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A Dynamic Programming Model of the Consumption Function

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Existing theories of the consumption function recognize that the allocation of income to saving and consumption by a rational consumer involves the element of uncertainty (or at any rate of risk) in an essential way [1], [2]. It is generally assumed that the consumer deals with risk by forming a consumption plan which essentially involves the entire stream of future incomes. But the sequential nature of his decision has not been emphasized. The fact that the consumer may look forward to another decision on his expenditure in the next income period is not explicitly introduced in these decision models. The purpose of this paper is to suggest such a model. This is not the most general Dynamic Programming model that might be constructed, but it may well be the simplest one.

Consider a person with stationary income prospects whose earned income is a random variable. In the simplest case his income is independently and identically distributed in all periods of his future life which we may idealize to be infinite. Let the consumer's utility function for current consumption be given and invariant through time, and let future utilities be discounted at compound discount rates. Assume that borrowing for consumption is impossible. The person's prospects in life are thus fixed by the probability distribution of his income, the nature of his utility of consumption function, the discount rate, and any assets he may own. Since only the assets are subject to change through saving or dissaving, it is a person's asset level that will determine the "state" of the system and his immediate action. The first question is this:

On the basis of an asset position at the beginning of a time period, what is the consumer's optimal strategy, i.e., the best level of consumption he should plan for the next period?

It is relatively simple to write out the implications of consumer's rationality if we introduce an auxiliary function, the "utility of wealth" function. We shall see that this function is completely determined by the utility of consumption function.

<u>Notation</u>

- y income
- x consumption
- s assets
- r interest factor (on assets)
- a discount factor (on utility)
- $u(x)$ utility function of consumption
- $v(s)$ utility of wealth function
- $F(y)$ probability distribution of income

We begin by deriving a utility of wealth function from the utility of consumption. This is defined implicitly

$$(1) \qquad v(s) = \max_{0 \leqq x \leqq s} \left[u(x) + a \int v(r(s-x) + y)\, dF(y)\right] .$$

That is, the utility of one's present asset position equals the maximum value of the utility of consumption in the next period plus the expected value of the utility of wealth at the end of the period, properly discounted. In the general theory of dynamic programming [3] it is shown that an equation of type (1) determines a unique function $v(s)$ as its solution.

1. We show first: If $u(x)$ is concave, so is $v(s)$; diminishing marginal utility of consumption implies decreasing marginal utility of wealth. Define a sequence of v function recursively

$$v_o(s) = u(s)$$

$$v_{n+1}(s) = \max_{0 \leqq x \leqq s} [u(x) + a \int v_n(r\cdot(s-x) + y))\, dF(y)]$$

We show by induction that all $v_n(s)$ are concave. For simplicity write

$$\varphi_n(s-x) = a\int v_n[r(s-x) + y]\, dF(y)$$

If $v_n(t)$ is concave so is $\varphi_n(t)$. Clearly $v_o(s)$ is concave. Suppose that $v_n(s)$ is concave then and let s_o, s_1 be two asset levels. Then

$$v_{n+1}(s_o) + v_{n+1}(s_1) = \max_{0 \leqq x_o \leqq s} [u(x_o) + \varphi_n(s_o - x)] + \max_{0 \leqq x_1 \leqq s} [u(x_1) + \varphi_n(s_1 - x_1)]$$

$$= \max_{0 \leqq x_o, x_1 \leqq s} [u(x_o) + u(x_1) + \varphi_n(s_o - x_o) + \varphi(s_1 - x_1)]$$

$$\leqq \max_{\substack{0 \leqq x_o \leqq s_o \\ 0 \leqq x_1 \leqq s_1}} [2u(\frac{x_o+x_1}{2}) + 2\,\varphi_n\,(\frac{s_o+s_1}{2} - \frac{x_o+x_1}{2})] \text{ by concavity of } u \text{ and } \varphi$$

$$\leqq 2 \cdot \max_{0 \leqq 2x \leqq s_o+s_1} [u(x) + \varphi_n\,(\frac{s_o+s_1}{2} - x)] = 2v_{n+1}(\frac{s_o+s_1}{2})$$

and this proves concavity of v_{n+1}.

To complete the proof one must show that the sequence of functions v_n converges. This is assured by fundamental theorems of dynamic programming [Bellman, Ch. IV].

In the following let u and v be piecewise differentiable functions.

2. We show next: If the utility function is concave $x = 0$ only when $s = 0$; consumption is positive except in the case of zero assets.

Proof: In order for consumption to be zero it is necessary that the derivative of the maximand in (1) be non-positive at $x = 0$

$$u'(0) - ar\int v'[rs + y]\, dF(y) \leqq 0$$

Suppose that s_o is the largest s for which this inequality holds. (Of course for s_o the $=$ sign applies.) Consider any asset position $s_o + x$. Since $u'(x) \leqq u'(0)$ by concavity of u it follows that

$$u'(x) - ar\int v'[r(s_o + x-x) + y] \leqq 0$$

so that consumption would not exceed x and assets would always remain above s_o .

This consumption strategy cannot be optimal since following this strategy and in addition consuming s_o initially would lead to an improvement. This shows that $s_o = 0$ and incidentally that

$$(2) \qquad u'(0) - ar\int v'(y)\, dF(y) \geqq 0 .$$

3. Throughout this section let u be strictly concave and u,v be twice piecewise differentiable. With this assumption we show: Consumption is an increasing function of assets $\frac{dx}{ds} > 0$.

1. Case: The maximizing $x = 0$. As shown this can happen only for $s = 0$ and because of (2) we are then in case 2.

2. Case: $x = s$. This occurs when $u'(s) \geqq ar\int v'[rs + y]\, dF(y)$. If the $>$ sign holds, then $x = s$ in a whole neighborhood of s and x is obviously an increasing function of s. If $=$ applies we are in Case 3.

3. Case: Interior maximum. Maximization of the right hand side of (1) implies

(3) $$u'(x) - ar\int v'[r(s-x)+y]\, dF(y) = 0$$

Regarding this as an implicit equation in x and s, $f(s,x) = 0$, we have

$$\frac{dx}{ds} = -\frac{f_s}{f_x}$$

Maximality implies that $f_x < 0$ and hence

$$\text{sign } \frac{dx}{ds} = \text{sign } f_s$$

and $$f_s = ar^2\int v''[r(s-x) + y]\, dF(y) < 0 .$$

4. Permanent shifts of income are not anticipated by the consumer in this model. But suppose they happen. It is tempting to conclude from (1) that the effect is the same as when assets were raised each time by the increase in income (multiplied by the factor a). But this is fallacious since it disregards the effect on the form of the unknown v function. That a permanent increase of average income is not equivalent to an increase of assets by an amount equal to the capitalized value of this additional income, is a result of the restrictions on borrowing: income may not be capitalized and disposed of like an asset. Hence the utility of wealth function is differently affected in either event.

5. A particularly transparent case is that in which the marginal utility of consumption is constant above a certain level -- that of subsistence, and the utility of consumption below the subsistence level is a large negative number. We recall our assumption of a rigid debt limit which we may set equal to zero* and we assume that the stochastic process under

* After a simple accounting transformation: unused credit is counted as an asset. This assumes that interest on debt equals interest on assets.

consideration stops whenever the individual is forced below his subsistence level or below his debt limit, in which event a large penalty -- the poor house -- ih incurred.

$$u(x) = \begin{cases} u_o + u_1 x & x \geq c \\ -U & x < c \end{cases}$$

$$v(s) = \max_{0 \leqq x \leq s} (u_o + u_1 x + a\int v[r(s-x) + y]\, dF(y))$$

$$v(o) = -V \qquad \text{where } U, V \text{ are large constants.}$$

Maximization now implies

$$x \begin{Bmatrix} \geqq \\ = \end{Bmatrix} c \quad \text{a.a.} \quad u_1 - ar\int v'[r(s-x) + y]\, dF(y) \begin{Bmatrix} = \\ < \end{Bmatrix} 0$$

The upper alternative applies whenever s exceeds a critical level, say S defined by

$$(4) \qquad ar\int v'[r(S - c) + y]\, dF(y) = u_1$$

The desired asset level S is determined as the solution of (4). To determine $v(s)$ one has the functional equation:

$$(5) \qquad v(s) = \begin{cases} u_o + u_1 (s-S+c) + ar\int v[r(S-c)+y]\, dF(y) & \text{if } s > S \\ u_o + u_1 c + ar\int v[r(s-c)+y]\, dF(y) & s \leqq S \end{cases}$$

To sum up, whenever $s \geqq S$

$$s - x = S - c$$

$$x = c + s - S$$

that is to say all income in excess of that which is required to maintain a desired asset level $S - c$ is consumed. The marginal propensity to consume out of assets or "transitory income" in excess of a fixed amount is unity.

On the other hand, when $s < S$ then

$$x = c$$

Consumption is at its subsistence level, and the marginal propensity to consume is zero.

6. Under what conditions do we obtain a linear consumption function? This question suggests considering a quadratic utility function. In addition we shall have to drop the constraint of the borrowing limit. Instead we may assume that the interest rate at which funds can be borrowed, r, exceeds the internal rate $\frac{1}{a}$.

Let the utility of consumption function be of the form

$$(6) \qquad u = u_o + u_1 x + u_2 x^2$$

In the absence of the borrowing limit one trivial solution is now possible. Let x_o be the maximizer of $u(x)$. Then there is a solution

$$x_o = \frac{u_o + u_1}{2\, u_2} = \text{constant}$$

$$v(s) = \frac{1}{1-a} \cdot u(x_o) = \text{constant}.$$

If average income falls short of x_o -- the typical case -- this solution implies an unlimited growth of debt. We may rule it out by requiring that the expected value of the debt at any time be less than a certain value, a modified version of a borrowing limit.

It can now be shown by induction -- as in section 1 -- that the utility of wealth function is also quadratic. Denote it by

(7) $$v = v_o + v_1\, s + v_2 s^2$$

A little arithmetic now produces

(8) $$x = \left(1 - \frac{1}{ar^2}\right) s + \frac{1 - \frac{1}{ar^2}}{r-1}\, \bar{y} + \frac{1 - \frac{1}{ar}}{r - 1}\; \frac{u_1}{2\, u_2}$$

The surprising statement is that the marginal propensity to consume out of assets (or "transitory income") $1 - \frac{1}{ar^2}$ is independent of the coefficients of the utility function. The same is true for the marginal propensity to consume out of "permanent" income $\bar{y}$ -- the mean of the y distribution. Only the constant of the consumption function depends at all on the utility function.

It may also be seen that the marginal propensity to consume out of transitory income is only $r-1$ times as large as that of consumption out of permanent income; it is the same fraction of the latter as the interest rate is of one ($r-1$ = interest rate).

Formula (11) is particularly simple when the internal discount factor $\underline{a}$ equals the external discount $\frac{1}{r}$. Then all dependence of consumption on the utility function ceases and we have:

$$(9) \qquad x = a\,\bar{y} + (1-a)\,s$$

Consumption is a weighted average of "permanent income" $\bar{y}$ and assets. The consumer tries to keep his consumption close to his expected income, but indulges a little when assets exceed "one year's average income" and skimps when his assets are below that magic level.

References

[1] M. Friedman: A Theory of the Consumption Function. Princeton University Press, Princeton 1957. In particular, pp. 14-17.

[2] F. Modigliani and R. Brumberg: Utility Analysis and the Consumption Function: An Interpretation of Cross-Section Data. K. Kurihara (ed.), Post Keynesian Economics.

[3] R. Bellman: Dynamic Programming, Princeton Univ. Press, Princeton, 1957.