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On "An Identity in Arithmetic"*

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On "An Identity in Arithmetic"*

H. D. Block and J. Marschak have presented in the Bulletin of the American Mathematical Society, 65, an identity which arose in a probability context. This note proves it by a probability theoretical argument.

Consider an experiment having the set of possible outcomes $N = \{1, \ldots, n\}$ with positive probabilities $(u_1, \ldots, u_n)$ and an infinite sequence of independent repetitions of that experiment. Let M be a subset of N and i be an element of M, and denote by $A(i,M)$ the event that i is the first element of M which occurs in an infinite sequence of outcomes. If $B_j(i,M)$ denotes the event that the first $(j-1)$st outcomes are not in M and the jth outcome is i, one has

$$A(i,M) = \bigcup_{j=1}^{\infty} B_j(i,M) .$$

Hence $\Pr[A(i,M)] = \sum_{j=1}^{\infty} \Pr[B_j(i,M)] = \sum_{j=1}^{\infty} (1-u_M)^{j-1} u_i = \frac{u_i}{u_M}$, writing u_M for $\sum_{j \in M} u_j$, and putting $0^0 = 1$ for the degenerate case $M = N$.

Let now r be a permutation of N and k_r be the element of N ranked kth by r. Let also $C(r)$ be the event that the first occurrences of the n elements of N in an infinite sequence of outcomes appear in the order $r = (1_r, \ldots n_r)$. If $D_{j_1, \ldots, j_n}(r)$ denotes the event that k_r occurs for

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the first time at the j_kth performance of the experiment (for every $k = 1, \ldots, n$), one has

$$C(r) = \bigcup_{j_1 < \ldots < j_n} D^{(r)}_{j_1, \ldots, j_n} .$$

Hence $\Pr[C(r)] = \sum_{j_1 < \ldots < j_n} \Pr[D^{(r)}_{j_1, \ldots, j_n}] =$

$$\sum_{j_1 < \ldots < j_n} u_{1_r}^{j_2 - j_1} u_{2_r} (u_{1_r} + u_{2_r})^{j_3 - j_2 - 1} u_{3_r} (u_{1_r} + u_{2_r} + u_{3_r})^{j_4 - j_3 - 1} \cdots u_{n_r} .$$

Putting $j_{k+1} - j_k - 1 = h_k$ and $u_{1_r} + \ldots + u_{k_r} = v_{k,r}$, one obtains

$$\Pr[C(r)] = \left(\prod_{j=1}^{n} u_j \right) \sum_{h_1, \ldots, h_{n-1} \geq 0} v_{1,r}^{h_1} v_{2,r}^{h_2} \cdots v_{n-1,r}^{h_{n-1}} = \left(\prod_{j=1}^{n} u_j \right) \prod_{k=1}^{n-1} (1 - v_{k,r})^{-1} .$$

Let finally $R(i,M)$ be the set of permutations of N for which i is ranked first among the elements of M. Since the event that some element of N never occurs has probability zero, one has

$$\Pr[A(i,M)] = \sum_{r \in R(i,M)} \Pr[C(r)] .$$

Hence the desired identity:

$$\frac{u_i}{\sum_{j \in M} u_j} = \left(\prod_{j=1}^{n} u_j \right) \sum_{r \in R(i,M)} \prod_{k=1}^{n} \left(\sum_{j=k}^{n} u_{j_r} \right)^{-1} .$$