be such that the range of indeterminacy characterized by the contract curve would narrow until in the limit a single point would be reached which would correspond to the market price under pure competition.

An Edgeworth market game consists of a set $I = M \cup N$ of players where the set of players M can be regarded as traders with initial assets of commodity 1 and the set of players N are traders with initial assets of commodity 2. Each player has a utility function consisting of a set of strictly convex indifference curves (i.e., an indifference map). The utility function of the i^{th} player is denoted by $\psi_i(x, y)$ where x and y are the quantities of the first and second commodity. It is assumed that:

$$\lim_{x\to\infty} \psi_i(x, y) < \infty \quad \text{and} \quad \lim_{y\to\infty} \psi_i(x, y) < \infty$$

and that the ψ_i are twice differentiable. The i^{th} player in the set M starts with initial resources $(a_i, 0)$ of goods 1 and 2; where the first number is the amount of the first good and the second, the amount of the second. The j^{th} player in set N has initially $(0, b_j)$. The characteristic function for the game is:

$$v(\Theta) = 0$$

$$v(S) = \max_{x_i, y_i} \left[\sum_{i\in S\cap M} \psi_i(a_i - x_i, y_i) + \sum_{j\in S\cap N} \psi_j(x_j, b_j - y_j) \right],$$

where $\Sigma x_i = \Sigma x_j$, $\Sigma y_i = \Sigma y_j$ and $x_j, y_i, \geq 0$.

In order to examine this game fully we would have to consider many special cases depending upon the shapes of the utility functions, the initial stocks of the traders and the number of traders. We limit ourselves to the simple case where all traders have identical utility functions $\psi_i(x, y) = \psi(x, y)$. Each trader belonging to M initially has $(a, 0)$ and each trader belonging to N has $(0, b)$.

Because of the above restrictions we can immediately write down the explicit characteristic function for any game of this type. If there is no joint gain to be obtained from trade, the game is inessential and the value of any coalition is the sum of the individual initial utilities. If the game is essential and trade takes place, then by the symmetry and convexity, at a position of joint maximization for any group of players all players will obtain equal quantities of the same good (at this point the marginal rate of substitution between the two commodities being traded will be the same for all players trading). The characteristic function is:

$$v(\Theta) = 0$$

$$v(S) = s\ \psi\left(\frac{as_m}{s}, \frac{bs_n}{s}\right)$$

where $s = |S|$, $s_m = |S \cap M|$ and $s_n = |S \cap N|$. The characteristic function for the two-person game becomes:

$$v(\Theta) = 0$$

$$v(\{1\}) = \psi(a, o)$$

$$v(\{2\}) = \psi(o, b)$$

$$v(\{1, 2\}) = 2\psi\left(\frac{a}{2}, \frac{b}{2}\right) .$$

The solution to this game consists of the set of imputations of the form:

$$\left(2p\ \psi\left(\frac{a}{2}, \frac{b}{2}\right) - p\ \psi(o, b) + (1 - p)\ \psi(a, o),\right.$$

$$\left. 2(1 - p)\ \psi\left(\frac{a}{2}, \frac{b}{2}\right) + p\ \psi(o, b) - (1 - p)\ \psi(a, o)\right) ,$$

$$0 \leq p \leq 1 .$$

It can be shown that if the number of traders in one commodity is the same as the number of traders in the other, then for any size of the market there is a solution analogous to the solution to the two-person game (although such a solution will no longer be the only solution).

There exists a set of imputations which form a solution such that all traders on the same side of the market always obtain an equal amount. The set of imputations is characterized by a parameter p (as in the two-person game) which can be interpreted as a market price. The range of this price remains constant for all games with the same number of players on each side of the market.

THEOREM 1 . An Edgeworth market game denoted by $[n, n]$ for any size n where the number of traders in one commodity is the same as the number of traders in the other commodity and the traders have the same preferences will have a solution consisting of all imputations of the form:

$$\left(2p\,\psi\left(\frac{a}{2}, \frac{b}{2}\right) - p\psi(o, b) + (1-p)\,\psi(a, o), \ \ldots, \ 2(1-p)\,\psi\left(\frac{a}{2}, \frac{b}{2}\right)\right.$$

$$\left. + p\psi(o, b) - (1-p)\,\psi(a, o), \ \ldots \right), \tag{1}$$

$$0 \leq p \leq 1 .$$

PROOF. No imputation of the form of (1) can dominate another imputation of that form. A set to be effective for any imputation requires members from both M and N , i.e., trade must take place; but no imputation of the form (1) dominates another of form (1) for a set containing members from both M and N .

Any imputation not of the form (1) has a pair $\gamma'_i + \gamma''_j < 2\psi(\frac{a}{2}, \frac{b}{2})$ where γ'_i is the amount a player $i \in M$ and γ''_j is the amount a player $j \in N$ obtains from this imputation. If we select a p such that

$$\frac{\gamma_i' - \psi(a,o)}{2\psi\left(\frac{a}{2},\frac{b}{2}\right) - \psi(a,o) - \psi(o,b)} < p < 1 - \frac{\gamma_j'' - \psi(o,b)}{2\psi\left(\frac{a}{2},\frac{b}{2}\right) - \psi(a,o) - \psi(o,b)}$$

then the imputation of form (1) for this value of p dominates the imputation γ. This completes the proof.

We have shown that a line in $2n$ dimensions which is the direct analogue of the contract curve is a solution to the $2n$ person game. Furthermore the range of this solution is the same as in the range in the two person game. This however is <u>not</u> the only solution to the $2n$ person game, although the contract curve is the only solution to the two person game. It appears that there will be many sets of imputations which satisfy the stability conditions. Thus as is indicated below there can be solutions in which subsets of players may fare relatively poorly as compared to others with the same initial economic strength.

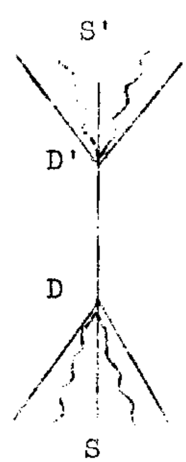

Figure 3

The vertical line in Figure 3 must be imagined as the contract curve in 2n dimensions. The lines departing from it represent sections of other solutions which include only part of the contract curve as members of the stable set. These may have very general shapes. No attempt is made here to calculate the totality of solutions which exist. For our purposes their intersection is of prime concern. In Figure 3 all the solutions have a common segment DD' . This segment is part of the contract curve, but for $n > 1$ in any of the 2n person games the segment DD' will not be the full length of the contract curve SS' . Under the appropriate conditions as the number of bargainers becomes large, we will show that the (multidimensional) segment DD' converges to a single point.

The intersection of all of the solutions to a game is known as its core. We give it an economic interpretation upon observing that it consists of a set of imputations which are always undominated under every circumstance. There is no effective set of players for any of these imputations. In economic terms, these imputations are stable against the economic power of any coalition. In the two person bilateral monopoly there is only one solution, hence the whole contract curve is the core of the game. The core appears to be determined by the technological conditions. Any imputation within the core gives to all players at least as much as they are able to enforce in any coalition. In any economy any imputation of wealth outside of the core implies that some group of individuals is profiting at the expense of another group.

The core to a game need not always exist. In particular this will hold true in an economy with decreasing returns. This implies that in such an economy no division of product can be made which is stable against all coalitions.

4. CORES AND COMPETITIVE EQUILIBRIA

The proofs of the following four theorems have appeared elsewhere [6] hence they will not be reproduced here. The discussion is limited to the economic interpretation of the results.

We consider the Edgeworth market game with one trader of the first type and examine the behavior of the core as the number of traders of the second type increases. The core approaches a single imputation at which the solitary trader of the first type acts as a perfectly discriminating monopolist and obtains all the gain from trading. The theorem may be stated as follows:

THEOREM 2. In an Edgeworth market game denoted by $[1, n]$, for any n the imputation:

$$\eta = \left((n+1)\,\psi\left(\frac{\frac{c}{a}}{n+1}, \frac{nb}{n+1}\right) - n\psi(o,b),\ \overbrace{\psi(o,b),\ \psi(o,b),\ \ldots,\ \psi(o,b)}^{n} \right)$$

lies in the core. For any ϵ there exists an $n(\epsilon)$ such that for all $n \geq n(\epsilon)$ no imputation ζ with $\zeta_1 = \eta_1 - \epsilon$ lies in the core.

In Theorem 3 a very specialized condition is introduced which results in the core of the game converging to the competitive equilibrium point of the no side-payments game.

THEOREM 3. In an Edgeworth market game denoted by $[m, n]$ such that $m = km'$, $n = kn'$ and

$$\psi\left(\frac{m'a}{m'+n'}, \frac{n'b}{m'+n'}\right) = \max \psi\left(\frac{s_m a}{s}, \frac{s_n b}{s}\right)$$

the imputation

$$\eta = \left(\frac{v(I)}{m+n}, \frac{v(I)}{m+n}, \ldots, \frac{v(I)}{m+n}\right)$$

is always in the core, and for any ϵ there exists a $k(\epsilon)$ such that for all $k \geq k(\epsilon)$ no imputation ζ with a component

$$\zeta_1 < \frac{v(I)}{m+n} - \epsilon$$

is in the core, $i = 1, \epsilon, \ldots, m + n$.

The proof of this theorem is similar to that of Theorem 2 . An interpretation of the condition that

$$\psi\left(\frac{m'a}{m'+n'}, \frac{n'b}{m'+n'}\right) = \max \psi\left(\frac{s_m a}{s}, \frac{s_n b}{s}\right)$$

is obtained by examining the nature of the function

$$s\psi\left(\frac{s_m a}{s}, \frac{s_n b}{s}\right).$$

It is homogeneous in the two variables, s_m and s_n . If s_m and s_n are increased in a constant ratio the value of the optimum trade increases in a constant manner. In economic terminology there are constant returns to scale.

The meaning of the optimum ratio can be seen for this rather specialized symmetric game by observing from the form of the characteristic function that in the process of obtaining the joint maximum each individual has as his final stock p units of a and $(1-p)$ units of b where $0 \leq p \leq 1$ depending upon the ratios of the different types of players. As can be seen in Figure 4 there will be an optimum size for p or in other words there will be an optimal coalition.

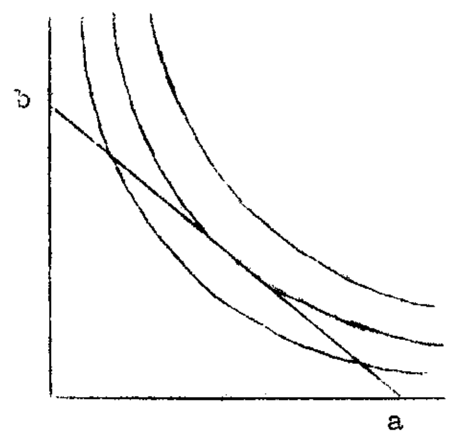

Figure 4

This size will be determined by the size of the initial stock of the players and the shape of the indifference maps. If the size of the game increases with the players remaining in the ratio given by p , then the core converges to the single point which is the point of tangency between the hyperplane representing the imputation space in the side-payment game and the Pareto optimal surface in the no side-payment situation. But this point also has the property of a competitive equilibrium. Given the initial stocks of the players there exists a price which clears the market and brings them to the Pareto optimal surface.

In this instance we have the property that the competitive equilibrium is the only undominated imputation in the limiting game.

A simple example of the optimum ratio for traders is given if we suppose that all members of the society maintain that the best proportion between the amount of tea and sugar used is 2 ounces of the first with 13 ounces of the second. If all players of the first type come to the market with 2 ounces of tea, and all of the second come with 6 ounces of sugar, then the optimal proportions between the players is 6 of the first type to 13 of the second.

There are two very simple theorems concerning the core of a game and increasing and decreasing returns.

> THEOREM 4. In an n-person economy with increasing returns everywhere a core will always exist for any size of n.

> THEOREM 5. In an n-person economy with decreasing returns for all coalitions with more than s^* members where $s^* < n$, no core exists.

5. CONCLUSIONS

One of the major reasons why it has been so hard for economists to even examine the relevance of the von Neumann and Morgenstern theory for the cooperative solution of games is that in their book very little attention was paid to constructing games with an economic content. In order to do so the characteristic function of the game must be obtained from economic considerations such as the initial assets of individuals, production functions and preference relations. Having given the characteristic function economic content it then appears to be reasonable to ask if the solutions appear to bear any relationship to the body of economic knowledge.

Although in the applications of welfare, utility comparisons are made daily and transfer payments are based upon them and the power of the coalitions involved; the pills of comparable and transferable utility are hard to swallow. This still does not rule out attempting to introduce the important role of coalitions into any solution concept. The neat formulation of the characteristic function with a single value for each calition must be abandoned and replaced by a "Pareto sub-surface" which indicates the restraints upon the payoffs between the players. Edgeworth was aware of the role of coalitions and pointed out that if they were permitted, the contract curve would not shrink as players became numerous 7/. Even without giving the formal definition for a solution to a game without transferable utilities, in the 2n person games described, making use of symmetry it is easy to see that the full contract curve is still a solution.

What does a unique point core mean? Is it a pure coincidence that the core and the equilibrium point coincided in the symmetric game discussed above? A unique point core means that there is only one possible imputation of wealth in that society which yields as much or more to all participants than every conceivable subgroup considered as a whole can offer to its members.

In Section 4 an example of an optimal combination of 6 tea traders with 13 sugar traders was given. The goods are infinitely divisible, the preferences of the players may be assumed to be continuous, however the players are not divisible. Thus although the price parameter or competitive equilibrium solution for the games $(6n, 13n)$ will always be the same, in the $(6,13)$ market game the coalitions are at loggerheads between any point on the contract curve. This is because of the fewness and indivisibility of the players. For a large n, there is always the possibility of finding a "more reasonable" optimal group of traders, thus the relative importance of the indivisibility caused by the presence of subgroups of optimum size diminishes. The possibility that large groups may still act together is reflected in the multiplicity of solutions; the narrowing of the core reflects the lessening effect of fewness as more combinations of players obtain power over the imputations.

FOOTNOTES

1/ Shapley, L. S., and M. Shubik, "Solutions of n-person games with ordinal Utilities" (Abstract) Econometrica 21 (1953) pp. 348-349.

2/ See for instance: Arrow, J. K., and G. Debreu, "Existence of an Equilibrium for a Competitive Economy" Econometrica 22 (1954), pp. 265-290, Wald, A., "On Some Systems of Equations of Mathematical Economics," Econometrica 19, (1951), pp. 368-403.

3/ Arrow, K. J., and L. Hurwicz, "On the Stability of the Competitive Equilibrium," Econometrica 26 (1958), pp. 522-552.

4/ Ibid., p. 525.

5/ Scarf, H., "Some Examples of Global Instability of the Competitive Equilibrium", CFDP No. 79, October, 1959.

6/ Shubik, M., "Edgeworth Market Games" in Tucker, A. W., and R. D. Luce, (eds.) Contributions to the Theory of Games V. IV (Princeton: Princeton University Press, 1959), pp. 267-278.

7/ Edgeworth, F. Y., Mathematical Psychics, (London: Kegan Paul, 1881), pp. 42-43.