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Economic Theory of Teams*

Chapter 5

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Chapter 5

BEST DECISION FUNCTIONS

1. Introduction

In this chapter we consider problems of determining the best team decision function for a given information structure. Since during the discussion of any particular problem the information structure will be fixed, we will use the notation

$$(1.1) \qquad \Omega(\alpha) = E\,\omega(x, \alpha[\eta(x)])$$

to denote the expected payoff for any particular decision function α. A decision function $\hat{\alpha}$ is optimal if it maximizes $\Omega(\alpha)$ in the set $\{\alpha\}$ of decision functions available to the team.

2. Person-by-person-optimality.

2.1 Introduction

A team decision function is person-by-person-optimal if it cannot be improved by changing any one component α_i alone. More formally, for any decision function $\tilde{\alpha}$, let

$$(2.1) \qquad \Omega_i(\alpha_i, \tilde{\alpha}) = \Omega(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_{i-1}, \alpha_i, \tilde{\alpha}_{i+1}, \ldots, \tilde{\alpha}_N);$$

then $\tilde{\alpha}$ is person-by-person (p.b.p.) optimal if, for every i,

$$(2.2) \qquad \Omega(\tilde{\alpha}) = \max_{\alpha_i} \Omega_i(\alpha_i, \tilde{\alpha}), \quad i=1, \ldots, N.$$

An optimal decision function is, a fortiori, p.b.p.-optimal but the converse is not in general true. The next sections describe a condition under which p.b.p.-optimality guarantees true optimality.

The concept of p.b.p.-optimality is useful in much the same way that it is useful to be able to characterize (when possible) the maximum of a function of several variables as being attained at a point at which the several partial derivatives vanish, i.e., at a point at which the function attains a maximum in each variable alone. In the case of the team problem, the "variables" in question are actually the several component decision functions.

The conditions (2.2) for p.b.p.-optimality further suggest a process of adjustment whereby one might hope, starting from any team decision function, to approach an optimal decision function by changing the team decision function one component at a time. The convergence of such a person-by-person adjustment process to an optimal team decision function is discussed in Section 2.4 of this chapter. Also discussed there are the usefulness and limitations of p.b.p. adjustment as a real organizational device to bring about improvement through somewhat decentralized individual efforts.

2.2. Maximizing Conditional Expectations.

In a one-person decision problem the action prescribed by an optimal decision function is an action that maximizes the conditional expected payoff, given the observation (see Chap. 2, Sec. 5).

In the case of a p.b.p.-optimal decision function, each condition of (2.2) can be thought of as referring to a one-person problem, namely, the problem that person i faces when the decision functions of all persons j different from i are fixed at $\tilde{\alpha}_j$. For any action a_i of person i, and any value y_i of his information variable, the conditional expected payoff is

$$\Psi_i(a_i, y_i) = E\left\{\omega[x, \tilde{\alpha}_1(y_1), \ldots, a_i, \ldots, \tilde{\alpha}_N(y_N)] \mid y_i\right\} \quad (2.3)$$

Thus $\tilde{\alpha}$ is p.b.p. maximal if and only if for every i and every y_i, $\tilde{\alpha}_i(y_i)$ maximizes $\Psi_i(a_i, y_i)$. In particular, if a_i is a real variable, and Ψ_i is differentiable in a_i, then the above maximum will occur at a value of a_i for which $\frac{\partial \Psi_i}{\partial a_i} = 0$. This motivates the following definition: a team decision function α is <u>stationary</u> if for every i and every y_i,

(2.4) $$\left.\frac{\partial \Psi_i(a_i, y_i)}{\partial a_i}\right|_{a_i = \alpha_i(y_i)} = 0.$$

2.3 A Condition under which Person-by-Person-Optimality Guarantees Optimality.

<u>Theorem</u> 1. For each i, let a_i be a real variable, and suppose that for every x, ω is a concave, differentiable function of $a=(a_1,\ldots,a_N)$; then any stationary team decision function is optimal.

<u>Sketch of Proof:</u>

Let α be stationary, let β be any other team decision function; define the function f by

$$f(t) = \Omega(\alpha + t\beta),$$

where t is any real number. Because the function ω is concave and differentiable in a, the function f is concave and differentiable in t; the derivative of f with respect to t is

$$f'(t) = \frac{d}{dt} E\,\omega[x, \alpha + t\beta] = \sum_{i=1}^{N} E \frac{\partial}{\partial a_i} \omega(x, \alpha + t\beta)\beta_i(y_i).$$

(The change in order of the operations of differentiation and expectation is valid here because $X, Y_1, \ldots, Y_N$ are finite.)

Hence

(2.5) $$f'(0) = \sum_{i=1}^{N} E \frac{\partial}{\partial a_i} \omega(x, \alpha)\ \beta_i(y_i).$$

But, for every i,

(2.6) $$E \frac{\partial}{\partial a_i} \omega(x, \alpha)\beta_i(y_i) = E[E\left[\frac{\partial}{\partial a_i} \omega(x, \alpha)\beta_i(y_i) \,\middle|\, y_i\right]]$$
$$= E[\beta_i(y_i)\, E\left[\frac{\partial}{\partial a_i} \omega(x, \alpha) \,\middle|\, y_i\right]].$$

Because α is stationary, for every i and y_i

$$E \frac{\partial}{\partial a_i}\left[\omega(x,\alpha) \mid y_i\right] = 0.$$

Substituting this in (2.6) gives

$$E \frac{\partial}{\partial a_i} \omega(x,\alpha)\beta i(y_i) = 0.$$

and substituting in turn in (2.5) gives

$$f'(0) = 0.$$

Hence zero is the value of t that maximizes $f(t)$; in particular, for any decision function $\gamma = \alpha + t\beta$,

$$\Omega(\gamma) = f(t) \leq f(0) = \Omega(\alpha).$$

Hence no other decision function is better than α.

A condition of differentiability seems to be essential to the above result, as the following example suggests.

Consider a team of two members, who payoff function is independent of x, with contour lines as on Figure 5.1, say

(2.7) $$\omega(a_1,a_2) = \min\left\{-a_1^2-(a_2-1)^2, -(a_1-1)^2 - a_2^2\right\}.$$

It is easily verified that any (a_1,a_2) for which $a_1 = a_2$ is p.b.p. optimal (e.g., the point P in the Figure), whereas the maximum of ω is attained only at $a_1 = a_2 = \frac{1}{2}$. Note that in this example ω is actually strictly concave.

For an extension of Theorem 1 to the case of an infinite X, see [ ].

2.4. Person-by-Person Adjustment

The person-by-person condition for optimality suggests at least one process of adjustment whereby a non-optimal team decision function would be improved by changing only one component decision function at a time. In this section we give a condition under which such a process would converge to a true optimum; this condition is very close to the condition that guarantees that a person-by-person optimal decision function is optimal.

The adjustment of decision rules one or a few components at a time is of course a process that goes on all the time in real organizations. In fact the simultaneous coordinated adjustment of all component decision functions would be a practical impossibility in any but the most rudimentary organizations. On the other hand, the example of the previous section shows, that a partial adjustment process could get "stuck" at a point that is not even a local optimum, much less an overall optimum. It is thus of interest to investigate the conditions under which this cannot occur.

Given any team decision function α^o, in some given problem, consider a sequence of team decision functions α^1, α^2, etc., constructed as follows: α^1 is the best team decision function that can be obtained by changing only the <u>first</u> component of α^o, α^2 is the best that can be obtained by changing only the <u>second</u> component of α^1, ...; α^{N+1} is the best that can be obtained by changing only the first component of α^N, etc. It is shown in the appendix that if the payoff function is concave and differentiable in the action variables, and attains a maximum value, for every state of nature, then the sequence of expected payoffs $\Omega(\alpha^N)$ will converge to the maximum possible expected payoff. (The conditions used in the appendix are slightly weaker than these.)

Similar results can be obtained for variations of the above adjustment process, in which the rule for determining the next component to be varied is changed, or the components are varied in groups, instead of singly.

Figure 5.2 shows a path of adjustment in the special case of constant state of nature, with two decision variables. In this special case the team problem reduces to that of maximizing a function ω in the two variables a_1 and a_2, which for this example was taken to be the quadratic function

$$(a_1, a_2) = -a_1^2 - a_1 a_2 - a_2^2 + 2a_1 + 3a_2.$$

Teams with Quadratic Payoff Functions

3.1 Introduction

The person-by-person principle provides a way of characterizing optimal decision functions in a limited, but still fairly broad class of cases. Further progress in characterizing optimal decision functions is possible if one considers even more limited classes of problems. These limitations can be on the payoff function, on the probability distribution, on the set of feasible decision functions, or on the information structure. In this section and the next we consider two special classes of payoff functions, quadratic and polyhedral. In the first case, the person-by-person principle applies, to give some fairly explicit characterizations of optimal decision functions. This will facilitate the study of the organizational consequences of changes in the various parameters of the situation. In the second case (polyhedral), the p.b.p. principle typically does not apply, but the techniques of linear programming provide a general method of solution; however, the effects of changes in the conditions are not as easily discerned.

In this section we will explore the consequences of assuming that, for every state of the world, the payoff is a quadratic function of the team action variables. Particular attention will be given to the case in which (1) the coefficients of the second degree terms do not depend upon the state of the world, and (2) the relevant random variables are normally distributed. In this case the optimal decision variables will be shown to be linear functions of the information variables, and an explicit algorithm for their computation will be given.

Such a quadratic payoff function may be thought of as an approximation, for each state of the world, to an arbitrary smooth payoff function in the neighborhood of the best team action, say $\gamma(x)$, corresponding to the state of the world x.

In a quadratic formulation, the variances and correlations of the information variables have an especially important role (whatever the probability distribution). In fact, the theory of the quadratic team has interesting connections with the statistical theory of regression; this subject will be pursued in chapter 7.

Almost all of the essential features of the "quadratic team" are present in the 2-person case. We will therefore first present a fairly complete treatment of this case, which will be followed by a brief discussion of the quadratic team with any number of members.

3.2 The Two-Person Case

Suppose that there are 2 real action variables, a_1 and a_2, and that the payoff function is given by

$$(3.1) \qquad (x,a) = \lambda(x) + 2\mu_1(x)a_1 + 2\mu_2(x)a_2 - \nu_{11}(x)a_1^2 - 2\nu_{12}(x)a_1a_2 - \nu_{22}(x)a_2^2,$$

where λ, μ_i and ν_{ij} are all real-valued functions of the state of the world x.

It is reasonable to confine our attention to situations in which there is a maximum payoff for every fixed x; for this reason we will make the assumption that the determinant

$$\begin{vmatrix} \nu_{11}(x) & \nu_{12}(x) \\ \nu_{12}(x) & \nu_{22}(x) \end{vmatrix}$$

is positive for every x.

Theorem 2. The optimal decision functions $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are determined by the following conditions:

$$(3.2) \qquad \begin{cases} \alpha_1(y_1)E(\nu_{11}|y_1) + E(\alpha_2 \nu_{12}|y_1) = E(\mu_1|y_1), \\ \\ E(\alpha_1 \nu_{21}|y_2) + \alpha_2(y_2)E(\nu_{22}|y_2) = E(\mu_2|y_2). \end{cases}$$

Sketch of proof: The conditions of Theorem 1 (Sec. 2.3) are satisfied here, so that the person-by-person rule can be applied, in its special form of stationarity (condition (2.5)). This last directly yields (3.2) above.

If the state of nature x can take on only a finite number of values, then (3.2) represents a system of linear equations as follows.

Suppose that

(1) the possible values of x are $x_1, \ldots, x_M$;

(2) for $i=1,2$ the possible values of the information variable y_i are $y_{i1}, \ldots, y_{iM_i}$;

(3) the conditional probability that $y_j = y_{jm}$ given that $y_i = y_{in}$ is q^i_{mnj}

(4) the value $\alpha_i(y_{im})$ is denoted by a_{im}.

Then, according to (3.2), the values $a_{11}, \ldots, a_{1M_1}$ and $a_{21}, \ldots, a_{2M_2}$ are to be chosen to satisfy the following $(M_1 + M_2)$ linear equations:

$$a_{1j}E(\nu_{11}|y_{1j}) + \sum_{m=1}^{M_2} a_{2m}E(\nu_{12}|y_{1j},y_{2m})q^1_{mj} = E(\mu_1|y_{1j})$$

(3.3)

$$\sum_{m=1}^{M_1} a_{1m}E(\nu_{21}|y_{1m},y_{2k})q^2_{mk} + a_{2k}E(\nu_{22}|y_{2k}) = E(\mu_2|y_{2k})$$

$$j=1,\dots,M_1;k=1,\dots,M_2.$$

Example (with constant coefficients of the quadratic terms). Consider again the production example already introduced in Chapter 3, Section 5. Here a_1 and a_2 denote inputs in a production process, x_1 and x_2 denote the prices of the inputs, the production function is quadratic in the inputs, and by suitable choices of origins and units the payoff function can be put in the form

(3.4) $$\omega(x,a) = U_0(x) - (a_1^2-2qa_1a_2 + a_2^2) - (a_1x_1+a_2x_2).$$

Comparing this with the general formulation in (3.1), we see that

(3.5) $$\begin{cases} \lambda(x) = U_0(x), \quad \mu_1(x) = \dfrac{-x_1}{2}, \quad \mu_2(x) = \dfrac{-x_2}{2}, \\ \\ \nu_{11}(x) = \nu_{22}(x) = 1, \qquad \nu_{12}(x) = -q. \end{cases}$$

Suppose, in this example, that each price x_i can have only one of M values x_{ij} $(j=1,\dots,M)$ and that the joint probability

(3.6) $$Pr(x_1=x_{1j}, x_2=x_{2k}) = P_{jk}.$$

Suppose further that the decision about each input is made only on the basis of knowledge about the corresponding price, i.e.,

(3.7) $$\eta_i(x) = x_i.$$

Each decision function α_i will have the form

(3.8) $$\alpha_i(x_{im}) = a_{im}, \qquad i=1,2;$$

i.e., a_{im} is the level of input i to be set by person i, if he finds out that the price of his input is x_{im}. The numbers a_{im} are to be determined so as to maximize the expected payoff; their optimal values are, by (3.3), determined by the following system of 2M equations:

$$a_{1j} - q \sum_{m=1}^{M} a_{2m} \Pr(x_{2m} | x_{1j}) = -x_{1j}/2, \qquad j=1,\ldots,M,$$

(3.9)

$$-q \sum_{m=1}^{M} a_{1m} \Pr(x_{1m} | x_{2k}) + a_{2k} = -x_{2k}/2, \qquad k=1,\ldots,M.$$

To specialize the example still further, suppose that each price x_i can take on only one of the two values +1 or -1, with the following joint probability distribution

(3.10)

x_2 \ x_1	-1	+1
-1	$\frac{1+r}{4}$	$\frac{1-r}{4}$
+1	$\frac{1-r}{4}$	$\frac{1+r}{4}$

Table 3.1. Joint distribution of x_1 and x_2.

Note that, in this simple distribution, x_1 and x_2 have the same marginal distributions, with the values +1 and -1 equally likely, and with means 0, variances 1, and correlation r.

To apply (3.9) we must first calculate

$$(3.11)\qquad \left.\begin{array}{l} \Pr(x_i = 1 \mid x_j = 1) \\ \\ \Pr(x_i = -1 \mid x_j = -1) \end{array}\right\} = \frac{1+r}{2}, \qquad i \neq j;$$

$$(3.12)\qquad \left.\begin{array}{l} \Pr(x_i = 1 \mid x_j = -1) \\ \\ \Pr(x_i = -1 \mid x_j = 1) \end{array}\right\} = \frac{1-r}{2}, \qquad i \neq j.$$

Substituting this in (3.9) gives the system

$$(3.13)\qquad \left\{\begin{array}{l} a_{11} - q(\frac{1+r}{2})a_{21} - q(\frac{1-r}{2})a_{22} = 1/2 \\ a_{12} - q(\frac{1-r}{2})a_{21} - q(\frac{1+r}{2})a_{22} = -1/2 \\ -q(\frac{1+r}{2})a_{11} - q(\frac{1-r}{2})a_{12} + a_{21} = 1/2 \\ -q(\frac{1-r}{2})a_{11} - q(\frac{1+r}{2})a_{12} + a_{22} = -1/2 \end{array}\right.$$

It can easily be verified that the solution to this system is

$$(3.14)\qquad \left\{\begin{array}{l} a_{11} = a_{21} = \frac{1}{2(1-rq)} \\ \\ a_{12} = a_{22} = \frac{-1}{2(1-rq)} \end{array}\right.$$

The information structure just considered might be called "Specialized information". It is interesting to compare these decision functions with those that would be optimal in the following two extreme situations:

(1) <u>full information</u>—each person knows the value of both x_1 and x_2 before taking action; i.e., $\eta_i(x) = (x_1, x_2)$.

(2) <u>no information</u>—each person takes action with only knowledge of the probability distribution of x_1 and x_2; i.e., $\eta_i(x)$ constant.

These are, of course, the situations discussed in Chapter 3, Section 5.

The solution for situation (1) is found by maximizing (3.4) for each pair of values of x_1 and x_2. This follows from Theorem 2, but of course can easily be seen directly. Setting the partial derivatives of (3.4) (with respect to a_1 and a_2) equal to zero—or, what amounts to the same thing, applying condition (3.2)—yields

$$(3.15)\qquad \begin{cases} \alpha_1(x_1,x_2) - q\alpha_2(x_1,x_2) = -x_1/2 \\ \\ -q\alpha_1(x_1,x_2) + \alpha_2(x_1,x_2) = -x_2/2 \end{cases}$$

Solving this for $\alpha_1(x_1,x_2)$ and $\alpha_2(x_1,x_2)$ gives

$$(3.16)\qquad \begin{cases} \alpha_1(x_1,x_2) = -\dfrac{x_1 - qx_1}{2(1-q^2)} \\ \\ \alpha_2(x_1,x_2) = -\dfrac{x_2 - qx_1}{2(1-q^2)} \end{cases}$$

In particular, for each of the four possible pairs of values of x_1 and x_2 we have the values of $\alpha_1(x_1,x_2)$ shown in Table 3.2.

(x_1, x_2)	$\alpha_1(x_1, x_2)$
(-1, -1)	$\frac{1}{2(1+q)}$
(-1, 1)	$\frac{1}{2(1-q)}$
(1, -1)	$\frac{1}{2(1-q)}$
(1, 1)	$\frac{1}{2(1+q)}$

Table 3.2.

A similar table can easily be constructed for α_2.

On the other hand, for the case of no information, application of (3.2) yields

$$(3.18) \qquad \begin{cases} a_1 - qa_2 = -E(x_1)/2, \\ \\ -qa_1 + a_2 = -E(x_2)/2. \end{cases}$$

Since the means of x_1 and x_2 are zero, the right side of (3.18) is zero, and the solution is

$$(3.19) \qquad a_1 = a_2 = 0.$$

The results of our discussion of this example are summarized in the Table 3.3, which shows for each person the action that is actually taken for each of the four possible states of the world.

	Full Information $\eta_i(x) = (x_1, x_2)$		Specialized Information $\eta_i(x) = x_i$		No Information η_i constant	
(x_1, x_2)	a_1	a_2	a_1	a_2	a_1	a_2
$(-1, -1)$	$\frac{1}{2(1+q)}$	$\frac{1}{2(1+q)}$	$\frac{1}{2(1-rq)}$	$\frac{1}{2(1-rq)}$	0	0
$(-1, 1)$	$\frac{1}{2(1-q)}$	$\frac{-1}{2(1-q)}$	$\frac{1}{2(1-rq)}$	$\frac{-1}{2(1-rq)}$	0	0
$(1, -1)$	$\frac{-1}{2(1-q)}$	$\frac{1}{2(1-q)}$	$\frac{-1}{2(1-rq)}$	$\frac{1}{2(1-rq)}$	0	0
$(1, 1)$	$\frac{-1}{2(1+q)}$	$\frac{-1}{2(1+q)}$	$\frac{-1}{2(1-rq)}$	$\frac{-1}{2(1-rq)}$	0	0

Table 3.3

3.3 The Two-Person Case Continued--Normality and Linear Decision Functions.

We will now show that if

(1) the coefficients of the quadratic terms in (3.1) are constant, independent of x, and

(2) the coefficients of the linear terms, and the information variables, are normally distributed,

then the optimal decision functions are linear.

This result will be used later to analyze the influence of the variability and correlation of the information variables on the choice of optimal information structures. 1/

According to assumption (1) above, the functions ν_{ij} in equation (3.1) are constant. Let $\nu_{12} = -q$, and let the units of a_1 and a_2 be chosen so that $\nu_{11} = \nu_{22} = 1$. This gives us the form of the payoff function

(3.20) $$\omega(x,a) = \lambda(x) + 2\mu_1(x)a_1 + 2\mu_2(x)a_2 - a_1^2 + 2qa_1a_2 - a_2^2.$$

The requirement that $\omega(x,a)$ have a maximum in a, for every x, is met by requiring that $q^2 < 1$. Indeed, by differentiating (3.20) with respect to a_1 and a_2, we get the conditions for a maximum as

(3.21) $$a_1 - qa_2 = \mu_1(x),$$
$$-qa_1 + a_2 = \mu_2(x).$$

The solution of this gives the best values of a_1 and a_2, for any given value of x, as

(3.22) $$\begin{cases} \gamma_1(x) = \dfrac{\mu_1(x) - q\mu_2(x)}{1 - q^2}, \\ \\ \gamma_2(x) = \dfrac{-q\mu_1(x) + \mu_2(x)}{1 - q^2} \end{cases}$$

(Compare this with equation (8) of Chapter 3.)

Now suppose that η_1 and η_2 are the information functions for persons 1 and 2, respectively. It will simplify the

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1/ The assumption of normality violates the assumption of finite X that we have been making all along; see [ ] for a rigorous proof.

exposition to assume that η_1 and η_2 are real-valued functions, but the extension to vector-valued functions (each person learning about several variables) is easy, and will be indicated later. Denoting the random variable $\eta_i(x)$ by y_i, let

$$\text{Co-variance } (y_1, y_2) = s_{12}. \tag{3.23}$$

There is no loss of generality is assuming that $Ey_i = 0$ and that Variance $(y_i) = 1$.

We now proceed to show that the optimal team decision functions are linear.

First, we shall write the person-by-person optimality condition for the case being considered (see equations (3.2)):

$$\begin{cases} \alpha_1(y_1) - qE(\alpha_2|y_1) = E(\mu_1|y_1), \\ \\ -qE(\alpha_1|y_2) + \alpha_2(y_2) = E(\mu_2|y_2). \end{cases} \tag{3.24}$$

We next show that changing μ_1 and μ_2 by adding constants, results in a change of the optimal α_1 and α_2 by adding constants. Let m_1 and m_2 be any two numbers, and let c_1 and c_2 be the solutions of

$$\begin{cases} c_1 - qc_2 = m_1, \\ \\ -qc_2 + c_2 = m_2. \end{cases} \tag{3.25}$$

It follows that if α_1 and α_2 satisfy (3.24) then

$$(3.26)\qquad \begin{cases} \alpha_1(y_1)+c_1 - qE(\alpha_2+c_2|y_1) = E(\mu_1+m_1|y_1). \\ \\ -qE(\alpha_1+c_1|y_2) + \alpha_2(y_2) + c_2 = E(\mu_2+m_2|y_2). \end{cases}$$

In other words, when the constants m_1 and m_2 are added to μ_1 and μ_2, respectively, the constants c_1 and c_2 must be added to α_1 and α_2, respectively, to obtain new optimal decision functions.

Therefore, there is no important loss of generality in assuming that $E\mu_1 = E\mu_2 = 0$, and this will now be done.

We next show that there exists a solution $(\hat{\alpha}_1,\hat{\alpha}_2)$ to (3.24) of the form

$$(3.27)\qquad \begin{cases} \alpha_1(y_1) = b_1y_1, \\ \\ \alpha_2(y_2) = b_2y_2. \end{cases}$$

In order to prove this, we substitute (3.27) into (3.24) and look for values of b_1 and b_2 that satisfy that condition;

$$(3.28)\qquad \begin{cases} b_1y_1 - qb_2E(y_2|y_1) = E(\mu_1|y_1), \\ \\ -qb_2E(y_2|y_1) + b_2 = E(\mu_2|y_2). \end{cases}$$

Since the μ_i and the y_i are normally distributed with zero means and unit variances,

$$(3.29)\quad \begin{cases} E(\mu_i | y_i) = d_i y_i, \\ E(y_i | y_j) = r y_j, \end{cases}$$

where

$$(3.30)\quad \begin{cases} d_i = \text{Covariance } (\mu_i, y_i), \\ r = \text{Correlation } (y_1, y_2), \end{cases}$$

(see [ ], p.).

Substituting (3.29) in (3.28) gives

$$(3.31)\quad \begin{cases} b_1 y_1 - q r b_2 y_1 = d_1 y_1, \\ -q r b_1 y_2 + b_2 y_2 = d_2 y_2, \end{cases}$$

for all y_1 and y_2. Therefore

$$(3.32)\quad \begin{cases} b_1 - q r b_2 = d_1 \\ -q r b_1 + b_2 = d_2, \end{cases}$$

which has the solution

$$(3.33)\quad \begin{cases} b_1 = \dfrac{d_1 + q r d_2}{1 - q^2 r^2}, \\ b_2 = \dfrac{d_2 + q r d_1}{1 - q^2 r^2}. \end{cases}$$

If we drop the assumption that $E(\mu_i) = 0$, and add now the solution of (3.25), with $m_i = E(\mu_i)$,

$$(3.34) \qquad \begin{cases} c_1 = \dfrac{E\mu_1 + qE\mu_2}{1 - q^2}, \\ \\ c_2 = \dfrac{E\mu_2 - qE\mu_1}{1 - q^2}, \end{cases}$$

we have, taking (3.33) and (3.34) together, the solution to our problem, as summarized in the following theorem.

Theorem 3. If the payoff function is given by (3.20) and if μ_1, μ_2, y_1 and y_2 are normally distributed, with $Ey_i = 0$, Var $y_i = 1$, $Ey_1y_2 = r$, and Cov $(\mu_i, y_i) = d_i$, then the optimal decision functions $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are linear,

$$(3.35) \qquad \hat{\alpha}_i(y_i) = c_i + b_i y_i,$$

and the coefficients b_i and c_i are given by equations (3.33) and (3.34).

Example. Consider the example of Sec. 3.2, except assume that x_1 and x_2 are normally distributed. Here x_1 and x_2 are the deviations of the input prices from their respective means, and a_1 and a_2 are the deviations of the levels of input from those values that would be best if the input prices had their mean values ($x_i = 0$). Denote the variance of x_i by s_i^2, and the correlation between x_1 and x_2 by r.

For the case of "specialized information" we take

$$(3.36) \qquad y_i = x_i/s_i,$$

in order to standardize the information variables as in the statement of Theorem 3. Then $d_i = s_i/2$, and the application of Theorem 3 immediately gives

$$(3.37)\qquad \begin{cases} \hat{\alpha}_1(y_1) = -\left[\dfrac{1 + qrs_2/s_1}{2(1-q^2r^2)}\right] x_1, \\[2em] \hat{\alpha}_2(y_2) = -\left[\dfrac{1 + qrs_2/s_2}{2(1-q^2r^2)}\right] x_2. \end{cases}$$

The solution to the "full information" case is of course given by (3.22), which in the present example looks like this:

$$(3.38)\qquad \begin{cases} \hat{\alpha}_1(x_1, x_2) = -\dfrac{x_1 - qx_2}{2(1-q^2)}, \\[2em] \hat{\alpha}_2(x_1, x_2) = -\dfrac{x_2 - qx_1}{2(1-q^2)}. \end{cases}$$

It is left to the reader to verify that the solution to the "no information" case is given by $\hat{\alpha}_1 = \hat{\alpha}_2 = 0$. The reader should compare the results of this example with those of the previous example (Table 3.3).

3.4. The general case

In this section we will generalize the results on the two-person case to the case of any (finite) number of team members. The reader who has difficulty with linear algebra may do well to skip this section and go on to the next, with the assurance that the two-person results are generalizable.

Suppose that there are N real action variables $a_1, \ldots, a_N$, and that the payoff function is

$$(3.39)\qquad \omega(x, a) = \lambda(x) + 2\sum_{i=1}^{N} \mu_i(x)a_i - \sum_{i,j=1}^{N} \nu_{ij}(x)a_i a_j.$$

where the λ, μ_i, and ν_{ij} are all real-valued functions of the state of the world x. As before, we will confine our attention to situations in which there is a maximum payoff for every fixed x; for this reason we will make the assumption that the matrix $((\nu_{ij}[x]))$ is positive definite for every x.

Theorem 4. If the payoff function is given by (3.39), and if $y_i = \eta_i(x)$ $(i=1,\cdots,N)$ are the information functions for the team, then the optimal decision functions $\alpha_1,\cdots,\alpha_N$ are determined (uniquely) by the following conditions:

$$(3.40)\quad \alpha_i(y_i)E(\nu_{ii}|y_i) + \sum_{j\neq i} E(\alpha_j \nu_{ij} \mid y_i) = E(\mu_i|y_i),\ i=1,\cdots,N.$$

Proof. The theorem follows directly from the sufficiency of stationarity for optimality in this case (see equation (2.5) and Theorem 1, in Section 2).

If the set X of the states of nature is finite (as we have been assuming, strictly speaking), then (3.40) represents a system of linear equations, whose unknowns are the values of the decision functions α_i for the various values of y_i. This has already been exemplified by equation (3.3), Section 3.2. More generally (3.40) represents a system of integral equations. That system may have no solution, but if it does, it will be unique. The mathematical questions associated with the case of an infinite set of states of nature are discussed in [ ].

One specific case of an infinite X is noteworthy, however, for the linearity of the optimal decision functions.

Theorem 5. If (1) the functions ν_{ij} in (3.39) are constants; q_{ij} (2) the information functions η_i are vector-valued, with

(3.41) $$\eta_i(x) = (y_{i1},\dots,y_{iM_i}) = y_i;$$

(3) the functions μ_i and η_i are jointly normally distributed, with

$$Ey_{ij} = 0,$$

$$\text{Var } y_{ij} = 1,$$

$$\text{Correlation } (y_{ij}, y_{hk}) = r^{ih}_{jk},$$

(3.42)

$$E\mu_i = m_i,$$

$$\text{Covariance } (\mu_i, y_{ij}) = d_{ij};$$

then the optimal decision functions $\alpha_1,\dots,\alpha_N$ are linear,

(3.43) $$\alpha_i(y_i) = \sum_{k=1}^{M_i} b_{ik}y_{ik} + c_i,$$

where the coefficients b_{ik} and c_i are determined by the systems of linear equations:

(3.44) $$\sum_{j=1}^{N} q_{ij} \sum_{k=1}^{M_j} r^{ji}_{kh}\, b_{jk} = d_{ih}, \qquad i=1,\dots,N, \quad h=1,\dots,M_i,$$

(3.45) $$\sum_{j=1}^{N} q_{ij}c_j = m_i, \qquad i=1,\dots,N.$$

Remark. There is no loss of generality in assuming that $Ey_{ij} = 0$, $\text{Var } y_{ij} = 1$, since the given function η_i can always be transformed by a 1-1 transformation into a function that has these properties.

Sketch of Proof.

The proof is based upon the person-by-person principle. (For a complete proof of the present theorem see [ ].) We first consider the special case in which $E\mu_i = 0$. Just as in the corresponding theorem for the 2-person case, we will show that there are linear functions in the form of (3.43), with $c_i = 0$, that satisfy the condition of stationarity (Sec. 2, equation (3.5)). We first note that

$$E(y_{jk}|y_i) = \sum_h r^{ji}_{kh} y_{ih},$$

(3.46)

$$E(\mu_i|y_i) = \sum_h d_{ih} y_{ih}.$$

Using (3.46), the condition of stationarity applied to decision functions of the form (3.43) immediately gives

$$\sum_j q_{ij} \sum_k b_{jk} \sum_h r^{ji}_{kh} y_{ih} = \sum_h d_{ih} y_{ih}, \tag{3.47}$$

for every $i=1,\ldots,N$, and every value of y_{ih}, $h=1,\ldots,M_i$. Equating the coefficients of y_{ih} on each side of (3.47) gives (3.44). It can be shown that if the matrix $((q_{ij}))$ is positive definite, then so is the matrix of coefficients of the unknowns b_{ij} in (3.44) (see [ ], p.) and therefore (3.44) has a unique solution. A second application of the stationarity condition now shows that if $\mu_i = m_i$, then the c_i must be given by (3.45).

4. Teams with Polyhedral Payoff Functions

4.1. Introduction

Much of the available data on firms leads naturally to the formulation of decision problems with payoff functions that are concave and polyhedral in the decision variables. The concept of a polyhedral function is a generalization to the case of several variables of the concept of a piecewise linear function of one variable. It is well recognized that the technique of linear programming provides a general computational method for solving one-person decision problems under certainty, with concave polyhedral payoff functions. We will show how such a problem, but within the team context with probabilistic uncertainty, is also amenable to linear programming.

We will begin with a discussion of a very simple example (Sec. 4.2). This will be followed by a general treatment (Sec. 4.3). The introduction of the complications of uncertainty and team structure into a programming problem tend to result in a substantial increase in the size of the problem. On the other hand, joint constraints on the actions of team numbers with different information may result in very restrictive constraints on the decision functions available to team. This is exemplified by the result presented in Sec. 4.4.

4.2. An Example: polyhedral case

Suppose that a firm has n salesmen, each of whom goes out at the beginning of the period to get orders. Assume that the ith salesman faces an unlimited demand, at price $(1+x_i)$; and on the basis of knowledge of that price, but not of the prices faced by the other salesmen, he must decide on the quantity a_i that he will accept in orders. The orders of all the salesmen are filled centrally and the unit cost depends upon the total quantity ordered. The unit cost is 1 if the total quantity ordered does not exceed a certain limit c; but for that amount that exceeds c, the unit cost is (1+d).

Thus the action variable for the ith salesman is a nonnegative real number a_i; the state of nature is specified by the n-tuple $x=(x_1,\dots,x_n)$; and the payoff function is

$$\omega(x,a) = \begin{cases} \sum_{i=1}^{n} a_i x_i, & \text{if } \sum_{i=1}^{n} a_i \leq c, \\ \sum_{i=1}^{n} a_i x_i - d\left(\sum_{i=1}^{n} a_i - c\right), & \text{if } \sum_{i=1}^{n} a_i \geq c, \end{cases}$$

(4.1)

$$= \min\left\{\Sigma a_i x_i,\ \Sigma a_i(x_i-d) + dc\right\}.$$

Figure 5.3 shows contours of ω for fixed x, and n=2.

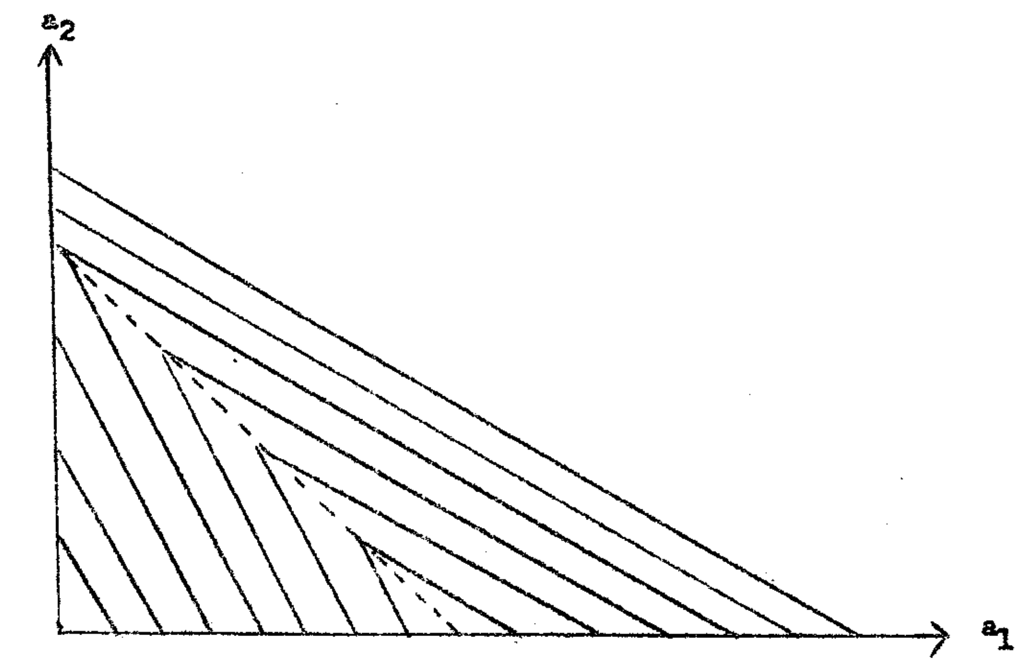

Figure 5.3

The state of nature x will be supposed to be subject to the probability function π.

The information structure is

$$\eta_i(x) = x_i. \tag{4.2}$$

The team decision problem can therefore be formulated as follows:

Problem I. Choose nonnegative functions $\alpha_1,\ldots,\alpha_n$ to maximize

$$E\,\omega(x,\alpha[\eta(x)]) = \sum_x \omega(x,\alpha_1[x_1],\ldots,\alpha_n[x_n])\pi(x).$$

We will now show that the optimal team decision functions can be found by solving the following associated linear programming problem. (Recall that x is assumed to be restricted to a finite set of possible values.)

Problem II. Choose nonnegative functions $\alpha_1,\ldots,\alpha_n$ and a nonnegative function β of x, to maximize

$$E\beta(x) = \sum_x \beta(x)\pi(x)$$

subject to the constraints

$$\left\{\begin{array}{l} \beta(x) \leq \sum_i \alpha_i(x_i)x_i \\ \\ \beta(x) \leq \sum_i \alpha_i(x_i)(x_i-d) + dc \end{array}\right. \qquad \text{for all } x. \tag{4.3}$$

(Note that, because x is restricted to a finite set, the functions β, $\alpha_1,\ldots,\alpha_n$ are each characterized by a finite sequence of numbers, so that Problem II is a finite dimensional linear programming problem.)

For an optimal team decision function $\hat{\alpha}$, consider the function

$$\hat{\beta}(x) = \omega(x,\hat{\alpha}[\eta(x)]). \tag{4.4}$$

By equation (4.1), for every x, $\hat{\beta}$ and $\hat{\alpha}$ satisfy (4.3). On the other hand, if α is any team decision function, and β is any nonnegative function of x, with both satisfying (4.3), then

(4.5) $$\beta(x) \leqq \omega(x, \alpha[\eta(x)]), \text{ for every } x,$$

and hence

(4.6) $$E\beta(x) \leqq E\omega(x, \alpha[\eta(x)]) \leqq E\omega(x, \hat{\alpha}[\eta(x)]) = E\hat{\beta}(x).$$

In other words, of all nonnegative functions β for which there exists a team decision function α such that β and α satisfy (4.3), the function $\hat{\beta}$ has the largest expected value.

On the other hand, if any such function β has an expected value as large as $E\hat{\beta}$, then it follows from (4.5) and (4.4) that the corresponding team decision function α is as good as $\hat{\alpha}$.

Thus we have shown that α is a solution of Problem I if and only if (α, β) is a solution of Problem II.

In the example just presented there were no other constraints on the action variables other than that of nonnegativity. Other linear constraints can easily be handled by the above method, by simply adding them to both maximization problems (I and II). This is so even if the constraints have random parameters. For example if the maximum total quantity that the hypothetical firm just described can supply, <u>at any cost</u>, is c' (possibly a random variable), and if the firm insists on making <u>sure</u> that <u>all</u> orders are filled, then the constraint

(4.7) $$\sum_i \alpha_i(x_i) \leqq c', \text{ for all } x \text{ (and possibly all } c')$$

is simply added to both problems I and II above.

4.3. <u>An Example: A simple linear case</u>

The simplest case of a polyhedral function is a linear function. In this case the original team decision problem is already in the linear programming form, and there is no need to go to the associated problem (the "Problem II" of Sec. 4.2.).

Consider the example of Sec. 4.2. , with the modification that supply is restricted absolutely to be no greater than c.

Thus

(4.8) $$\omega(x,a) = \Sigma\, a_i x_i,$$

(4.9) $$a_i \geqq 0, \quad \Sigma\, a_i \leqq c.$$

Here the team decision problem is:

Choose nonnegative functions $\alpha_1,\cdots,\alpha_n$ to maximize

(4.10) $$E\,\omega(x,\alpha[\eta(x)]) = \sum_x \sum_i \alpha_i(x_i)x_i\pi(x)$$

subject to

(4.11) $$\sum_i \alpha_i(x_i) \leqq c, \quad \text{for every } x.$$

Note that (4.10) and (4.11) are linear in the functions $\alpha_1,\cdots,\alpha_n$.

It may help the reader to see in more detail the special case in which $n=2$, and in which x_1 and x_2 can each take on one of two values, x_{i1} and x_{i2}. Let the probability that $x_i = x_{im}$ be denoted by p_{im}, and let $\alpha_i(x_{im})$ be denoted by $a_{im}(i,m=1,2)$; then this team decision problem is:

Choose nonnegative numbers a_{im} $(i,m=1,2)$ to maximize

(4.12) $$\sum_i \sum_m a_{im}\, x_{im}\, p_{im}$$

subject to the constraints

$$
(4.13)\qquad
\begin{array}{llll}
a_{11} & + a_{21} & & \leq c \\
a_{11} & & + a_{22} & \leq c \\
 & a_{12} + a_{21} & & \leq c \\
 & a_{12} & + a_{22} & \leq c
\end{array}
$$

4.4. Use of Person-by-Person Optimality

As was mentioned earlier, the person-by-person optimality principle does not typically provide a sufficient condition for an optimal team decision function when the payoff function is polyhedral. However, the fact that p.b.p. optimality is a necessary condition for optimality is often helpful.

In the example just discussed (Sec. 4.3.), the p.b.p. principle leads almost immediately to an explicit solution if one further assumption is made.

For any fixed i, the conditional expectation of $\omega(x,\alpha[\eta(x)])$ given x_i is (see (4.8))

$$(4.14)\qquad \alpha_i(x_i)x_i + \sum_{j\neq i} E[\alpha_j(x_j)x_j \mid x_i].$$

Hence, for α to be p.b.p. optimal, $\alpha_i(x_i) = 0$ if $x_i < 0$; and if $x_i > 0$, then $\alpha_i(x_i)$ must be equal to the largest value of a_i for which

$$(4.15)\qquad a_i + \sum_{j\neq i} \alpha_j(x_j) \leq c$$

for every x such that x_i has the given value. <u>If it is assumed that the range of every variable x_k is independent of</u> the values of the other variables, then in (4.15) all combinations of x_j's are possible, so that $\alpha_i(x_i)$ must equal some one number $\bar{a}_i$ for all $x_i > 0$. Hence, in order for α to be p.b.p. optimal it must be of the form

$$(4.16) \qquad \alpha_i(x_i) = \begin{cases} 0, & \text{if } x_i < 0 \\ \bar{a}_i, & \text{if } x_i > 0 \end{cases}, \quad i=1,\dots,n,$$

where $\bar{a}_1,\dots,\bar{a}_n$ are nonnegative numbers satisfying

$$(4.17) \qquad \sum_i \bar{a}_i \leq c.$$

For such a team decision function

$$(4.18) \qquad \Omega(\alpha) = \sum_i \bar{a}_i E(x_i | x_i > 0).$$

Hence for the optimal team decision function α, $\bar{a}_i = c$ for some one i for which $E(x_i|x_i > 0)$ is a maximum, and $\bar{a}_j = 0$ for all $j \neq i$. In other words, in the optimal team decision function, under the constraint that has been assumed, only one salesman ever accepts any orders (the others are "fired"!); he accepts an order equal to the full capacity of the firm whenever the price exceeds the unit cost, and otherwise accepts no orders at all.

4.5 General Formulation

A function f of n real variables will be called concave-polyhedral if there exist linear functions $f_1,\dots,f_M$ such that

$$(4.19) \qquad f = \min_{1 \leq m \leq M} f_m.$$

An example is given by equation (4.1), in which ω, for every fixed

x, is a concave-polyhedral function of a, with $M=2$. A concave-polyhedral function is a generalization to the case of several variables of a concave piecewise linear function of one variable.

Consider a team decision problem such that, for every state of nature, the payoff function is concave-polyhedral in the action variables, i.e., for every x

$$(4.20) \qquad \omega(x,a) = \min_{1\leq m\leq M} \lambda_m(x,a),$$

where $\lambda_1,\ldots,\lambda_M$ are, for every x, linear functions of a. Suppose further that the action variables are constrained by the conditions

$$(4.21) \qquad a_i \geq 0 \qquad i=1,\ldots,n$$

$$(4.22) \qquad \sum_{i=1}^{n} \delta_{ki}(x)a_i \leq \gamma_k(x), \quad \text{for every } x, \quad k=1,\ldots,K,$$

where δ_{ki} and γ_k are given functions of the state of nature x.

Problem I (team decision problem))

Choose nonnegative functions $\alpha_1,\ldots,\alpha_n$ (where α_i is a function of y_i) to maximize

$$(4.23) \qquad E \min_{1\leq m\leq M} \lambda_m(x,\alpha[\eta(x)])$$

subject to

$$(4.24) \qquad \sum_{i=1}^{n} \delta_{ki}(x)\alpha_i[\eta_i(x)] \leq \gamma_k(x), \quad \text{for every } x, \ k=1,\ldots,K.$$

Associated with Problem I, consider another problem.

Problem II (associated linear programming problem):

Choose nonnegative functions β, $\alpha_1,\ldots,\alpha_n$ (where β is a function of x, and α_i is a function of y_i to maximize $E\beta(x)$ subject to the constraints

(4.25) $\beta(x) \leq \lambda_m(x,\alpha[\eta(x)])$, for every x, $m=1,\ldots,M$;

(4.26) $\sum_{i=1}^{n} \delta_{ki}(x)\alpha_i[\eta_i(x)] \leq \gamma_k(x)$, for every x, $k=1,\ldots,K$.

Theorem 6. A team decision function α is optimal (i.e., is a solution of Problem I) if and only if, there is a function β such that (α,β) is a solution of Problem II.

The proof of Theorem 6 follows the line of reasoning of the example of Sec. 4.2. so closely that it will be omitted.

Note that if the space X has e elements, and for each i, the space Y_i has e_i elements, then the number of "unknowns" in Problem II equals $(e + \sum_i e_i)$, which is not greater than $e(N+1)$, and the number of individual constraints in (4.25) and (4.26) together is at most $e(M+K)$.

In the special case in which $M=1$, i.e., in which the payoff function is linear in the action variables, for every x, Problem I is already in the linear programming form, and there is no need to go to Problem II.

4.6. The Case of Information Variables with Independent Ranges

If the ranges of variation of the different information variables are "independent," in a sense that will be made precise, and if the payoff function is linear in the action variables, then there will generally follow from this a reduction in the "size" of the resulting linear programming problem of finding an optimal team decision function.

Assuming that the set of states of the world is finite, the functions $\eta_1,\ldots,\eta_n$ are said to have independent ranges if every n-tuple of values $(y_1,\ldots,y_n)$ has positive probability. If

$\eta_1, \ldots, \eta_n$ are the information functions, then this means that any one team member cannot rule out, on the basis of his own information, any combination of values of the information variables of other team members that was a priori possible.

Consider the team decision problem: maximize

$$(4.27) \qquad E \sum_i \lambda_i(x)\alpha_i(\eta_i[x])$$

subject to the constraint that, for every x $\alpha(\eta[x]) = (\alpha_1(\eta_1[x]), \ldots, \alpha_n(\eta_n[x]))$ lies in the (convex) set $k(x)$ defined by the linear inequalities (4.24), and the condition that $\alpha_i \geq 0$.

Suppose that the functions k, $\eta_1, \ldots, \eta_n$ have independent ranges. For any α, and for each i, let $[\underline{a}_i, \bar{a}_i]$ be the smallest closed interval such that

$$(4.28) \qquad \text{Prob}\left\{\underline{a}_i \leq \alpha_i \leq \bar{a}_i\right\} = 1,$$

and let $I(\alpha)$ be the cartesian product of the intervals $[\underline{a}_i, \bar{a}_i]$, $i=1,\ldots,n$. (For $n=2$, I is a rectangle, for $n=3$, a rectangular parallelopiped, etc.).

It follows from the convexity of $k(x)$, and the independence of the ranges of k, $\eta_1, \ldots, \eta_n$ that the requirement that $\alpha(\eta[x])$ be in $k(x)$ for every x is equivalent to the requirement that $I(\alpha)$ be contained in

$$(4.29) \qquad \bar{k} = \bigcap_x k(x).$$

(i.e., $\bar{k}$ is the largest set that is contained in all the sets $k(x)$).

Given any particular rectangle I in N-dimensional space one may ask, what is the best α such that $I(\alpha) = I$? Applying the person-by-person optimality condition, α must satisfy:

$$(4.30)\qquad \alpha_i(y_i) = \left\{ \begin{matrix} \bar{a}_i \\ \\ \underline{a}_i \end{matrix} \right\} \text{ as } E(\lambda_i | y_i) \left\{ \begin{matrix} > \\ \\ < \end{matrix} \right\} 0.$$

Hence the best expected payoff corresponding to the rectangle I is

$$(4.31)\qquad \Omega^*(I) = \sum_i (\bar{a}_i \bar{d}_i + \underline{a}_i \underline{d}_i),$$

where

$$\bar{d}_i = E(\lambda_i | E(\lambda_i | y_i) > 0) \quad \text{Prob}(E|\lambda_i|y_i| > 0),$$

$$(4.32)$$

$$\underline{d}_i = E(\lambda_i | E(\lambda_i | y_i) < 0) \ \text{Prob}(E|\lambda_i|y_i| < 0).$$

Thus the original problem has been reduced to one of maximizing $\Omega^*(I)$, which is linear in the $\bar{a}_i$ and $\underline{a}_i$, subject to the condition that I be contained in $\bar{K}$.

If in the original constraints (4.24) the functions δ_{ki} and γ_k are all nonnegative, then the above problem is even more greatly simplified. In that case it is easy to see that all the $\underline{a}_i$ must be zero; furthermore, if the $\underline{a}_i$ are zero, then $I(\alpha)$ is in $\bar{K}$ if and only of $(\bar{a}_1, \cdots, \bar{a}_n)$ is in $\bar{K}$.
Hence the problem reduces to one of choosing $\bar{a}_1, \cdots, \bar{a}_n$ so as to maximize

$$\Sigma \bar{d}_i \bar{a}_i$$

subject to the constraint that $(\bar{a}_1, \cdots, \bar{a}_n)$ be in $\bar{K}$.

Appendix to Chapter 5

In this appendix we sketch a proof of the fact that the condition given in Section 2.3 guarantees that person-by-person adjustment results in a sequence of expected payoffs that converges to the maximum value. The theorem as formulated below applies to successive variation of groups of component decision functions, as well as to variation one component at a time.

For each $i = 1, \cdots, N$ let A_i be a given finite dimensional Euclidean space, and let $A \equiv \prod_i A_i$ be the direct product of these. Let f be a real valued function on A, and for each real number r define

(A.1) $$S_r \equiv \left\{ x \middle| x \in A, f(x) \geqq r \right\}$$

In the team situation, A_i is to be interpreted as the set of decision functions available to person i, given his information function, and f as the expected payoff. (It is assumed that the set of states of nature is finite.) We are making the following assumptions about f:

A1. f is differentiable.

A2. $\max_x f(x) \equiv M$ is attained in A.

A3. For every r, S_r is convex and bounded.

A4. If $r < M$ and x is in the interior of S_r, then $f(x) > r$.

We define the functions ξ_i and μ_i as follows. For any x^0 in A let

(A.2) $$\mu_i(x^0) \equiv \max_{x_i} f(x_1^0, \ldots, x_{i-1}^0, x_i, x_{i+1}^0, \ldots, x_N^0)$$

where x_j denotes an element of A_j; and let $\xi_i(x^o)$ be the set of all $\bar{x}_i$ in A_i such that

(A.3) $f(x_1^o, \dots, x_{i-1}^o, \bar{x}_i, x_{i+1}^o, \dots, x_N^o) = \mu_i(x^o)$

Finally, let $\xi(x^o)$ be the set of all $\bar{x}$ such that

$$(A.4) \quad \begin{cases} \bar{x}_1 \in \xi_1(x^o), \\ \bar{x}_2 \in \xi_2(\bar{x}_1, x_2^o, \dots, x_N^o) \\ \vdots \\ \bar{x}_N \in \xi_N(\bar{x}_1, \dots, \bar{x}_{N-1}, x_N^o) \end{cases}$$

<u>Theorem</u>. Let $x^o, x^1, \dots$ be any sequence such that $x^n \in \xi(x^{n-1}), n \geq 1$; then under assumptions A1 - A4

$$\lim_{n \to \infty} f(x^n) = M$$

<u>Proof</u>. The sequence $\{f(x^n)\}$ is increasing and bounded above, and therefore has a limit. In particular,

$$\lim [f(x^n) - f(x^{n-1})] = 0.$$

By A3, The sequence $\{x^n\}$ is bounded, and therefore has a convergent subsequence. Thus it is sufficient to show that the function δ defined by

$$(A.5) \quad \delta(x) = \inf_{\bar{x} \in \xi(x)} [f(\bar{x}) - f(x)]$$

is lower semi-continuous, and is positive for every x such that

$f(x) < M$.

Since $\xi(x)$ is bounded for every x, and f is continuous, it follows that to prove the lower semi-continuity of δ it suffices to prove that ξ is upper semi-continuous (i.e. that $x^n \to \bar{x}$, $y^n \in$ (x^n), and $y^n \to \bar{y}$ imply $\bar{y} \in \xi(\bar{x})$). To prove the latter it suffices to show this for each ξ_i, as will now be done for (say) i=1. For any $x = (x_1, \cdots, x_N)$ let $y = (x_2, \cdots, x_N)$; then on any bounded rectangle in A, the family $\{f(x_1, y)\}$ is an equicontinuous family of functions of the variable y. Hence μ_1 (as defined by equation (A.2)) is continuous, which implies that ξ_1 is upper semi-continuous.

It remains to show that $\delta(x)$ is positive if $f(x) < M$. Suppose, to the contrary, that there exists x^o and $\bar{x}$ such that $f(x^o) = f(\bar{x}) = r < M$, and $\bar{x}$ is in $\xi(x^o)$. Let

$$x^1 = (\bar{x}_1, x_2^o, \cdots, x_n^o),$$
$$\vdots$$
$$x^{n-1} = (\bar{x}_1, \cdots, \bar{x}_{n-1}, x^o_n)$$
$$x^n = \bar{x}$$

as in equation (A.4); then

(A.6) $\mu_i(x^{i-1}) = r,\ 1 \leq i \leq N.$

Let $\widetilde{A_i}$ denote the internal linear subspace of A corresponding to A_i $(1 \leq i \leq N)$. By (A.6) and assumption A.4 the linear variety $(x^1 + \widetilde{A_1})$ supports the convex set S_r at the point x^1; hence there exists a hyperplane H containing $(x^1 + \widetilde{A_1})$ that supports S_r at x^1. Similarly, there is a hyperplane containing $(x^1 + \widetilde{A_2})$ that supports S_r at x^1; but the differentiability of f implies that there is only one supporting hyperplane for S_r at x^1. Hence $(x^1 + \widetilde{A_2}) \subset H$. It follows that x^2 is in H and that $(x^2 + \widetilde{A_1})$ and $(x^2 + A_2)$ are both contained in H. But, by (A.6), and assumption A4, x^2 is on the boundary of S_r; hence H supports

S_r at x^2. A continuation of this line of reasoning leads to the conclusion that $(x^{n-1} + \widetilde{A}_i) \subset H$ for $1 \leq i \leq N$, which contradicts the fact that the $\widetilde{A}_i$ are mutually orthogonal, and span the space A. Thus we have shown that if $f(x^0) < M$ and $\bar{x} \in \xi(x^0)$, then $f(\bar{x}) > f(x^0)$. But $\xi(x^0)$ is closed and bounded; hence

$$\inf_{\bar{x} \in \xi(x^0)} f(\bar{x}) > f(x^0)$$

i.e. $\delta(x^0) > 0$, which completes the proof.

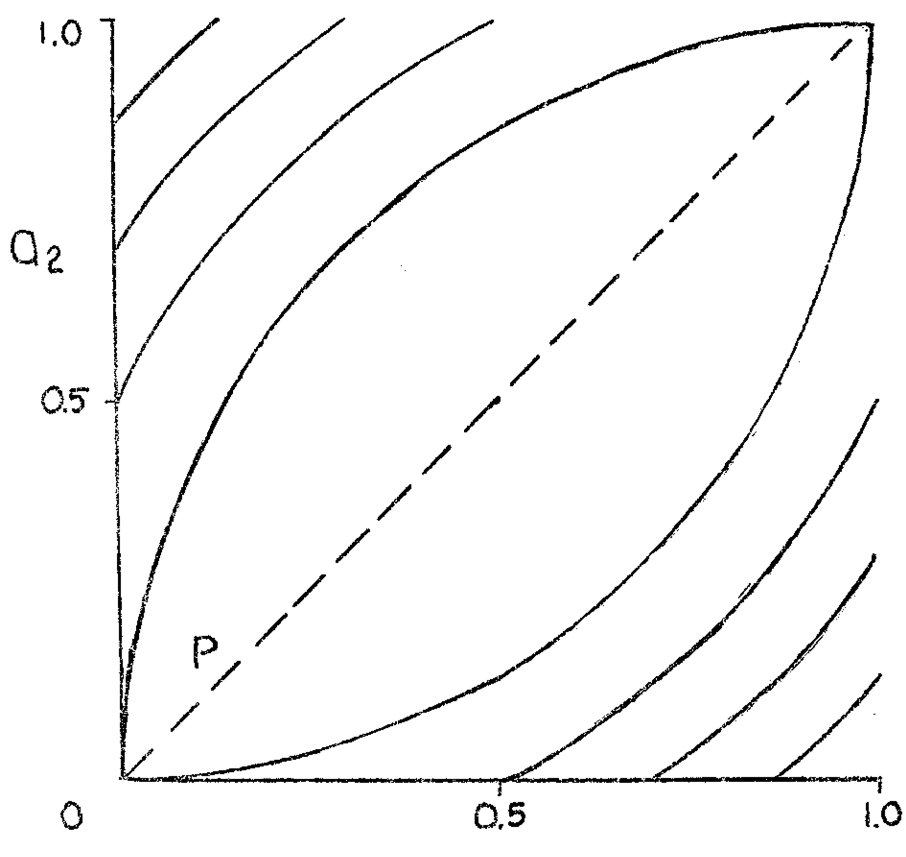

Fig. 1.

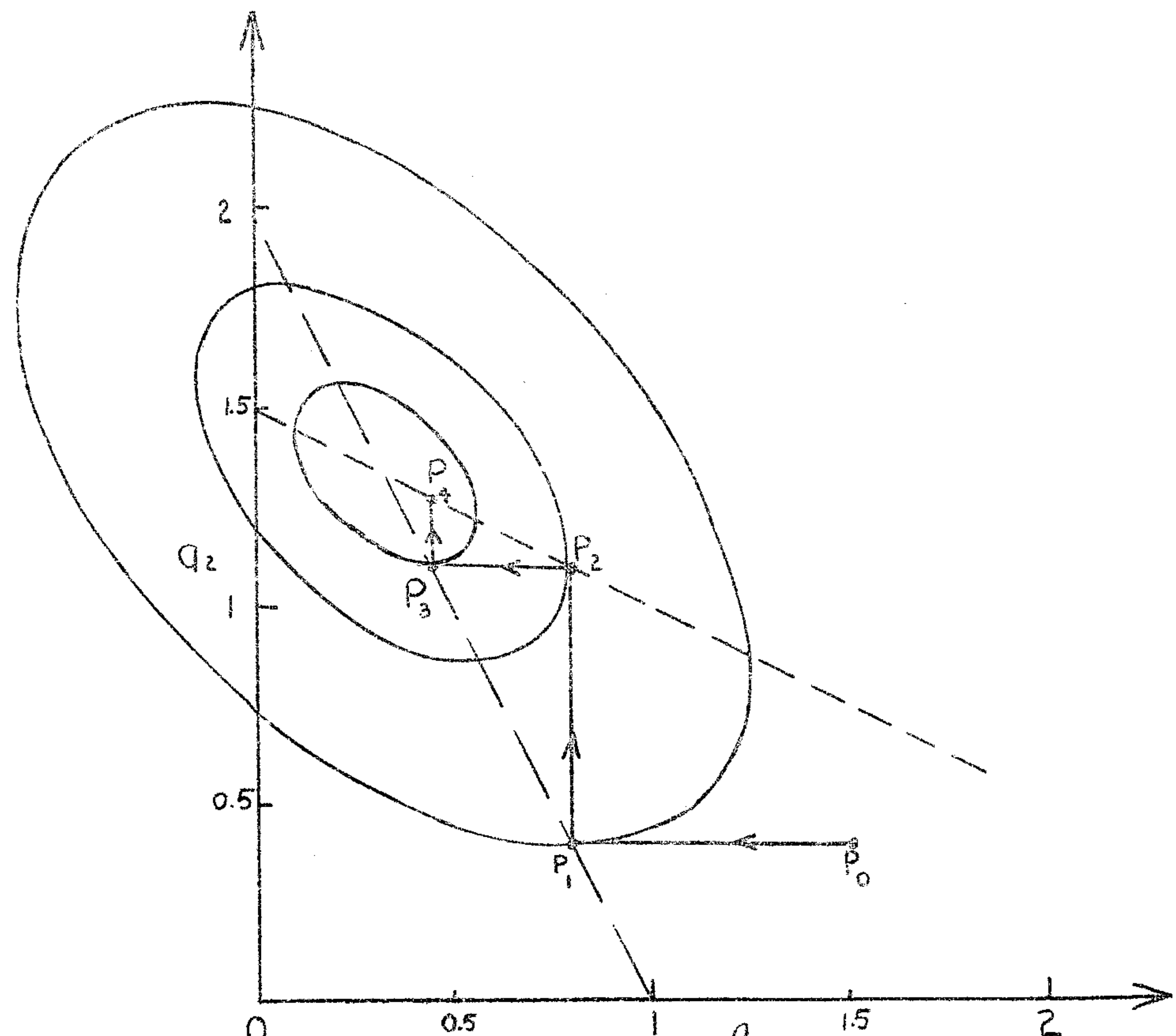

Fig. 2