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PARTIAL TRACE CORRELATION THEORY

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1. INTRODUCTION

This paper is a sequel to [4] in which a generalized correlation coefficient (the trace correlation) for multi-equation** models was presented.

** By multi-equation models we mean any model which contains more than one equation, be it of the simultaneous or recursive type. In this paper we shall discuss our problem within the context of simultaneous equation models, although with minor changes the results are equally valid for the recursive type models.

Here we are concerned with the development of partial trace correlation theory. The relation between partial trace correlations and the trace correlation is analogous to the relation between partial correlation coefficients and the multiple correlation coefficient in a single-equation model.*** In the latter the multiple correlation coefficient measures the extent to which the regression relationship on all of the independent variables accounts for the observed variation in the dependent variable. In addition there are partial correlation coefficients which measure the extent to which the regression on a particular independent variable explains the observed variation in the dependent variable, when the influence of the remaining independent variables

*** For a clear account of this relationship see Anderson [1, pp. 27-34].

* I am indebted to Professors K. J. Arrow and H. Chernoff for their constructive comments on the contents of this paper. Any errors which remain are the sole responsibility of the author. This paper is based upon Chapter IV in [3].

is held constant. Similarly, in multi-equation systems there is the trace correlation* which is a measure of the extent to which the regression rela-

* The reader is referred to [4] for the development of the trace correlation.

tionship explains the variation in the set of jointly dependent variables. The purpose of this paper is to develop the analogue of partial correlation coefficients for multi-equation models, i.e., to develop a statistic which measures the degree to which the regression relationship on a subset of the independent variables accounts for the variation in the jointly dependent variables, while holding the influence of the other independent variables constant.

As in the case of the trace correlation, the basic concepts are developed through the use of canonical correlation theory. This is done in Section 2, while in Section 3 partial trace correlations are defined. In Section 4 the asymptotic sampling variances are given and Section 5 contains a numerical example.

2. CANONICAL CORRELATION THEORY APPLIED TO CONDITIONAL SETS OF VARIABLES

In order to discuss partial trace correlation coefficients we must first develop canonical partial correlation theory** which is concerned with the

** For a short account of canonical partial correlation theory see [7]. The results in this section were developed before this account in [7] was known to me. In any case it is felt that it is desirable to have a more extended development available.

relationships between two sets of variables. For our purpose let these two sets of variables be $y_1, \ldots, y_M$ and $X_1, \ldots, X_\Lambda$, where it is assumed that T observations on each of these variables are available, so that the aggregate set of data can be written in the form of a $T \times M$ matrix Y and a $T \times \Lambda$ matrix X.* We first partition the X matrix into

$$X = [X_1 \; X_2] \,, \tag{2.1}$$

* We also assume that Y is of rank M and X of rank Λ, and that all variables are measured as deviates from their means so that the sums of the T rows of Y and X are zero row vectors.

where X_1 has Λ_1 columns of T elements each and X_2 has Λ_2 columns $(\Lambda_1 + \Lambda_2 = \Lambda)$. We now consider a particular pair of canonical variates, η and ξ, which are obtained by forming the linear combinations

$$Yk = \eta \; ; \; X_1 h = \xi \,, \tag{2.2}$$

where ξ and η are column vectors of T observations on the two canonical variates and h and k are unknown coefficient vectors of Λ_1 and M elements respectively. We also consider the linear regression of Y on X_2 and X_1 on X_2 which can be represented as

$$\begin{aligned} Y &= X_2 B_1 + \overline{V}_1 \\ X_1 &= X_2 B_2 + \overline{V}_2 \,, \end{aligned} \tag{2.3}$$

where B_1 and B_2 are unknown parameter matrices of order $\Lambda_2 \times M$ and $\Lambda_2 \times \Lambda_1$ respectively, and where $\overline{V}_1 = v_{t\mu}$ and $\overline{V}_2 = v_{t\lambda_1}$ are

$T \times M$ and $T \times \Lambda_1$ matrices of parent disturbances.* We then find that the

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* It is further assumed that the random elements of these disturbance matrices have the following usual properties:

(i) $E\left(v_{t\mu}\right) = E\left(v_{t\lambda_1}\right) = 0$;

(ii) $E\left(v_{t\mu} v_{t'\mu'}\right) = \sigma_{\mu\mu'}$, $t = t'$ and $E\left(V_{t\lambda_1} V_{t\lambda_1'}\right) = \sigma_{\lambda\lambda_1'}$, $t = t'$
$= 0$, $t \neq t'$ $= 0$, $t \neq t'$,

for $t, t' = 1, \ldots, T$; $\mu, \mu' = 1, \ldots, M$; and $\lambda_1, \lambda_1' = 1, \ldots, \Lambda_1$.

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means of the canonical variates are

(2.4)
$$E\eta = EYk = X_2 B_1 k$$
$$E\xi = EX_1 h = X_2 B_2 h \ .$$

We now impose the restriction that the sum of squared deviations from the means are unity and we have

(2.5)
$$(\eta - E\eta)'(\eta - E\eta) = (Yk - X_2 B_1 k)'(Yk - X_2 B_1 k) = k'C_1 k = 1$$
$$(\xi - E\xi)'(\xi - E\xi) = (X_1 h - X_2 B_2 h)'(X_1 h - X_2 B_2 h) = h'C_2 h = 1 \ ,$$

where

(2.6)
$$C_1 = [Y'Y - Y'X_2(X_2'X_2)^{-1}X_2'Y]$$
$$C_2 = [X_1'X_1 - X_1'X_2(X_2'X_2)^{-1}X_2'X_1] \ .$$

The correlation, r , between a pair of canonical variates, η and ξ , is

(2.7)
$$r = (\xi - E\xi)'(\eta - E\eta) = (X_1 h - X_2 B_2 h)'(Yk - X_2 B_1 k) = h'Wk \ ,$$

where

(2.8)
$$W = [X_1'Y - X_1'X_2(X_2'X_2)^{-1}X_2'Y] \ .$$

The expression in (2.7) should be stationary for variations in h and k subject to the restrictions in (2.5). So we consider the unconditional stationary value of

$$h'Wk - (1/2)\lambda_1(k'C_1k - 1) - (1/2)\lambda_2(h'C_2h - 1) , \tag{2.9}$$

where the λ's are scalar Lagrange multipliers. Differentiating with respect to the elements of h and k and setting the results equal to zero we obtain

$$\begin{aligned} Wk - \lambda_2C_2h &= 0 \\ W'h - \lambda_1C_1k &= 0 . \end{aligned} \tag{2.10}$$

Premultiply the first set of Λ_1 equations by h' and the second set of M equations by k' and we have, making use of (2.5) and (2.7), that

$$\lambda_1 = \lambda_2 = r . \tag{2.11}$$

Combining the first set of equations in (2.10) with (2.11) gives

$$h = (1/r)C_2^{-1} Wk \tag{2.12}$$

and combining this result with the second set of equations in (2.10) gives

$$(C_1^{-1} W'C_2^{-1} W - r^2I)k = 0 . \tag{2.13}$$

This result implies that the squared canonical correlations, all of which lie between zero and unity,* are the latent roots of the matrix $C_1^{-1}W'C_2^{-1} W$.

* Cf. [7] for a statement of these same results.

To determine what these results mean within the context of simultaneous equation systems we consider the following reduced-form system of equations:

$$Y = X\Pi + \overline{V} , \tag{2.14}$$

where Π is the matrix of parent reduced-form coefficients and $\bar{V}$ the $T \times M$ matrix of parent reduced-form disturbances. We partition the Π matrix to conform to the partitioning of the X matrix and so we obtain

$$(2.15) \qquad Y = X_1\Pi_1 + X_2\Pi_2 + \bar{V} ,$$

where Π_1 and Π_2 are $\Lambda_1 \times M$ and $\Lambda_2 \times M$ matrices of reduced-form coefficients, respectively. The estimates of these coefficients as obtained by least squares are

$$(2.16) \qquad \begin{aligned} P_1 &= C_2^{-1} W \\ P_2 &= (X_2'X_2)^{-1}[X_2'Y - (X_2'X_1)C_2^{-1}W] . \end{aligned}$$

The estimated reduced-form equations can then be written as

$$(2.17) \qquad Y = X_1P_1 + X_2P_2 + V ,$$

where V is a $T \times M$ matrix of calculated residuals. It then follows after multiplication and some simplification that the estimated moment matrix of the jointly dependent variables can be written as

$$(2.18a) \qquad Y'Y = W'C_2^{-1} W + Y'X_2(X_2'X_2)^{-1}X_2'Y + V'V$$

or *

$$(2.18b) \qquad C_1 = W'C_2^{-1} W + V'V$$

The left-hand side of (2.18b), C_1 , is the estimated conditional (conditioned on X_2) moment matrix of the jointly dependent variables which means that it is the variation left unexplained by the regression of Y on X_2 . We notice from the right-hand side of (2.18b) that this unexplained

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* See (2.6) for the definition of C_1 .

variation in the y's given X_2 can be divided into two parts, viz.; $W'C_2^{-1}W$ which can be interpreted as that part of the variation of the variables in C_1 which would be explained by the regression of C_1 on C_2 and $V'V$, which is the observed moment matrix of unexplained residuals. However C_2 is also a conditional moment matrix since it represents that part of the variation in the X_1's which is not accounted for by the regression of X_1 on X_2. Thus we see that $W'C_2^{-1}W$ represents in matrix form that part of the variation in the jointly dependent variables which is explained by the regression relationship on the exogenous variables contained in the set X_1, after the influence of the exogenous variables contained in the set X_2 has been eliminated from both Y and X_1.

It now easily follows from (2.18b) that

(2.19) $$C_1^{-1}W'C_2^{-1}W = I - F \quad \text{, say}$$

where I is the unit matrix of order M and F is defined as

(2.20) $$F = C_1^{-1}V'V \quad .$$

Thus the matrix whose latent roots are considered in (2.13) is simply the inverse of the conditional moment matrix of the jointly dependent variables, postmultiplied by the estimated conditional moment matrix of the systematic part of the reduced form, i.e., $[X_1 - X_2(X_2'X_2)^{-1}X_2'X_1]C_2^{-1}W$ which results from the regression of $[Y - X_2(X_2'X_2)^{-1}X_2'Y]$ on $[X_1 - X_2(X_2'X_2)^{-1}X_2'X_1]$. So we have

(2.21) $$|(I - F) - r^2 I| = 0$$

and from (2.19) and (2.20) that

(2.22) $$|F - (1 - r^2)I| = 0$$

The matrix $C_1^{-1} W' C_2^{-1} W = I - F$ may be regarded as a matrix generalization of a product of the conditional variances of the single dependent variable and an independent variable to the squared conditional covariance of these variables in a single equation model. This ratio is equal to $r^2_{yj \cdot q}$ where $r^2_{yj \cdot q}$ is the partial correlation coefficient between y, the dependent variable, and the jth independent variable, holding the other q independent variables fixed.* However it follows from (2.21) that the latent roots of $I - F$ are

* See [1, p. 29].

r^2_μ where r^2_μ is a canonical partial correlation between the y's and the x_1's . So we find that there is not only a matrix generalization of $r^2_{yj \cdot q}$ in the case of simultaneous equations, i.e., the matrix $I - F$, but also a vector generalization, i.e., the vector $[r^2_\mu]$ of M or Λ_1 elements (M if $M \leq \Lambda_1$ and Λ_1 if $M \geq \Lambda_1$) which is the vector of latent roots of $I - F$. Similarly the matrix $F = C_1^{-1} V'V$ may be regarded as the matrix generalization of $1 - r^2_{yj \cdot q}$ in a single equation system and the vector of latent roots of F, i.e., $[1 - r^2_\mu]$, may be interpreted as the vector generalization.

3. PARTIAL TRACE CORRELATIONS

We shall use the same function, i.e., the trace, to take the matrices F and $I - F$ into scalars as we did for the trace correlation. So we define the partial trace correlation coefficient as the positive square root of $\bar{r}^2_{yx_1 \cdot x_2}$ where

$$(3.1) \qquad \bar{r}^2_{yx_1 \cdot x_2} = (1/M)\mathrm{tr}(I - F) = (1/M) \sum_{\mu=1}^{M} r^2_\mu \quad .$$

We also have that

$$(3.2) \qquad 1 - \bar{r}^2_{yx_1 \cdot x_2} = (1/M)\mathrm{tr}\ F = (1/M) \sum_{\mu=1}^{M} (1 - r^2_\mu)$$

Considering the discussion of $[r^2_\mu]$ as given in Section 2 we may conclude that $\bar{r}^2_{yx_1 \cdot x_2}$ can be interpreted as a scalar measure of that part of the variation in the set of conditional jointly dependent variables $(Y|X_2)$, say, that is explained by the systematic part of the regression of $(Y|X_2)$ on a subset of conditional independent variables $(X_1|X_2)$. Similarly $1 - \bar{r}^2_{yx_1 \cdot x_2}$ can be interpreted as a measure of that part of the variation in $(Y|X_2)$ which is left unexplained by the regression of $(Y|X_2)$ on $(X_1|X_2)$. The squared partial trace correlation also possesses the same desirable properties of the trace correlation, viz., $\bar{r}^2_{yx_1 \cdot x_2} + (1 - \bar{r}^2_{yx_1 \cdot x_2}) \equiv 1$, $0 \leq \bar{r}^2_{yx_1 \cdot x_2} \leq 1$, and $\bar{r}^2_{yx_1 \cdot x_2}$ is invariant to the units in which the variables are measured.

Some special cases of the partial trace correlation coefficient are of interest.* When $M = \Lambda_1 = 1$, $\Lambda_2 \geq 1$ the partial trace correlation

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* Cf. [7] for a listing of these cases.

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becomes the single equation partial correlation coefficient. When $\Lambda_2 = 0$

the partial trace correlation becomes the trace correlation and then for $M = \Lambda_1 = 1$ we have the zero-order correlation coefficient (disregarding sign) and for $\Lambda > M = 1$ we have the multiple correlation coefficient. For the case of $M = 1$, $\Lambda_1 > 1$, $\Lambda_2 \geq 1$ we have the multiple partial correlation coefficient.*

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* See [7]. This is the term used by these authors. There seems to be very little discussion of this potentially useful concept in the literature.

One situation in which the partial trace correlation might prove useful is as an aid in the selection of exogenous variables to be used in the model. This is the situation in which either $\Lambda > T$, in which case $(X'X)$ is singular so that neither the reduced-form or the structural parameters can be estimated, or of the case when $T - \Lambda$ is positive but small so that there is but a small number of degrees of freedom for estimating the parameters.**

** An excellent discussion of this problem is given in [6].

In these situations the usual procedure is to perform the estimation using only some subset of the original set of exogenous variables. So the problem becomes that of selecting which exogenous variables are to be used. Although there are several ways to select these variables, all to some extent arbitrary, it would seem that one way which possesses some merit is to use the partial trace correlations in conjunction with the trace correlation in order to determine those exogenous variables which contribute the most to the explanation of the observed variation in the jointly dependent variables. Then in descending order of importance enough exogenous variables can be selected so as to make the estimation of the parameters possible (when $\Lambda > T$) or more effective (when $T - \Lambda > 0$ but small).

There are also other situations in which one must neglect some of the exogenous variables* and in these situations the partial trace correlations

* See [2, pp. 203-204].

and the full trace correlation can be used to determine which variables can be omitted so as to make the unexplained variation $(V'V)$ as 'small' as possible.**

** Cf. [2, p. 104] for a discussion of the 'size' of $(V'V)$.

4. ASYMPTOTIC SAMPLING VARIANCES

In order to obtain some idea of the significance of a particular partial trace correlation computed from a sample, it is desirable to have, at least, the asymptotic sampling variance. In this section we shall derive this variance to the order of T^{-1} under the assumption that each of the T-rows of $[Y\ X_1\ X_2]$ are independent random drawings from a $[M + \Lambda]$-dimensional normal parent distribution with zero means. We shall also assume that there are no multiple roots in the population (except possibly zero multiple roots caused by the fact that $\Lambda_1 < M$, which are allowed).

From the definition of the partial trace correlation (3.1) we have

$$M\bar{r}^2_{yx_1\cdot x_2} = \sum_{\mu=1}^{M} r_\mu^2 \tag{4.1}$$

Taking differentials we obtain

$$M\bar{r}_{yx_1\cdot x_2}\, d\bar{r}_{yx_1\cdot x_2} = \sum_{\mu=1}^{M} r_\mu dr_\mu \quad . \tag{4.2}$$

Squaring and taking expected values we find for large samples that

$$M^2\bar{\rho}^2_{yx_1\cdot x_2} \operatorname{var} \bar{r}_{yx_1\cdot x_2} = \sum_{\mu,\mu'} \rho_\mu \rho_{\mu'} \operatorname{cov}(r_\mu, r_{\mu'}) \ . \tag{4.3}$$

where $\bar{\rho}^2_{yx_1\cdot x_2} = (1/M)\mathrm{tr}(I - \Phi)$ being the square of the parent partial trace correlation. Now the problem is solved if we can find $\operatorname{var} r_\mu$ and $\operatorname{cov}(r_\mu, r_{\mu'})$. These can be derived to the order of T^{-1} but we can eliminate this tedious derivation by noticing that the two sets of variables $V_1 = Y - X_2(X_2'X_2)^{-1}X_2'Y$ and $V_2 = X_1 - X_2(X_2'X_2)^{-1}X_2'X_1$, between which we have determined the canonical correlations have the same properties as unconditioned sets of variables. This means that the <u>large sample</u> variances and covariances of the canonical correlations between V_1 and V_2 are the same as those between unconditioned sets of variables and so we have the well-known result* that

$$\operatorname{var} r_\mu = (1/T)(1 - \rho_\mu^2)^2 \ ; \ \operatorname{cov}(r_\mu, r_{\mu'}) = 0 \quad \text{for} \quad \mu \neq \mu' \ , \tag{4.4}$$

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* See [5, p. 340].

to the order* of T^{-1} . So we obtain

(4.5) $$\operatorname{var} \bar{r}_{yx_1 \cdot x_2} = \frac{1}{TM^2 \bar{\rho}^2_{yx_1 \cdot x_2}} \sum_{\mu=1}^{M} \rho_\mu^2 (1 - \rho_\mu^2)^2$$

to our degree of approximation. Furthermore, since $\operatorname{var} \bar{r}^2_{yx_1 \cdot x_2} = 4\bar{\rho}^2_{yx_1 \cdot x_2} \operatorname{var} \bar{r}_{yx_1 \cdot x_2}$, we also obtain

(4.6) $$\operatorname{var} \bar{r}^2_{yx_1 \cdot x_2} = \frac{4}{TM^2} \sum_{\mu=1}^{M} \rho_\mu^2 (1 - \rho_\mu^2)^2 .$$

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* The fact that the large sample variance of canonical partial correlations is the same as that of canonical correlations should cause no surprise since the same result holds as between the finite sample distributions of canonical partial and canonical correlations. More precisely, the distributions of canonical partial correlations in random samples of size $T + 1$ from a $(M + \Lambda_1 + \Lambda_2)$-variate normal population are of the same form as those of canonical correlations in random samples of size $T + 1 - \Lambda_2$. See [8] for this result. So one might expect that the divisor in (4.4) should be $T - \Lambda_2$. However as (4.4) is only true to T^{-1} the divisor in (4.4) causes no error.

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5. FURTHER COMMENTS ON PARTIAL TRACE CORRELATIONS

The concept of the partial trace correlation is but a logical extension of the trace correlation. Thus the same problem of non-uniqueness in the case of the presence of definitional equations and of using more efficient methods of estimation than unrestricted least-squares, as discussed in Section 6 of [3] occur. As for the latter problem it can be shown that the partical trace correlation can be extended to take account of a method of estimation such as limited-information or two-state least-squares.*

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* See Section 5, Chapter II of [3] for this extension. This remark applies only to the estimation of partial trace correlations and not to their distributions.

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6. AN EXAMPLE

In this section we shall apply the results obtained in the previous sections to an econometric model. For this purpose we shall use the same model** as used in Section 7 of [4].

** This model is Tintner's model of the American meat market which consists of the following two equations:

(i) $y_1 = \beta y_2 + \alpha_1 x_1 + \mu_1$,

(ii) $y_1 = \beta' y_2 + \alpha_2 x_2 + \alpha_3 x_3 + \mu_2$,

where (i) is the demand equation and (ii) is the supply equation, y_1 is per capita consumption of meat in pounds, y_2 the retail price of meat, x_1 is real per capita disposable income in dollars, x_2 the cost of processing meat, and x_3 the cost of producing agricultural products. There are 23 annual observations from 1919-1941. Cf. Tintner [9, pp. 169-172] for a complete description of the model.

There are, of course, many partial trace correlation coefficients (six in this case) which can be computed. We shall compute one in some detail and give the results for the others. For the example in detail we shall put the variable x_1 in the matrix X_1 and the variables x_2 and x_3 in the matrix X_2 . In this case $\bar{r}^2_{yx_1 \cdot x_2} = \bar{r}^2_{y\bar{x}_1 \cdot \bar{x}_2 \bar{x}_3}$ will be a measure of the amount of variation in the jointly dependent variables as explained by $\bar{x}_1$, after the influence of* $\bar{x}_2$ and $\bar{x}_3$ has been eliminated in a linear manner. We then obtain that

* To distinguish between matrices and variables in this section we shall place a bar over the variables.

$$
(6.1)\qquad
\begin{array}{ccccccc}
Y'Y & = & W'C_2^{-1}W & + & Y'X_2(X_2'X_2)^{-1}X_2'Y & + & V'V \\
\begin{bmatrix} 1{,}369.54 & -352.55 \\ --- & 1{,}581.49 \end{bmatrix} & = & \begin{bmatrix} 25.67 & -78.15 \\ --- & 246.46 \end{bmatrix} & + & \begin{bmatrix} 649.75 & 286.79 \\ --- & 691.47 \end{bmatrix} & + & \begin{bmatrix} 694.12 & -560.19 \\ --- & 643.56 \end{bmatrix}
\end{array}
$$

So we have

$$
(6.2)\qquad C_1 = \begin{bmatrix} 719.79 & -639.34 \\ --- & 890.02 \end{bmatrix} \quad \text{and} \quad C_1^{-1} = \begin{bmatrix} .00383841 & .00275730 \\ --- & .00310426 \end{bmatrix}
$$

which leads to

$$
(6.3)\qquad C_1^{-1} W'C_2^{-1} W = \begin{bmatrix} -.119705 & .375751 \\ -.174922 & .546834 \end{bmatrix} = I - F
$$

and

$$
(6.4)\qquad C_1^{-1}(V'V) = \begin{bmatrix} 1.119705 & -.375751 \\ .174922 & .453166 \end{bmatrix} = F \, .
$$

We obtain finally that the squared partial trace correlation is

$$(6.5) \qquad \bar{r}^2_{y\bar{x}_1 \cdot \bar{x}_2 \bar{x}_3} = (1/M)\mathrm{tr}(I - F) = .2135645$$

and

$$(6.6) \qquad 1 - \bar{r}^2_{y\bar{x}_1 \cdot \bar{x}_2 \bar{x}_3} = (1/M)\mathrm{tr}\, F = .7864355 .$$

One interpretation of the result in (6.5) is that approximately 21 per cent of the variation which remains in the jointly dependent variables after the influence of the variables $\bar{x}_2$ and $\bar{x}_3$ has been held constant, is explained by the variable $\bar{x}_1$, after the influence of $\bar{x}_2$ and $\bar{x}_3$ are eliminated from it.

For the other partial trace correlations we shall give the matrices C_1 and $I - F$. We have

$$(6.7) \qquad \bar{r}^2_{y\bar{x}_2 \cdot \bar{x}_1 \bar{x}_3} = .156994 \; ; \quad C_1 = \begin{bmatrix} 970.83 & -727.48 \\ --- & 744.68 \end{bmatrix} ; \; I - F = \begin{bmatrix} .435447 & -.263324 \\ .200750 & -.121459 \end{bmatrix}$$

$$(6.8) \qquad \bar{r}^2_{y\bar{x}_3 \cdot \bar{x}_1 \bar{x}_2} = .288478 \; ; \quad C_1 = \begin{bmatrix} 1{,}004.60 & -577.35 \\ --- & 644.51 \end{bmatrix} ; \; I - F = \begin{bmatrix} .605459 & -.033459 \\ .515747 & -.028503 \end{bmatrix}$$

$$(6.9) \qquad \bar{r}^2_{y\bar{x}_1 \bar{x}_2 \cdot \bar{x}_3} = .3683855 \; ; \quad C_1 = \begin{bmatrix} 1{,}001.06 & -815.37 \\ --- & 1{,}000.18 \end{bmatrix} ; \; I - F = \begin{bmatrix} .294069 & .105680 \\ -.0153951 & .442702 \end{bmatrix}$$

$$(6.10) \qquad \bar{r}^2_{y\bar{x}_1 \bar{x}_3 \cdot \bar{x}_2} = .526150 \; ; \quad C_1 = \begin{bmatrix} 1{,}256.00 & -172.58 \\ --- & 1{,}296.24 \end{bmatrix} ; \; I - F = \begin{bmatrix} .497546 & .384832 \\ .365269 & .554754 \end{bmatrix}$$

$$(6.11) \qquad \bar{r}^2_{y\bar{x}_2 \bar{x}_3 \cdot \bar{x}_1} = .360343 \; ; \quad C_1 = \begin{bmatrix} 1{,}207.94 & -720.24 \\ --- & 744.91 \end{bmatrix} ; \; I - F = \begin{bmatrix} .701923 & -.121310 \\ .463819 & .018763 \end{bmatrix}$$

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