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Stationary Ordinal Utility and Impatience

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1. Introduction. Ever since the appearance of Böhm Bawerk's "Positive Theorie des Kapitales," the idea of a preference for advancing the timing of future satisfaction has been widely used in economic theory. However, the question how to define this idea precisely has been given insufficient attention. If the idea of preference for early timing is to be applied also in a world of changing prices, money expenditure on consumption is not a suitable measure of "satisfaction level," and money expenditure divided by a consumers' goods price index is at best an approximate measure, useful for econometric work but not providing the sharp distinctions that theory requires. It seems better to try to define preference for advanced timing entirely in terms of a utility function. Moreover, if the idea of preference for early timing is to be expressed independently of assumptions that have made the construction of cardinal utility possible[**] (such as choice between uncertain prospects, or

** For a recent discussion, see Debreu, Topological Methods in Cardinal Utility Theory, CFDP. No. 76.

stochastic choice, or independence of commodity groups in the preference structure) it will be necessary to express it in terms of an ordinal utility function, that is, a function that retains its meaning under a monotonic (increasing) transformation. It would seem that this can be done only if one postulates a certain persistency over time in the structure of preference.

This study started out as an attempt to formulate postulates permitting a sharp definition of impatience, the short term Irving Fisher has introduced for preference

* I am indebted to Gerard Debreu and Herbert Scarf for extremely valuable suggestions on the subject and methods of this paper.

for advanced timing of satisfaction. To avoid complications connected with the advancing age and finite life span of the individual consumer, these postulates were set up for a (continuous) utility function of a consumption program extending over an infinite future period. The surprising result was that only a slight strengthening of the continuity postulate (incorporated in Postulate 1 below) permits one to conclude from the existence of a utility function satisfying the postulates to the presence of impatience in a central area of the commodity space. In other words, conditions hardly stronger than those that appear needed to <u>define</u> impatience are sufficient to <u>prove</u> that there is a central zone of impatience. Intuitively, the reason is that if there is in all circumstances a preference for postponing satisfaction -- or even neutrality toward timing -- then there is not enough room in the set of real numbers to accommodate and label numerically all the different satisfaction levels that may occur in relation to consumption programs for an infinite future.

This paper then is a study of the implications of continuous and stationary (see Postulate 3) ordering of infinite programs. Flexibility of interpretation remains as to whether this ordering may serve as a first approximation to the preferences of an individual consumer, or may perhaps be an "impersonal" result of the aggregation of somewhat similar individual preferences (interpreting "consumption" as "consumption per head" in the case of a growing population), or finally may guide choices in a centrally planned economy. In each of these interpretations further modifications and refinements may be called for.

Two levels of discussion are separated in what follows. The contents and findings of each section are first stated in general terms. Then where needed the more technical stipulations, proofs and discussions are given in a starred section bearing the same number. The starred sections can be passed up by readers interested primarily in the results.

2. The Commodity Space. Notation. A program for an infinite future will be denoted

$$(1) \qquad {}_1x = (x_1,\ x_2,\ x_3,\ \ldots,\ x_t,\ \ldots) = (x_1,\ {}_2x) = \text{etc.}$$

Each symbol x_t, $t = 1, 2, \ldots$, represents a vector (bundle)

$$(2) \qquad x_t = (x_{t1}, x_{t2},\ \ldots,\ x_{tn})$$

of the nonnegative amounts of n listed commodities to be consumed in the period t. Subvectors of (1) consisting of several consecutive vectors (2) will be denoted

$$(3) \qquad {}_tx_{t'} = (x_t,\ x_{t+1},\ \ldots,\ x_{t'}) ,$$

where omission of the right subscript t' of ${}_tx_{t'}$ indicates that $t' = \infty$. The subscript t of x_t is called the timing of the consumption vector x_t, the subscript s of ${}_sx = (x_s,\ x_{s+1},\ \ldots)$ the time of choice between ${}_sx$ and its alternatives ${}_sx'$, ${}_sx''$, ... A constant program is denoted

$$(4) \qquad {}_{con}x = (x,\ x,\ x,\ \ldots) .$$

2*. Each consumption vector x_t is to be selected from a closed and connected subset X of the nonnegative orthant in n-dimensional space, which may be the entire orthant, and which we take to be the same for all t. Hence ${}_tx = (x_t,\ x_{t+1},\ \ldots)$ belongs to the cartesian product ${}_1X$ of an infinite sequence of identical sets X. Expressions such as "for some x_t," "for all ${}_tx$," etc., will in what follows always mean "for some $x_t \epsilon X$," "for all ${}_tx \epsilon {}_1X$," etc., and all functions of x_t or ${}_tx$ are to be thought of as defined on X or ${}_1X$, respectively.

3. Existence of a Continuous Utility Function. Before stating the basic postulate asserting this existence, the meaning of continuity needs to be clarified. Continuity of a function $f(y)$ of a vector y means that, for every y one can make the absolute difference $|f(y') - f(y)|$ as small as desired by making the distance $d(y',y)$ between y' and y sufficiently small, regardless of the direction of approach of y' to y. For vectors $y = (y_1, \ldots, y_n)$ with a finite number n of components there is a wide choice of definitions of the distance function $d(y',y)$, all of which establish the same continuity concept, and euclidean distance

$$(5) \qquad d(y', y) = |y' - y| \equiv \sqrt{\sum_{k=1}^{n} (y'_k - y_k)^2}$$

is as good as any other. But in an infinite-dimensional space the continuity concept is sensitive to the choice of the distance function used. In what follows we shall employ as a "distance" between two programs ${}_1x'$, ${}_1x$

$$(6) \qquad d({}_1x', {}_1x) \equiv \sup_t |x'_t - x_t| ,$$

which is the maximum distance in the sense of (5) between any two corresponding one-period consumption vectors x'_t, x_t, whenever such a maximum exists. This definition treats all future periods alike, and if anything has a bias toward neutrality with regard to the timing of satisfaction.

Postulate 1. There exists a utility function $U_1({}_1x)$ with the continuity property that, if U is any one of the values assumed by that function, ~~and $U' < U < U''$~~, for any U' and U'' such that $U' < U < U''$, then there exists a positive distance δ such that the utility $U_1({}_1x')$ of every program ${}_1x'$ having a distance $d({}_1x', {}_1x) \leq \delta$ from some program ${}_1x$ with utility $U_1({}_1x) = U$ satisfies $U' \leqq U_1({}_1x') \leqq U''$.

Comparison with the above definition of continuity of a function $f(y)$ will show that we are here making a slightly stronger requirement. For any U' and

U'' bracketing the given U, we want the same maximum distance δ between ${}_1x'$ and ${}_1x$ to guarantee that $U' \leqq U_1({}_1x') \leqq U''$ regardless of which is the member ${}_1x$ of the class of all programs with utility equal to U, to which the program ${}_1x'$ has a distance $\leqq \delta$.

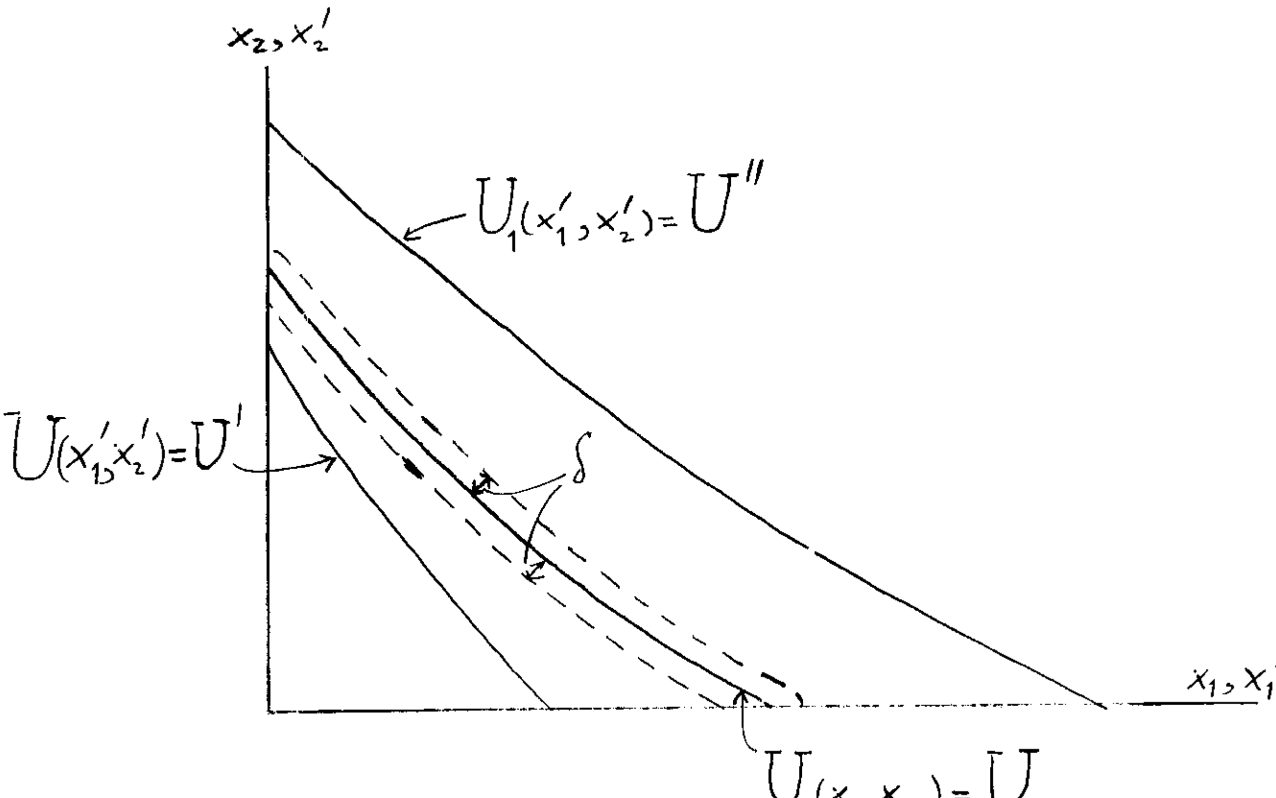

Figure 1.

In diagram 1, showing a simplified case where ${}_1x$ has only two scalar components x_1 and x_2, we require that there be a band consisting of all points no further than δ away from some point of the indifference curve $U_1(x_1, x_2) = U$, which is to fall entirely within the zone $U' \leqq U_1(x_1', x_2') \leqq U''$. Essentially, then we are requiring that the utility function not be infinitely more sensitive to changes in the quantities of one program than it is to any such changes in another <u>equivalent</u> program.

4. Period-wise Aggregation. Having rejected expenditure on consumption as a measure for the satisfaction levels reached in particular periods, we must find another means of labeling such levels. This can be done if we are willing to postulate, essentially, that the particular bundle of commodities to be consumed in the first period has no effect on the preference between alternative sequences of bundles in the remaining future, and conversely. One cannot claim a high degree of realism for such a postulate, because there is no clear reason why complementarity of goods could not extend over more than one time-period. It may be surmised, however, that weaker forms of this postulate would still allow similar results to be reached. The purpose of the present form is to set the simplest possible stage for a study of the effect of timing alone on preference.

Postulate 2(2a and 2b). For all $x_1, x_1', {}_2x, {}_2x'$,

2a) $$U_1(x_1, {}_2x) \geqq U_1(x_1', {}_2x) \text{ implies } U_1(x_1, {}_2x') \geqq U_1(x_1', {}_2x'),$$

2b) $$U_1(x_1, {}_2x) \geqq U_1(x_1, {}_2x') \text{ implies } U_1(x_1', {}_2x) \geqq U_1(x_1', {}_2x').$$

We shall show that, as a consequence of Postulate 2, the utility function can be written in the form

$$U_1({}_1x) = V_1(u_1(x_1), U_2({}_2x)), \tag{7}$$

where $V_1(u_1, U_2)$ is a continuous and increasing function of its two variables u_1, U_2, and where both $u_1(x_1)$ and $U_2({}_2x)$ have the stronger continuity property attributed to $U_1({}_1x)$ in Postulate 1. We shall call $u_1(x_1)$ immediate utility or one-period utility (at time t=1), interpreting it as a numerical indicator of the satisfaction level associated with the consumption vector x_1 in period 1. $U_2({}_2x)$ will be called prospective utility (as from time t = 2), with a similar interpretation with regard to the remaining future. While this suggests calling $U_1({}_1x)$ prospective utility as from time 1, we shall for contrast call it aggregate utility (aggregated, that is, over all future time periods). Finally, the function $V_1(u_1, U_2)$, to be called the aggregator, indicates how any given pair of utility levels, immediate (u_1) and prospective (U_2) stacks up against any other pair in making choices for the entire future.

4* Since ${}_1x$ and ${}_1x'$ as well as ${}_2x$ and ${}_2x'$ can be interchanged in Postulate 2a, and since "$>$" means "$\geqq$ and not $\leqq$" and " $=$ " means " $\geqq$ and $\leqq$ ", Postulate 2a implies that, for all x_1, x_1', ${}_2x$, ${}_2x'$,

$$(8 >) \qquad U_1(x_1, {}_2x) > U_1(x_1', {}_2x) \text{ implies } U_1(x_1, {}_2x') > U_1(x_1', {}_2x') \ ,$$

$$(8 =) \qquad U_1(x_1, {}_2x) = U_1(x_1', {}_2x) \text{ implies } U_1(x_1, {}_2x') = U_1(x_1', {}_2x') \ .$$

If we assign to ${}_2x$ a particular value ${}_2x^0$ and define

$$(9) \qquad u_1(x_1) \equiv U_1(x_1, {}_2x^0)$$

we read from (8 =) that

$$u_1(x_1) = u_1(x_1') \text{ implies } U_1(x_1, {}_2x') = U_1(x_1', {}_2x') \text{ for all } {}_2x' \ .$$

Writing again ${}_2x$ for ${}_2x'$ this means that

$$U_1(x_1, {}_2x) = F_1\big(u_1(x_1), {}_2x\big) \ .$$

Applying a similar argument to Postulate 2b and defining

$$(10) \qquad U_2({}_2x) \equiv U_1(x_1^0, {}_2x)$$

we obtain for $U_1({}_1x)$ the form (7). It follows from the definitions (9) and (10) that $u_1(x_1)$ and $U_2({}_2x)$ have the same continuity property as $U_1({}_1x)$.

From (8 >) and (9) we see that $V_1(u_1, U_2)$ is increasing in u_1, and, by similar reasoning from Postulate 2b, in U_2.

The proof of the continuity of $V_1(u_1, U_2)$ is slightly complicated by the possibility of zones of indifference, which necessitates separate consideration of the four quadrants of a neighborhood of a given point (u_1^0, U_2^0). Assume first that

$$u_1^0 = u_1(x_1^0) < M_{u_1} \equiv \text{lub}\left\{u_1(x_1)\,|\,x_1 \epsilon X\right\}$$

$$U_2^0 = U_2({}_2x^0) < M_{U_2} \equiv \text{lub}\left\{U_2({}_2x)\,|\,{}_2x \epsilon {}_1X\right\}$$

Then there exist points x_1^1 and ${}_2x^1$ such that

$$u_1(x_1^1) > u_1(x_1^0) , \qquad U_2({}_2x^1) > U_2({}_2x^0) ,$$

and, because X is connected, continuous curves $x_1(\lambda)$, ${}_2x(\Lambda)$ such that

$$x_1(0) = x_1^0 , \qquad x_1(1) = x_1^1 , \qquad x_1(\lambda) \epsilon X \quad \text{for} \quad 0 \leqq \lambda \leqq 1 ,$$

$${}_2x(0) = {}_2x^0 , \qquad {}_2x(1) = {}_2x^1 , \qquad {}_2x(\Lambda) \epsilon_1 X \quad \text{for} \quad 0 \leqq \Lambda \leqq 1 ,$$

and finally points x_1', ${}_2x'$ on these curves such that

$$x_1' = x_1(\lambda') , \quad 0 \leqq \lambda' \equiv \text{lub}\left\{\lambda\,|\,u_1\big(x_1(\lambda)\big) \leqq u_1^0\right\} < 1, \text{ hence } \; u_1(x_1') = u_1^0 ,$$

$${}_2x' = {}_2x(\Lambda'), \quad 0 \leqq \Lambda' \equiv \text{lub}\left\{\Lambda\,|\,U_2\big({}_2x(\Lambda)\big) \leqq U_2^0\right\} < 1, \text{ hence } \; U_2({}_2x') = U_2^0 .$$

Now

$$V_1(u_1^0 , U_2^0) = V_1\big(u_1(x_1'), U_2({}_2x')\big) = U_1(x_1', {}_2x') = U_1({}_1x') ,$$

and, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\max_t \, |x_t - x_t'| \leqq \delta \quad \text{implies} \, |U_1({}_1x) - U_1({}_1x')| \leqq \epsilon .$$

Finally, there exist λ'', Λ'' such that $\lambda' < \lambda'' \leqq 1$, $\Lambda' < \Lambda'' \leqq 1$, and

$u_1'' \equiv u_1\big(x_1(\lambda'')\big) > u_1^0$, $\quad |x_1(\lambda) - x_1(\lambda')| \leqq \delta \quad$ for $\lambda' \leqq \lambda \leqq \lambda''$,

$U_2'' \equiv U_2\big({}_2x(\Lambda'')\big) > U_2^0$, $\quad \max_t |x_t(\Lambda) - x_t(\Lambda')| \leqq \delta \quad$ for $\Lambda' \leqq \Lambda \leqq \Lambda''$.

Since $u_1\big(x_1(\lambda)\big)$, $U_2\big({}_2x(\Lambda)\big)$ are continuous in λ, Λ, respectively, every value u_1 such that $u_1^0 \leqq u_1 \leqq u_1''$ is attained by $u_1(x_1(\lambda))$ for some λ such that $\lambda' \leqq \lambda \leqq \lambda''$, and a similar statement applies to $U_2\big({}_2x(\Lambda)\big)$. Hence

$$u_1^0 \leqq u_1 \leqq u_1'',\ U_2^0 \leqq U_2 \leqq U_2'' \text{ imply } |V_1(u_1,U_2) - V_1(u_1^0,U_2^0)|$$

$$= \big|U_1\big(x_1(\lambda),\ {}_2x(\Lambda)\big) - U_1(x_1^0,\ {}_2x^0)\big| \leqq \epsilon\,,$$

the essential point being, of course, that $u_1^0 < u_1''$, $U_2^0 < U_2''$. If, on the other hand, $u_1^0 = M_{u_1}$ or $U_2^0 = M_{U_2}$, the particular quadrant does not need to be considered. Similar reasoning applies to the other three quadrants.

5. <u>Stationarity</u>. Postulate 2b says that the preference ordering within a class of programs ${}_1x$ with a common first-period consumption vector x_1 does not depend on what that vector x_1 is. We now go a step further and require that that preference ordering be the same as the ordering of corresponding programs obtained by advancing the timing of each future consumption vector by one period (and, of course, forgetting about the common first-period vector originally stipulated). This expresses the idea that the passage of time does not have an effect on preferences.

<u>Postulate</u> 3. <u>For some</u> x_1 <u>and all</u> ${}_2x$, ${}_2x'$,

$$U_1(x_1,\ {}_2x) \geqq U_1(x_1,\ {}_2x') \ \underline{\text{if and only if}}\ U_1({}_2x) \geqq U_1({}_2x')\ .$$

In the light of (7) and the fact that $V_1(u_1,\ U_2)$ increases with U_2, this is equivalent to

$$U_2({}_2x) \geqq U_2({}_2x') \ \underline{\text{if and only if}}\ U_1({}_2x) \geqq U_1({}_2x')\ .$$

By reasoning similar to that in section 4* it follows that

$$U_2({}_2x) = F_2\big(U_1({}_2x)\big)$$

where $F_2(U_1)$ is a continuous increasing function of U_1. If $U_1 = F_2^{-1}(U_2)$ denotes the inverse function of $U_2 = F_2(U_1)$, it follows that the transformation (which is in part only a change of notation)

$$U({}_1x) \equiv U_1({}_1x) = F_2^{-1}\big(U_2({}_1x)\big) , \qquad u(x_1) \equiv u_1(x_1) ,$$

$$V(u, U) \equiv V_1\big(u, F_2(U)\big) ,$$

preserves the ordering originally represented by $U_1({}_1x)$, and brings (7) into the simple form

$$(11) \qquad U({}_1x) = V\big(u(x_1) , U({}_2x)\big) .$$

This relation will be the point of departure for all further reasoning. It says that the ordering of pairs of utility levels - immediate, $u(x_1)$, and prospective, $U({}_2x)$ - defined by the aggregator $V(u, U)$ is such as to produce an ordering of programs for all future, identical but for a shift in time with the ordering of programs from the second period on. Time subscripts have been omitted from the function symbols u, U, V because ${}_2x$ can again be substituted for ${}_1x$ in (11), giving $U({}_2x) = V\big(u(x_2), U({}_3x)\big)$, and so on. The function $V(u, U)$ is again continuous and increasing in its arguments u, U. Since both $u(x_1)$ and $U({}_2x)$ are continuous, the arguments u, U of $V(u, U)$ can take any value in an interval I_u, I_U, respectively, and the values attained by $V(u, U)$ fill the interval I_U. Since we are dealing with ordinal utility, there is still freedom to apply separate increasing transformations to $u(x_1)$ and to $U({}_2x)$, which can be used to make both I_u and I_U coincide with the unit interval*

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* Unless I_u or I_U or both degenerate to a single point. This will be ruled out in Postulate 4 below.

extending from 0 to 1. The aggregator $V(u, U)$ can then be represented, though incompletely, by its niveau lines in the unit square, which are decreasing to the right, as shown in Figure 2.

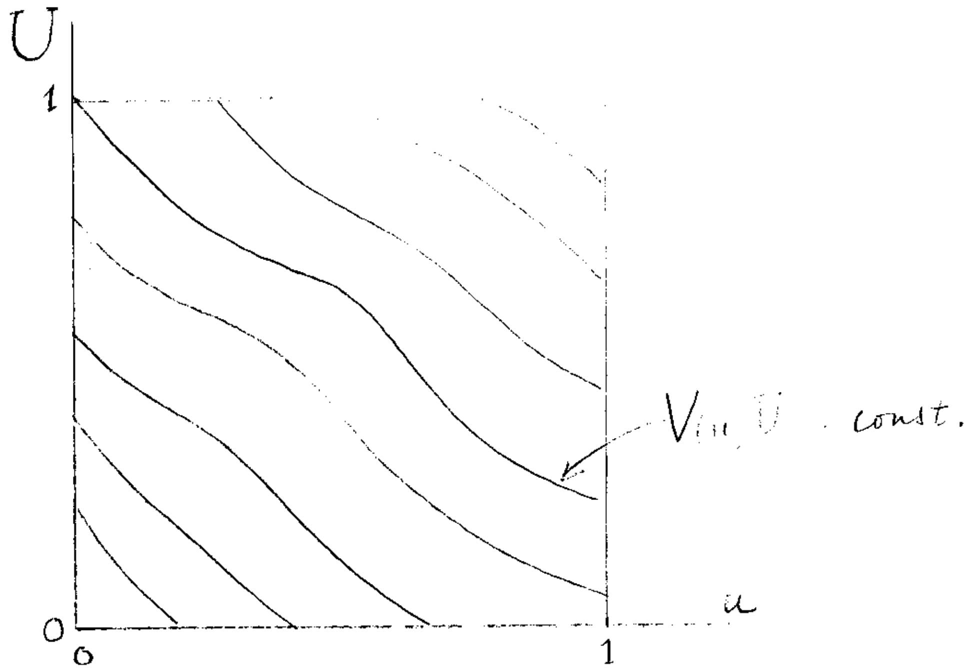

Figure 2

The representation is incomplete in that one still has to associate with each niveau line a numerical value of the function, which is to be referred to the vertical scale. It is also somewhat arbitrary in that separate increasing transformations of u and U that preserve the end points 0,1 are still permitted. The information conveyed by $V(u, U)$ is therefore as yet somewhat hidden in those interrelations between the niveau lines, the verticals, the horizontals, and the numerical niveaus themselves, which are invariant under such transformations.

6. <u>Extreme programs</u>. In order to sidestep a mathematical complication, we shall consider only the case in which there exist a best program $_1\bar{x}$ and a

worst program ${}_1\underline{x}$.

<u>Postulate 4</u>. <u>There exist</u> ${}_1\underline{x}$, ${}_1\bar{x}$ <u>such that</u> $U({}_1\underline{x}) < U({}_1\bar{x})$ <u>and</u>,

$$U({}_1\underline{x}) \leqq U({}_1x) \leqq U({}_1\bar{x}) \quad \underline{\text{for all}} \quad {}_1x$$

(where we have now written $U({}_1x)$ for $U_1({}_1x)$).

As a result of the transformation already applied, we must then have

$$(12) \qquad U({}_1\underline{x}) = 0, \qquad U({}_1\bar{x}) = 1 .$$

Furthermore, if ${}_1\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots)$, we must also have

$$u(\bar{x}_t) = u(\bar{x}_1) = \bar{u} \text{ , say, for all } t \text{ ,}$$

because, if we had $u(\bar{x}_{t'}) < u(\bar{x}_t)$ for some t , t' , the program ${}_1\bar{x}'$ defined by

$$\bar{x}'_{t'} = \bar{x}_t \text{ ,} \qquad \bar{x}'_t = \bar{x}_t \quad \text{for all} \quad t \neq t'$$

would be a better one, in view of (11) and the monotonicity of $V(u, U)$. For the same reason, $\bar{u} = u(\bar{x}_1)$ must be the maximum attainable immediate utility. From this and similar reasoning for the worst program ${}_1\underline{x}$ we have

$$(13) \qquad 0 = u(\underline{x}_1) \leqq u(x) \leqq u(\bar{x}_1) = 1 \quad \text{for all} \quad x \text{ .}$$

It follows that in the present case the intervals $I_u = I_U$ contain both end points 0, 1. Finally, if ${}_1\bar{x}$ is a best (${}_1\underline{x}$ a worst) program, it follows from (11) and the monotonicity of $V(u, U)$ that ${}_2\bar{x}$ (or ${}_2\underline{x}$) is likewise a best (worst) program. Hence, by inserting ${}_1\underline{x}$ and ${}_1\bar{x}$ successively into (11) and using (12) and (13),

$$(14) \qquad V(0, 0) = 0 \text{ ,} \qquad V(1, 1) = 1 \text{ .}$$

7. Corresponding levels of immediate and prospective utility. In this section, we shall study the question whether, if one of the two utilities, immediate (u) or prospective (U) is given, one can find a value for the other one that equates prospective and aggregate utility,

$$V(u, U) = U \quad (15)$$

A pair (u, U) that satisfies this condition will be called a pair of corresponding (immediate and prospective) utility levels. One interpretation of this correspondence is that the immediate utility level u just compensates for the postponement of a program with aggregate utility U by one period. Another still simpler interpretation will be given later.

The existence of a prospective utility U corresponding to a given immediate utility u is readily established. Let u be a point of I_u. Then there exists a one-period consumption vector x such that $u(x) = u$. The aggregate utility $U({}_{con}x)$ of the constant program in which x is repeated indefinitely then satisfies, by (11),

$$U({}_{con}x) = V\big(u(x), U({}_{con}x)\big) , \quad (16)$$

because a shift in time does not modify the program. Hence $U = U({}_{con}x)$ meets the condition (15).

We shall now prove that for each u there is only one corresponding U, which represents a continuous increasing function

$$U = W(u) \text{ , with } W(0) = 0 , \quad W(1) = 1 , \quad (17)$$

of u, to be called the correspondence function. It follows from this that, conversely, to each U there is one and only one corresponding u. Figure 3 illustrates the connection between $V(u, U)$ and $W(u)$.

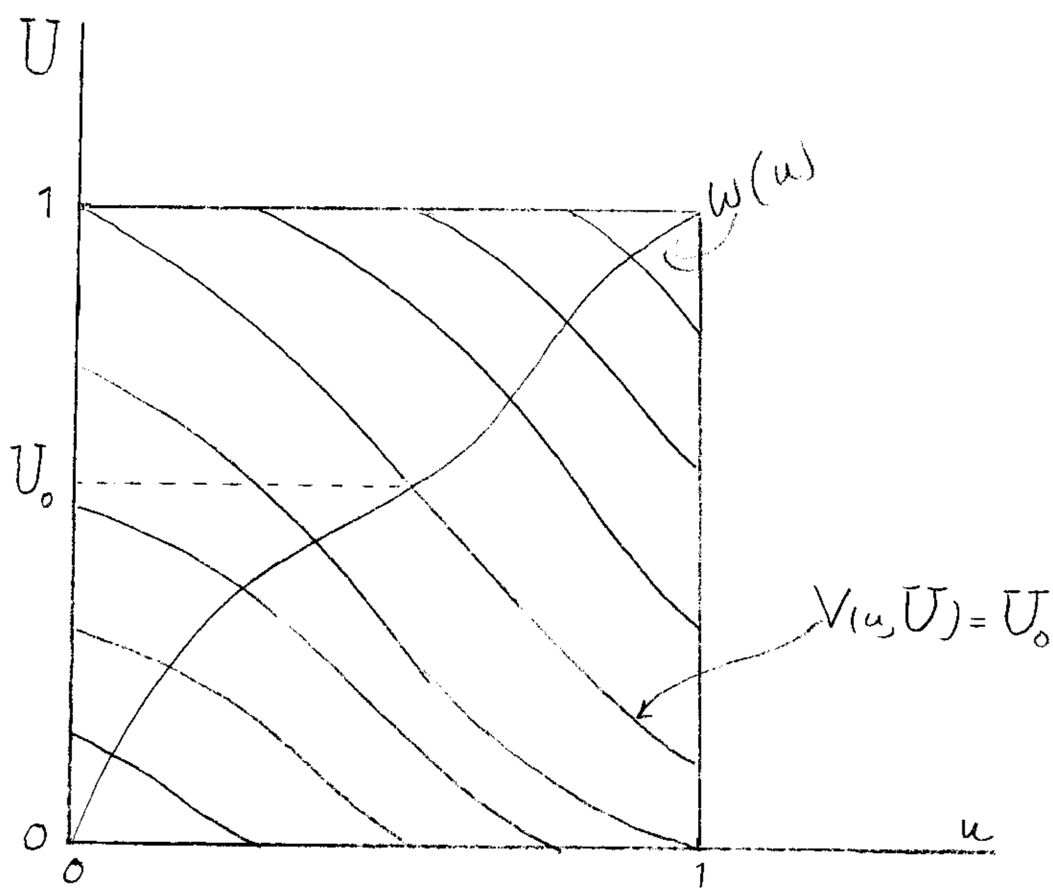

Figure 3

7* We proceed by a sequence of lemmas. With a view to possible later study of the case where no best or worst program exists, Postulate 4 is not assumed* in this section 7* (unless otherwise stated). Reference to the reasoning of section 6 shows that in any case $I_u = I_U$, regardless of whether one or both of the endpoints 0, 1 of I_U are included in I_U.

Lemma 1a. Let $u \epsilon I_u$, $U \epsilon I_U$ satisfy (15) with $u < 1$. Then there exists no $U'' \epsilon I_U$ such that $U'' > U$ and

$$V(u, U') - U' \geqq 0 \quad \text{for all } U' \text{ such that } U < U' \leqq U''.$$

* However, we do assume that I_u and I_U are proper intervals, even if open or half-open.

Proof: Suppose there were such a U''. There exist a vector x and a program $_1x$ such that

$$u(x) = u , \qquad U(_1x) = U .$$

Since $u < 1$, the reasoning of section 4* shows that we can indeed choose x in such a way that every neighborhood of x in X contains points x' with $u(x') > u$. Consider the programs

$$(18) \qquad \begin{cases} {}_1x^{(\tau)} \equiv (\overbrace{x, x, \ldots, x}^{\tau \text{ components}}, {}_1x) \\ {}_1x'^{(\tau)} \equiv (x', x', \ldots, x', {}_1x) \end{cases}$$

Because of (15),

$$U(_1x^{(\tau)}) = U(_1x^{(\tau-1)}) = \ldots = U(_1x) = U \quad \text{for all} \quad \tau .$$

Choosing U''', U^{IV} such that $U < U''' < U^{IV} < U''$, we can therefore because of Postulate 1 choose $\delta > 0$ such that, for all τ,

$$\max_t |x'_t - x_t^{(\tau)}| \leqq \delta \text{ implies } U(_1x') \leqq U'''.$$

Choosing next x' such that $|x' - x| \leqq \delta$ and $u' \equiv u(x') > u$, we have in particular

$$(19) \qquad U(_1x'^{(\tau)}) \leqq U''' \quad \text{for all} \quad \tau .$$

Since $u' > u$ the function $V(u', U') - V(u, U')$ is positive. As it is also continuous for $U \leqq U' \leqq U''$, we have

$$\epsilon' \equiv \min_{U \leqq U' \leqq U''} (V(u', U') - V(u, U')) > 0 ,$$

and

$$\epsilon \equiv \min \left\{ \epsilon', \ U'' - U^{IV} \right\} > 0 \ .$$

Using, with regard to any program ${}_1x$, the notations

$$(20) \quad \begin{cases} {}_1u \equiv u({}_1x) \equiv \big(u(x_1),\ u(x_2),\ \ldots\big) \equiv (u_1,\ u_2,\ \ldots) \\ V_\tau({}_1u,\cdot U) \equiv V\big(u_1,\ V(u_2,\ \ldots,\ V(u_\tau,\ U)\ \ldots)\big) \ , \end{cases}$$

we then have, as long as $\tau\epsilon \leqq U'' - U$, and if ${}_{con}u' \equiv (u',\ u',\ \ldots)$,

$$\begin{aligned} U({}_1x'^{(\tau)}) &= V_\tau({}_{con}u';\ U) = V_{\tau-1}\big({}_{con}u';\ \ V(u',\ U)\big) \geqq \\ &\geqq V_{\tau-1}\big({}_{con}u';\ V(u,\ U)\ +\epsilon\big) = V_{\tau-1}({}_{con}u';\ U\ +\epsilon) = \\ &= V_{\tau-2}\big({}_{con}u';\ V(u',\ U\ +\ \epsilon)\big) \geqq V_{\tau-2}\big({}_{con}u';\ V(u,\ U\ +\ \epsilon)\ +\ \epsilon\big) \geqq \\ &\geqq V_{\tau-2}({}_{con}u';\ U\ +\ 2\epsilon) \geqq \ldots \geqq V\big(u',\ U\ +\ (\tau-1)\epsilon\big) \geqq U\ +\ \tau\epsilon. \end{aligned}$$

But then we can choose τ such that $U + \tau\epsilon \leqq U''$ but

$$U({}_1x'^{(\tau)}) \geqq U\ +\ \tau\epsilon \ \ \geqq U^{IV} \ ,$$

a contradiction with (19) which proves Lemma 1 . The reasoning is illustrated in Figure 4, where the locus $\left\{(u',\ U',\ \middle|\ V(u',\ U') = U'\right\}$ is drawn in a manner proved impossible in Lemma 1.

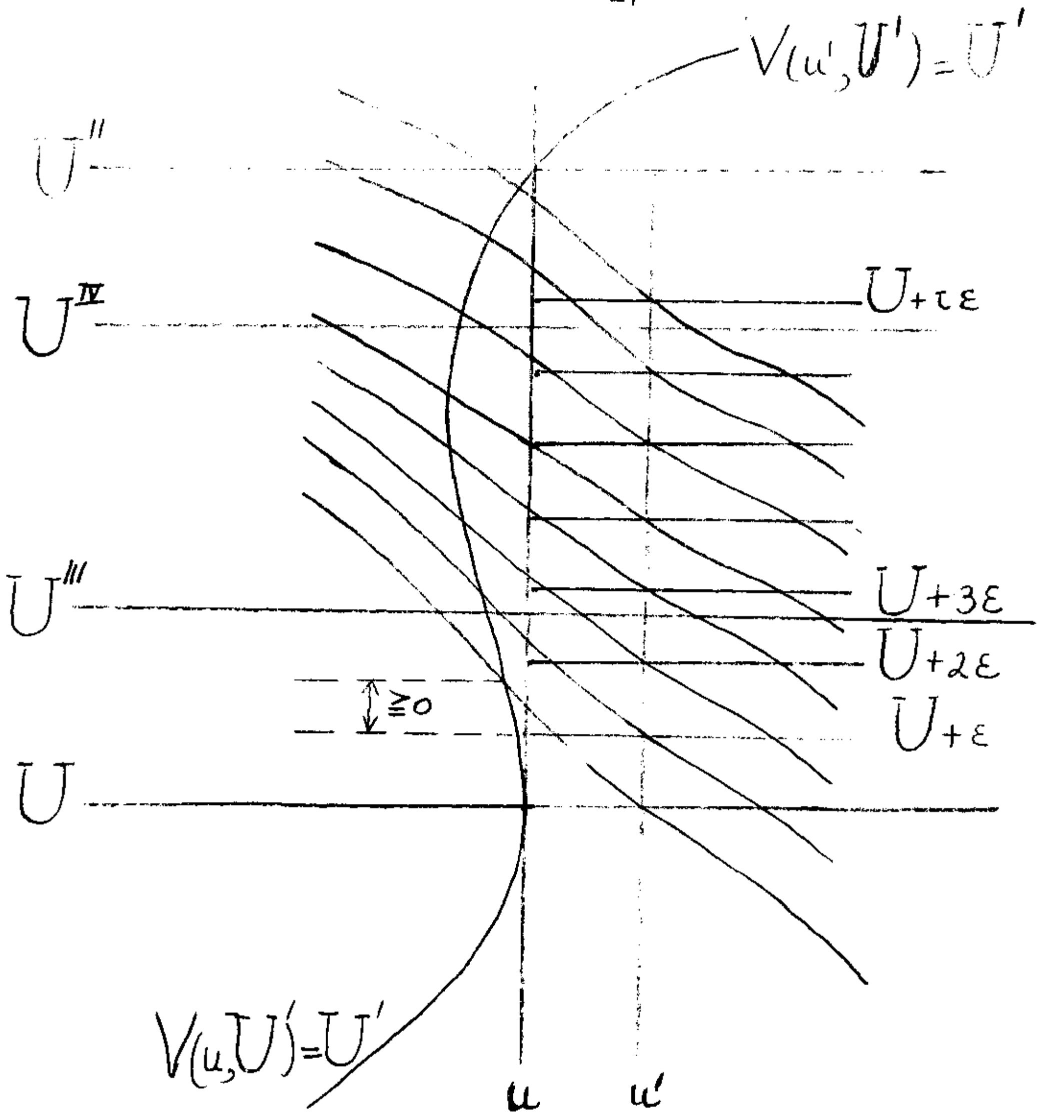

Figure 4.

Symmetrically, we have

Lemma 1b. Let $u \epsilon I_u$, $U \epsilon I_U$ satisfy (15) with $u > 0$. Then there exists no $U \epsilon I_U$ such that $U'' < U$ and

$$V(u, U') - U' \leqq 0 \quad \text{for all } U' \text{ such } U'' \leqq U' < U .$$

We can now prove, if $\overline{I}_u$ denotes the closure of I_u ,

Lemma 2. Let $u \epsilon I_u$, $U \epsilon I_U$ satisfy (15) with $0 < u < 1$. Then

(21) $\quad V(u', U') - U' < 0$ <u>for all</u> $u' \epsilon \bar{I}_u$, $U' \epsilon I_U$ <u>with</u> $u' \leqq u$, $U' \geqq U$,
<u>except</u> $(u', U') = (u, U)$.

(22) $\quad V(u', U') - U' > 0$ <u>for all</u> $u' \epsilon \bar{I}_u$, $U' \epsilon I_U$ <u>with</u> $u' \geqq u$, $U' \leqq U$,
<u>except</u> $(u', U') = (u, U)$.

Proof (see Figure 5): We first prove (21) with $u' = u$ by considering its negation. This says that there exists $U'' \epsilon I_U$ with $U'' > U$ such that $V(u, U'') - U'' \geqq 0$. But this implies by Lemma 1a that there exists U''' with $U < U''' < U''$ such that $V(u, U''') - U''' < 0$, and by the continuity of $V(u, U') - U'$ with respect to U' that there exists a U^{IV} with $U''' < U^{IV} \leqq U''$ such that $V(u, U^{IV}) - U^{IV} = 0$ and $V(u, U') = U' < 0$ for $U''' \leqq U' < U^{IV}$. Inserting U^{IV} for U and U''' for U'' in Lemma 1b we find these statements in contradiction with Lemma 1b. This proves (21) with $u' = u$. The remaining cases with $u' < u$, $U' \geqq U$ follow from the increasing property of $V(u', U')$ with respect to u' . The proof of (22) is symmetric to that of (21).

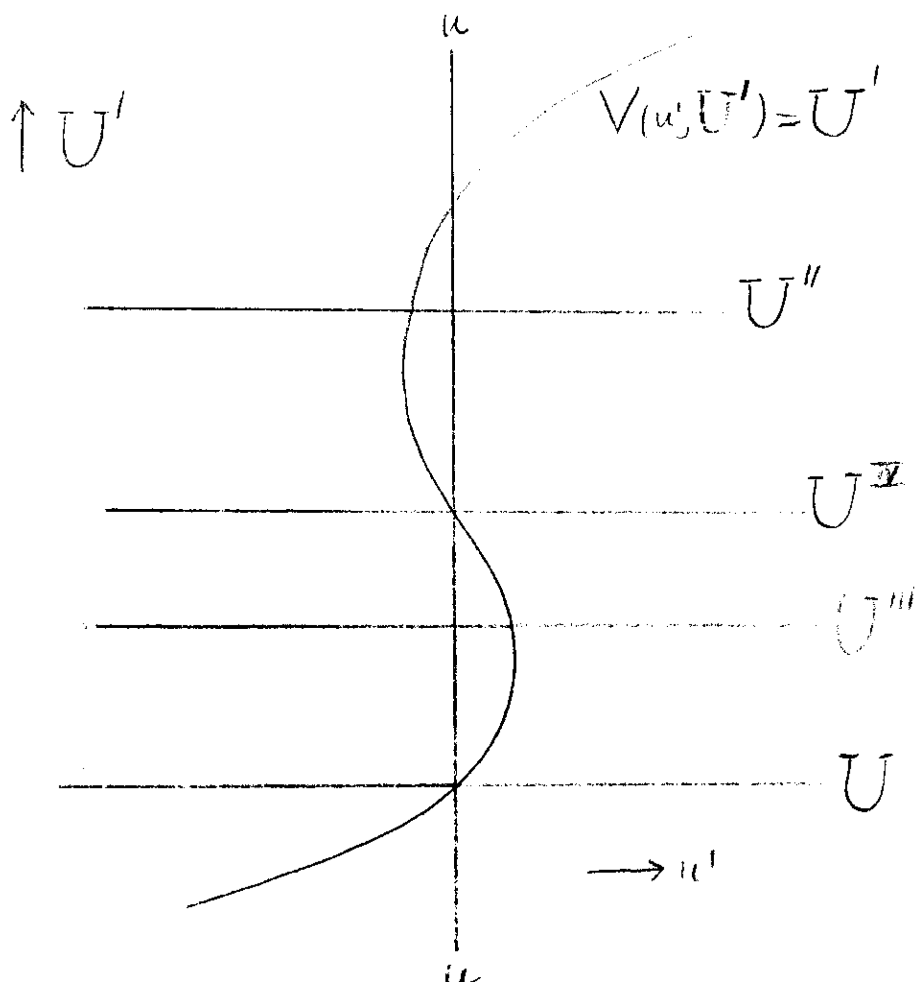

Figure 5.

Since we know already that there exists for each $u \epsilon I_u$ at least one corresponding U, it follows from Lemma 2 that if $0 < u < 1$ there exists precisely one, to be denoted $W(u)$, and that $W(u)$ increases with u. Moreover, if for $0 < u < 1$ we had

$$W(u) < \lim_{u' \to u+0} W(u') \equiv W(u + 0)$$

the continuity of $V(u, U)$ would entail the existence of two prospective utility levels, $W(u)$ and $W(u + 0)$, corresponding to the immediate utility level u, contrary to Lemma 2. Hence $W(u)$ is continuous for $0 < u < 1$, and since $0 \leqq W(u) \leqq 1$ can be extended by

$$W(0) \equiv \lim_{u \to 0} W(u), \qquad W(1) \equiv \lim_{u \to 1} W(u)$$

to be continuous and increasing for $0 \leqq u \leqq 1$.

Now if $0 \epsilon I_U$ and hence $0 \epsilon I_u$, we must have $W(0) = 0$, because $W(0) > 0$ would create a contradiction between (14) and Lemma 1a (with 0 substituted for U, and U for U''), since $V(0, U') - U' < 0$ for any U' such that $0 < U' \leqq W(0)$ is precluded by Lemma 2 and the continuity of $V(u, U')$. A similar reasoning for the case $1 \epsilon I_u$ completes the proof of (17).

8. <u>Equivalent constant program</u>. Now that the correspondence of utility levels u, U has been shown to be unique and reversible, another interpretation is available. Given an aggregate utility level U, find the corresponding immediate utility u, and a one-period consumption vector x for which it is attained, $u(x) = u$. Then we can reinterpret (16) to mean that the program ${}_{con}x$ obtained by indefinite repetition of the vector x again has the given aggregate utility $U({}_{con}x) = U$. The correspondence (17) therefore gives us a means to associate with any program a constant program of the same aggregate utility.

8* If Postulate 4 is not assumed, the possibility exists of a program ${}_1x$ with successive one-period utility levels $u(x_t)$ increasing (or decreasing) with t in such a way that no equivalent constant program, and no compensation for a postponement of ${}_1x$ by one period, exist.

9. Equating corresponding utility levels. The correspondence function $W(u)$ can be used to change the scale of one of the two utility types, for instance of u, in such a way as to equate corresponding utility levels. The appropriate increasing transformation is defined by

$$(23) \qquad \begin{aligned} u^*(x) &\equiv W(u(x)) , \qquad U^*({}_1x) \equiv U({}_1x) , \\ V^*(u^*,U^*) &\equiv V(W^{-1}(u^*),U^*) , \end{aligned}$$

where $u = W^{-1}(u^*)$ is the inverse of $u^* = W(u)$. If now u^* and U^* represent corresponding utility levels on the new scales, we have

$$0 = V^*(u^*, U^*) - U^* = V(W^{-1}(u^*), U) - U ,$$

and hence, by the definition of $W(u)$,

$$U^* = U = W(W^{-1}(u^*)) = u^* .$$

Hence the new correspondence function $U^* = W^*(u^*)$ is simply the identity $U^* = u^*$, represented in the new form of Figure 3 by the diagonal connecting (0,0) with (1, 1). Although this change of scale is not essential for any of the reasoning that follows, we shall make it in order to simplify formulae and diagrams. Dropping asterisks again, the correspondence relation (15) now takes the form

$$(24) \qquad V(U, U) = U .$$

10. Repeating programs. A program in which a given sequence ${}_1x_\tau$ of τ one-period vectors $x_1, x_2, \ldots, x_\tau$ is repeated indefinitely will be called a repeating program, to be denoted

$$ {}_{rep}x_\tau \equiv ({}_1x_\tau, {}_1x_\tau, \ldots) . $$

The sequence ${}_1x_\tau$ will be called the theme of the repeating program, τ its span, provided no $\tau' < \tau$ exists permitting the same form. We shall use the notations

$$ {}_{rep}u_\tau \equiv ({}_1u_\tau, {}_1u_\tau, \ldots) $$

$$ {}_1u_\tau \equiv u({}_1x_\tau) \equiv (u(x_1), \ldots, u(x_\tau)) \equiv (u_1, \ldots, u_\tau) $$

for the corresponding sequences of one-period utility levels, and call ${}_1u_\tau$ the utility theme corresponding to ${}_1x_\tau$. The function

(25) $$ V({}_1u_\tau ; U) \equiv V(u_1, V(u_2) \ldots, V(u_\tau, U) \ldots) $$

then indicates how the utility level U of any program is modified if that program is postponed by τ periods and a theme with corresponding utility theme ${}_1u_\tau$ is inserted to precede it.

Given a utility theme ${}_1u_\tau = u({}_1x_\tau)$, we can now ask whether there is a utility level U which is not affected by such a postponement,

(26) $$ V({}_1u_\tau; U) = U . $$

Obviously, the utility level

(27) $$ U = U({}_{rep}x_\tau) $$

meets this requirement, because the program ${}_{rep}x_\tau$ itself it not modified by such postponement. By an analysis entirely analogous to that already given for the case $\tau = 1$, one can show that this utility level is unique and hence is a function *

(28) $$U = W({}_1u_\tau)$$

of the utility theme; and that this function is continuous and increasing with respect to each of the variables $u_1, \ldots, u_\tau$. Finally, as before in the case $\tau = 1$,

(29) $$U \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} V({}_1u_\tau;\ U) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} W({}_1u_\tau) \quad \text{if} \quad U \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} W({}_1u_\tau) .$$

10* The uniqueness of the solution of (26), and the first set of inequalities in (29), are proved by having an arbitrary one of the variables $u_1, \ldots, u_\tau$ play the role performed by u in section 7*. To prove continuity and monotonicity of $W({}_1u_\tau)$, that role is assigned successively to each of these variables. The second set of inequalities in (29) then follows from (26), (28) and the fact that $V({}_1u_\tau;\ U)$ increases with U.

To obtain one further interesting result we revert to the notation (20). By repeated application of (29) we have, for $n = 1, 2, \ldots,$

(30) $$\begin{gathered} U'' < U = W({}_1u_\tau) < U' \quad \text{implies} \\ V_{n\tau}({}_{rep}u_\tau;\ U'') < V_{n\tau}({}_{rep}u_\tau;\ U) = U < V_{n\tau}({}_{rep}u_\tau;\ U') , \end{gathered}$$

where $V_{n\tau}({}_{rep}u_\tau, U''')$ is increasing with n if $U''' < U$, decreasing if $U''' > U$. It follows that

(31) $$\lim_{n \to \infty} V_{n\tau}({}_{rep}u_\tau;\ U''')$$

exists for all $U''' \epsilon I_U$. But for any such U''' insertion of (31) for U in (26) satisfies that condition, which we know to be satisfied by U only. Hence,

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* The function $W({}_1u_\tau)$ is a generalized correspondence function, interpretable either as the aggregate utility of any program, the postponement of which by τ (cont.)

by (28),

(32) $$\lim_{n \to \infty} V_{n\tau}({}_{rep}u_\tau;\, U''') \equiv V_\infty({}_{rep}u_\tau) = W({}_1u_\tau) \quad \text{for all} \quad U'''\epsilon I_U \,.$$

11. Alternating programs and impatience. A repeating program with a span $\tau = 2$ will be called an alternating program. Its one-period utility sequence alternates between two different levels, u' and u'', say, which we shall always choose such that

(33) $$u' > u'' \,.$$

If we write $w' \equiv (u', u'')$, $w'' \equiv (u'', u')$ for the two possible utility themes, the two possible alternating programs have the respective utility sequences

(34) $$\begin{cases} (34') & {}_{rep}w' \equiv (u', u'', u', u'', \ldots) \\ (34'') & {}_{rep}w'' \equiv (u'', u', u'', u', \ldots) \end{cases}$$

The implications of the preceding analysis for this type of program are illustrated in Figure 6. The aggregate utility level U' corresponding to (34'),

(35) $$U' \equiv W(w') = V\Big(u',\, V\big(u'',\, V(u', \ldots)\big)\Big) \,,$$

satisfies the condition

(36) $$\Phi'(U') \equiv V\big(u',\, V(u'', U')\big) - U' = 0 \,.$$

Hence U' can be read off, as indicated in Figure 6, from a quadrilateral consisting of two horizontals and two niveau lines (drawn solid), with two vertices on the diagonal of the unit square, the other two vertices on the verticals at

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(cont.) periods can just be compensated by insertion of a sequence ${}_1x_\tau$ with $u({}_1x_\tau) = {}_1u_\tau$, or as the aggregate utility of the repeating program ${}_{rep}({}_1x_\tau)$, where again $u({}_1x_\tau) = {}_1u_\tau$.

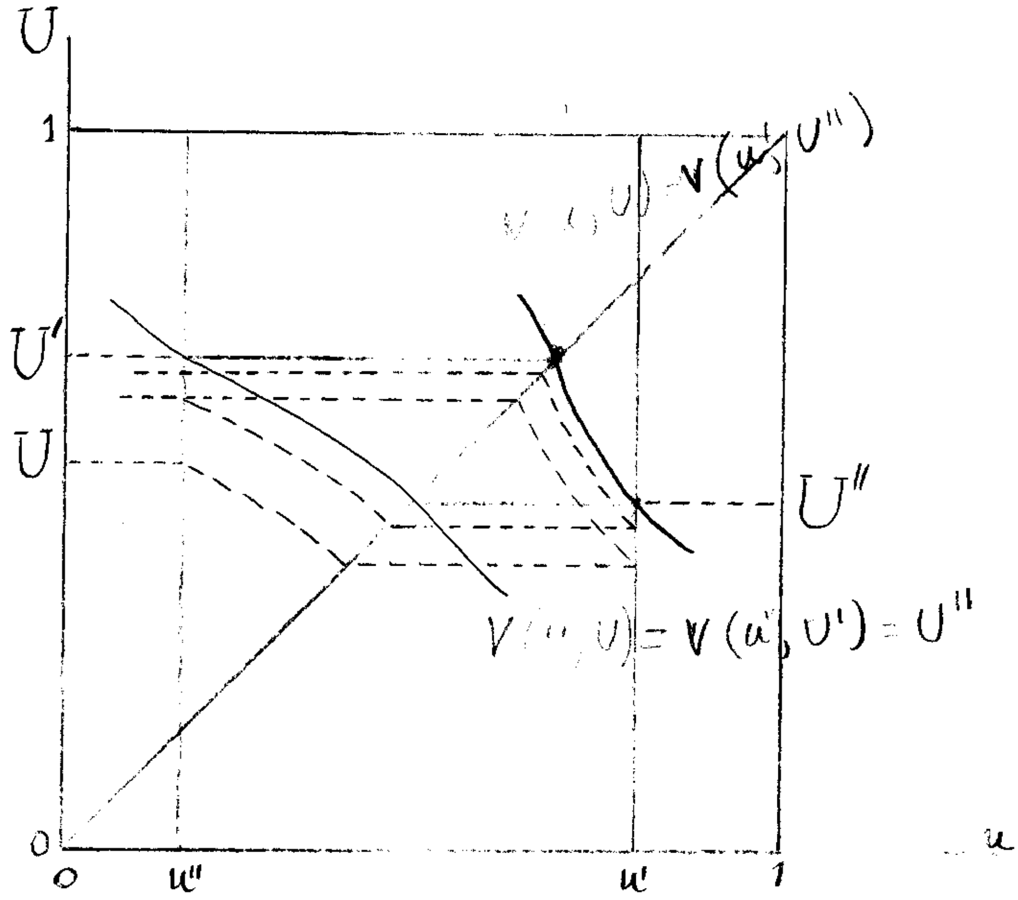

Figure 6

u = u' and u = u'' , respectively. Enlarging on (36), we also have from (29)

$$(37) \qquad 0 \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} \Phi'(U) \equiv V(u', V(u'', U)) - U \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} U' - U \quad \text{if} \quad U \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} U' \ .$$

Hence, for any program with an aggregate utility $U \neq U'$, postponement by two periods with insertion of the utility theme (u', u'') in the first two periods thereby vacated, will bring the aggregate utility closer to U', without overshooting, and indefinite repetition of this operation will make the aggregate utility approach U' as a limit see dotted lines for a case with $U < U'$) .

Symmetrically to (37), we have

$$(38) \qquad 0 \left\{\begin{matrix} < \\ = \\ > \end{matrix}\right\} \Phi''(U) \equiv V(u'', V(u', U)) - U \begin{matrix} < \\ = \\ > \end{matrix} U'' - U \quad \text{if} \quad U \left\{\begin{matrix} < \\ = \\ > \end{matrix}\right\} U'' ,$$

with similar interpretations, and where U'' is related to U', u'' and u' by

$$(39) \qquad u'' < U'' = V(u'', U') < U' = V(u', U'') < u' ,$$

as indicated in Figure 6, and proved in detail below.

We are now ready to state a definition of impatience, and to draw inferences about the presence of impatience in certain parts of the utility space.

Definition 1. A program ${}_1x$ with the first two one-period utility levels $u(x_1) = u'$ and $u(x_2) = u''$ (where $u' > u''$), and a subsequent prospective utility level $U({}_3x) = U$, will be said to meet the impatience condition if

$$(40) \qquad \Phi(U) \equiv V\big(u', V(u'', U)\big) - V\big(u'', V(u', U)\big) > 0 .$$

Essentially, this says that, if the one-period utility of the first-period consumption vector x_1 exceeds that of the second-period vector x_2, an interchange of the two vectors leads to a decrease in aggregate utility.

We note that

$$(41) \qquad \Phi(U) = \Phi'(U) - \Phi''(U) .$$

Reference to (37) and (38), or to Figure 7 in which the implications of (37) and (38) are exhibited, shows that, since $\Phi'(U) > 0$ for $U'' \leqq U < U'$ and $\Phi''(U) < 0$ for $U'' < U \leqq U'$, we have

$$(42) \qquad \Phi(U) > 0 \quad \text{for} \quad U'' \leqq U \leqq U' .$$

This proves the presence of impatience in a central zone of the space of the utility triples (u', u'', U), as illustrated in Figure 8. It is to be noted that the

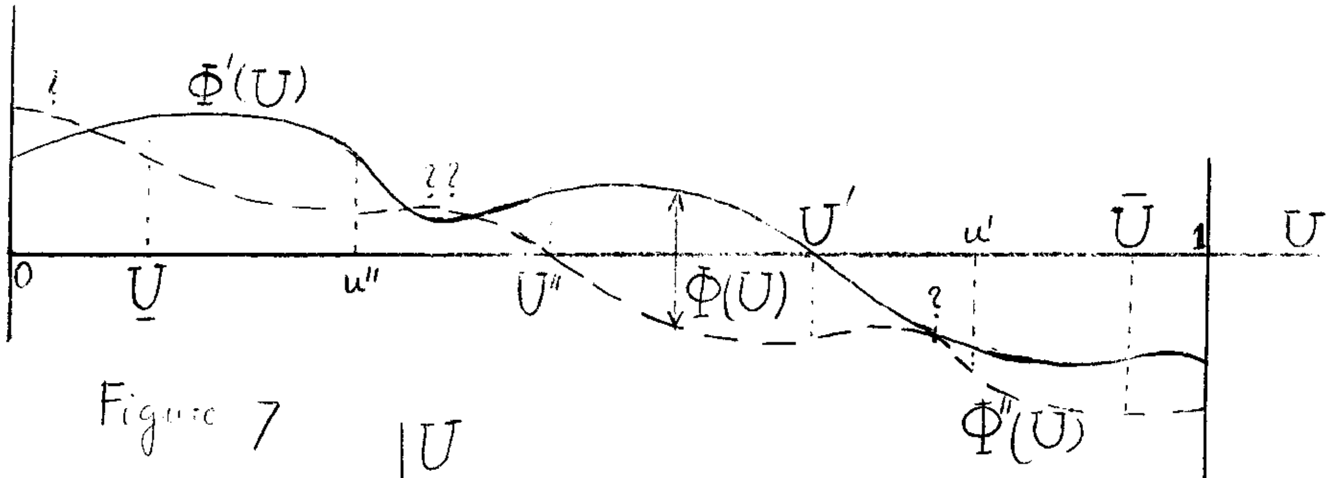

Figure 7

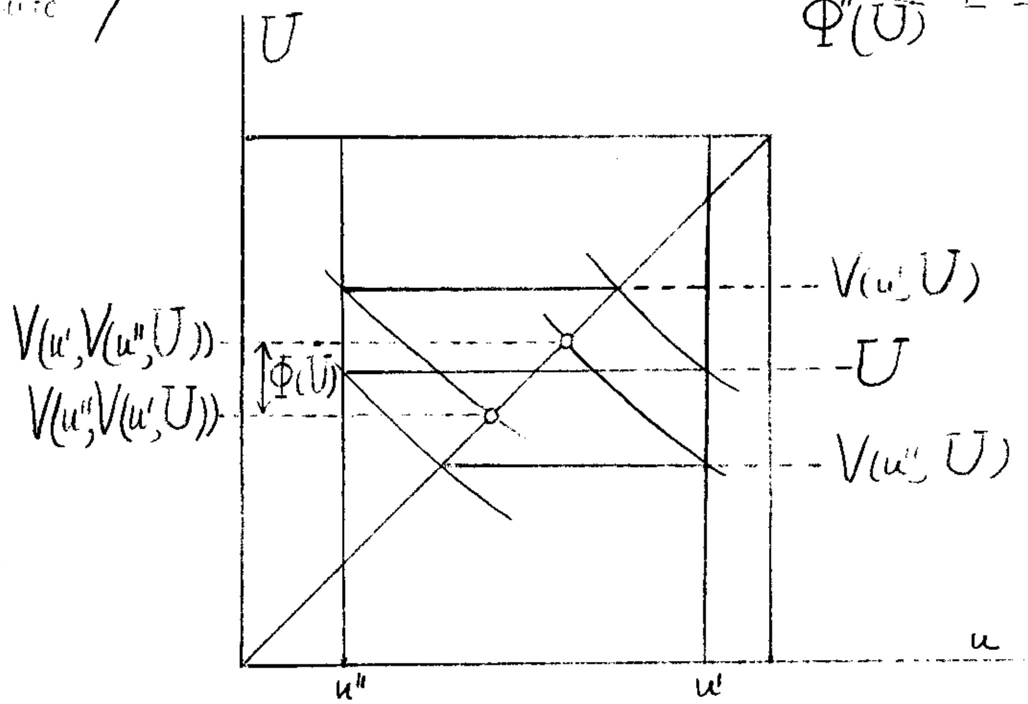

Figure 8

result (42) is obtained as long as the two marked points do not fall on the same side of the horizontal at U. This is the case precisely if $U'' \leqq U \leqq U'$.

Two other zones can be added to this one, on the basis of the monotonicity of $V(u, U)$ with respect to U. If we define $\underline{U}$, $\bar{U}$ by

$$V(u', \underline{U}) = u'', \qquad V(u'', \bar{U}) = u', \tag{43}$$

if solutions of these equations exist, and by $\underline{U} = 0$, and/or $\bar{U} = 1$ otherwise, Figure 9 suggests that

(44) $\Phi(U) > 0$ for $\underline{U} \leqq U \leqq u''$ and for $u' \leqq U \leqq \bar{U}$.

A detailed proof is given below.

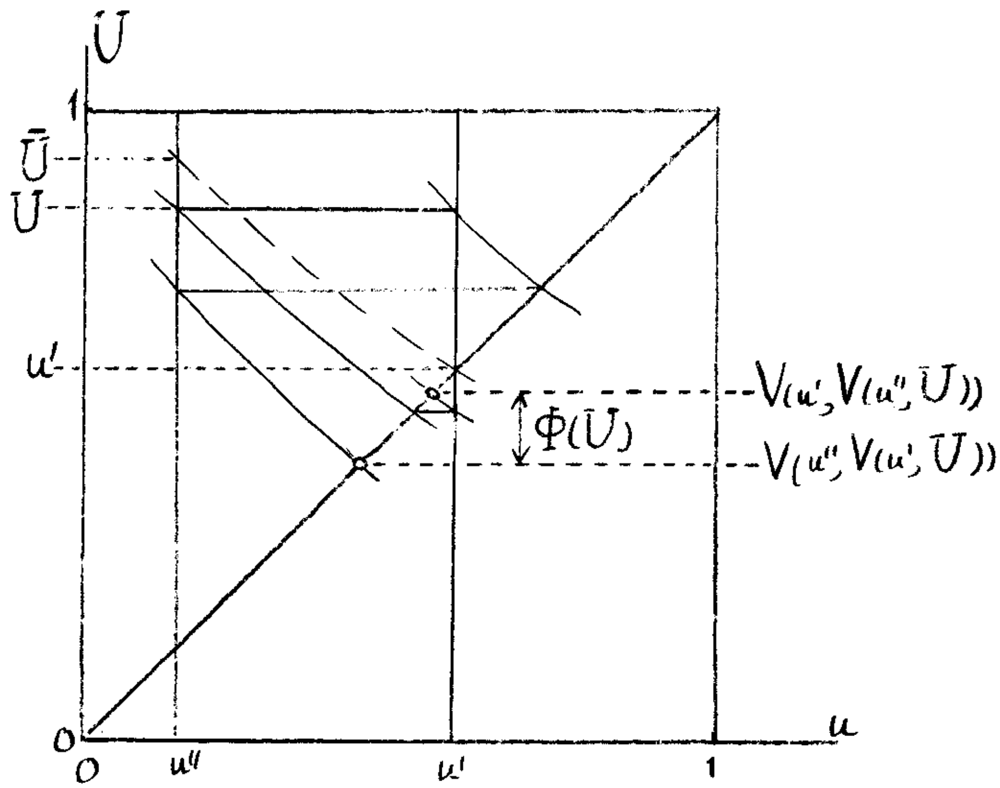

Figure 9.

There are indications that in the intermediate zones, $u'' < U < U''$ and $U' < U < u'$, impatience is the general rule, neutrality toward timing a conceivable exception. The behavior of $\Phi(U)$ in these zones will not be analyzed further in this paper, in the hope that an argument simpler than that which has furnished these indications may still be found.

For the sake of generality of expression, we shall state the present results in a form that does not presuppose the, convenient but inessential, transformation introduced in section 8 to equate corresponding utility levels.

Theorem 1. If Postulates 1, 2, 3 and 4 are satisfied, a program ${}_1x$ with first- and second-period utilities $u' = u(x_1)$ and $u'' = u(x_2)$ such that $u' > u''$ and with prospective utility as-from-the-third-period $U = U({}_3x)$ meets the condition (40) of impatience in each of the following three zones:

(a) If U equals or exceeds the utility of a constant program indefinitely repeating the vector x_1, provided U is not so high (if that should be possible) that the utility of the program $(x_2, {}_3x)$ exceeds that of the constant program $(x_1, x_1, x_1, \ldots)$,

(b) If U equals the utility of either of the alternating programs

$$(x_1, x_2, x_1, x_2, \ldots)$$
$$(x_2, x_1, x_2, x_1, \ldots)$$

or falls between these two utility levels,

(c) If U equals or falls below the utility of the constant program $(x_2, x_2, x_2, \ldots)$, provided U is not so low (if that should be possible) that the utility of the program $(x_1, {}_3x)$ falls below that of the constant program $(x_2, x_2, x_2, \ldots)$.

This is in a way a surprosing result. The phenomenon of impatience was introduced by Boehm Bawerk as a psychological characteristic of human economic preference indecisions concerning (presumably) a finite time horizon. It now appears that impatience is also, at least in one central and two outlying zones of the space of programs, a necessary logical consequence of more elementary properties of a preference ordering of programs with an infinite time horizon: continuity (uniform on each equivalence class), period-wise aggregation, independence of calendar time (stationarity), and the existence of extreme programs.

11* In order to prove the relations (39) and (44) on which Theorem 1 depends, without reference to a diagram, we lift from the already proved statements (37) and (38) the defining relations

$$(45'') \text{ and } (45') \qquad V\big(u'', V(u', U'')\big) = U'' , \qquad V\big(u', V(u'', U')\big) = U' ,$$

of U'' and U', respectively. From (45') we read that $V\Big(u'', V\big(u', V(u'', U')\big)\Big) =$ $= V(u'', U')$, showing that $V(u'', U')$ satisfies the defining relation (45'') of U''. This, and an argument symmetric to it, establish the equalities in (39). Now assume first that $U'' < U'$. In that case, because $V(u, U)$ increases with U,

$$0 = V(u', U'') - U' < V(u', U') - U' ,$$

whence $U' < u'$ by Lemma 2, since $V(u', u') - u' = 0$. By similar reasoning, $U'' > u''$, establishing the inequalities in (39) for the present case. But the same reasoning applied to the assumption $U'' \geqq U'$ would entail $u'' \geqq U'' \geqq U' \geqq u'$, which is contradicted by the datum that $u' > u''$. This completes the proof of (39).

To prove (44) we note that, given u', u'' with $u' > u''$,

$$\text{if } \left\{\begin{matrix} U = u' \\ u' < U < \overline{U} \\ U = \overline{U} \end{matrix}\right\} \text{ then } \; V(u',U)\left\{\begin{matrix} = \\ < \\ < \end{matrix}\right\}U, \quad V\big(u'',V(u',U)\big)\left\{\begin{matrix} = \\ < \\ < \end{matrix}\right\}V(u'',U)\left\{\begin{matrix} < \\ < \\ = \end{matrix}\right\}u' ,$$

using in succession (24), Lemma 2, the monotonicity of $V(u,U)$ with respect to U, (43). But then also

$$V\big(u', V(u'', U)\big)\left\{\begin{matrix} > \\ > \\ = \end{matrix}\right\}V(u'', U) ,$$

using again (24) and Lemma 2. A comparison of these results establishes (44).

The forms here given to the proofs of (39) and (44) have been chosen so that they may carry over by mere reinterpretation to a more general case to be considered in a later paper.

12. Period-wise Independence. It might seem only a small additional step if to Postulate 2 we add the further

Postulate 2' (2'a and 2'b). For all $x_1, x_2, {}_3x, x'_1, x'_2, {}_3x'$,

(2'a) $U(x_1, x_2, {}_3x) \geqq U(x'_1, x'_2, {}_3x)$ implies $U(x_1, x_2, {}_3x') \geqq U(x'_1, x'_2, {}_3x')$,

(2'b) $U(x_1, x_2, {}_3x) \geqq U(x'_1, x_2, {}_3x')$ implies $U(x_1, x'_2\ {}_3x) \geqq U(x'_1, x'_2, {}_3x')$.

In fact, it follows from a result of Debreu [1959], that this would have quite drastic implications. Postulates 1-4 and 2' together satisfy the premises of a theorem* which, translated in our notation and terminology, says that one can

* l.c., section 3

find a monotonic transformation of $U({}_1x)$ such that

(46) $$U({}_1x) = u_1(x_1) + u_2(x_2) + U_3({}_3x)$$

Taken in combination with the stationarity Postulate 3, this would leave only the possibility that

(47) $$U({}_1x) = \sum_{t=1}^{\infty} \alpha^{t-1}\, u(x_t)\ , \qquad 0 < \alpha < 1 .$$

that is, aggregate utility is a disconnected sum of all future one-period utilities, with a constant discount factor α . This form has been used extensively in the literature.** Since the form (47) is destroyed by any other transformations than

** See, for instance, Ramsay [1928], Samuelson and Solow [1956], Strotz [1957]. The first two publications find a way to make $\alpha = 1$.

increasing linear ones, one can look on Postulate 2' (as Debreu does) as a basis (in conjunction with the other postulates) for defining a cardinal utility function (47). While this in itself is not objectionable, the constant discount rate seems too rigid to describe important aspects of choice over time. If for the sake of argument we assume that our function $V(u, U)$ is differentiable, it is easily seen that the discount factor

$$\left(\frac{\partial V(u, U)}{\partial U}\right)_{U=u}$$

is invariant for differentiable monotonic transformations, but can take different values for different common values of $U = u$. The main purpose of the system of postulates of this paper therefore is to clarify behavior assumptions that will permit the relative weight given to the future as against the present to vary with the level of all-over satisfaction attained - a consideration which can already be found in the work of Irving Fisher [1930].

13. Correction to Section 6. It is necessary to split Postulate 4 into two separate Postulates 4 and 5, as follows:

Postulate 4 (Sensitivity). There exist first-period consumption vectors x_1, x_1' and a program ${}_2x$ from-the-second-period-on, such that

$$U(x_1, {}_2x) > U(x_1', {}_2x)$$

Postulate 5 (Extreme Programs). There exist ${}_1\underline{x}$, ${}_1\bar{x}$ such that

$$U({}_1\underline{x}) \leqq U({}_1x) \leqq U({}_1\bar{x}) \quad \text{for all } {}_1x .$$

The difference with the old Postulate 4 is that sensitivity is now postulated specifically with respect to the first-period consumption vector, rather than merely with respect to the program as a whole. Without that, the interval I_u may shrink to a single point, as shown by an example suggested by Herbert Scarf:

$$U({}_1x) = \lim_{\tau \to \infty} \sup_{t \geqq \tau} (1 - e^{-x_t}), \quad x_t \text{ scalar and } \geqq 0 .$$

REFERENCES

Böhm-Bawerk, E. von [1912], Positive Theorie des Kapitales, Dritte Auflage, 1912, especially Buch IV, Abschnitt I, Gegenwart und Zukunft in der Wirtschaft, pp. 426-486.

Debreu, G., "Topological Methods in Cardinal Utility Theory," Cowles Foundation Discussion Paper No. 76, to appear in Mathematical Methods in the Social Sciences, 1959, Stanford University Press.

Fisher, Irving, The Theory of Interest, New York, 1930, Chapter IV, especially §§ 3 and 6.

Ramsay, F. P., "A Mathematical Theory of Saving," Economic Journal, December 1928, pp. 543-559.

Samuelson, P. A. and Solow, R., "A Complete Capital Model Involving Heterogeneous Capital Goods," Quarterly Journal of Economics, November, 1956, pp. 537-562.

Strotz, R. H., "Myopia and Inconsistency in Dynamic Utility Maximization," Review Economic Studies, XXIII, 3 (1957), pp.