COWLES FOUNDATION DISCUSSION PAPER NO. 42

Random Orderings

H. D. Block and Jacob Marschak *)

October 17, 1957

*) Research undertaken at the Department of Mathematics, Cornell University, under contract Nonr-401(03) of the Department of Industrial Engineering, Columbia University with the Office of Naval Research and at the Cowles Commission for Research in Economics under contract Nonr-358(01), NR 047-006 with the Office of Naval Research.

RANDOM ORDERING

H. D. Block and Jacob Marschak[1]

§0. Introduction

The present study was motivated by the need to substitute "stochastic consistency of choices" for the "absolute consistency of choices." The latter is usually assumed in economic theory but is not too well supported by experience and is, in fact, not assumed in empirical econometrics and psychology. We set up various types and degrees of stochastic consistency and analyze the logical relations between them, in Sections 1 and 2 of this Report. Section 3 deals in a tentative way with the testing of the hypothesis that a certain type of stochastic consistency is to be ascribed to a given person.

Let A be the set of all alternatives (actions; bundles of goods). In the non-stochastic theory of choice we say that $x \underset{i}{>} y$ (x is preferred to y by the individual i) if and only if the individual i, having to choose exactly one element of any subset $B \subseteq A$ containing x and y, never chooses y. We say $x \underset{i}{\doteq} y$ (i is indifferent between x and y) if neither $x \underset{i}{>} y$ nor $y \underset{i}{>} x$. We write $x \underset{i}{\geq} y$ if $x \underset{i}{>} y$ or $x \underset{i}{\doteq} y$. (The subscript i will be often omitted). The individual i is said to be absolutely consistent if the relation $\underset{i}{\geq}$ is transitive; it will then induce a complete weak ordering on A. Given a (so called "offered" or "feasible") set $B \subseteq A$, the subset $B_1 \subseteq B$ is called its "optimal" subset if it consists of all elements x_1 of B such that $x_1 \underset{i}{\geq} x$ for all x in B.

Under certain weak restrictions on A, satisfied by all finite sets and by an important class of infinite sets (see Debreu [1]) the existence of the relation $\geq_i$ implies the existence of a real valued function f^i on A such that $f^i(x) \geq f^i(y)$ if $x \geq_i y$. The function f^i (called taste-function or utility-index function of the individual i) is unique up to a monotone transformation. Given a feasible set $B \subseteq A$, the optimal set B_i consists of all elements $\hat{x}_B^i$ in B such that

$$\max_{x \varepsilon B} f^i(x) = f^i(\hat{x}_B^i).$$

To take an economic example: the individual's monetary wealth and the set of market prices of consumers' goods determine the feasible set B, consisting of all combinations of goods that he can buy. His taste function f^i determines then the optimal subset B_i. Note that f^i is not directly revealed by the individual's observable actions: the latter consist, rather, in the actual purchases made. Using certain assumptions on f^i Abraham Wald [3] has suggested a method to evaluate f^i from such observations, using the non-stochastic model just outlined and not attempting to develop a method of statistical estimation.

If A is finite, with elements labeled arbitrarily $1,\ldots,n$, and if B_i has to consist of only one element (i.e., ties are excluded) then each of the n! permutations (rankings) $r = (r_1,\ldots,r_n)$ of the first n integers can be regarded as a distinct taste-function, viz., $f(k) = r_k$; and precisely one of these taste-functions will be f^i, the taste function of the individual i, so that $f^i = r^i = r_1^i,\ldots,r_n^i)$. Again, the permutation r^i characterizing the individual's taste will not be revealed by his actions directly. Although it is possible to ask him to rank verbally the elements of A, or of any subset $B \subset A$, according to his preferences, this verbal response may or may not be consistent with his actual choices. What he will actually choose, will be only the most preferred element of an offered subset B of A. It is these "first choices" (for varying subsets B) that constitute the observable data. In the non-stochastic model, the individual i is characterized by a constant permutation r^i such that if k is in B then $r^i_k \geq r^i_j$ for all j in B.

Proceeding now to stochastic models: one way to weaken absolute consistency into stochastic consistency is to assume for the individual i the existence of a probability distribution F^i (or F, for brevity) on the set of all real-valued functions on A. We may call F^i the taste-distribution. For any subset $B \subseteq A$, the distribution F^i will generate a distribution G^i_B of the optimal subsets B_i. If, in addition, each B_i consists of one element only, or if all elements of B_i are assumed equiprobable, F^i will generate a distribution H^i_B of all the first choices $\hat{x}^i_B$ made out of the subset B. Suppose now that for certain given subsets B', B'', ... of A one knows the distributions $H^i_{B'}$, $H^i_{B''}$, ... of the first choices made. What can be then inferred about the underlying taste distribution F^i ?

In economic terms: suppose again that we have observed the variations of commodity prices and of a given individual's monetary wealth, and hence know the various feasible subsets B', B'', ..., that have been accessible to him--each subset possibly observed several times, thus permitting one to estimate the distributions $H^i_{B'}$, $H^i_{B''}$, ..., i.e., the probabilities with which he allocates his given resources (at given prices) in any particular fashion. What can be inferred about the underlying distribution F^i of his tastes?

This problem can be easily reformulated, and existing economic statistics (consumers' surveys) used, for the case when F^i is assumed to be the same for all individuals.

Preliminary to such investigations, one has to ask the question: which properties of the (observable, in principle) distributions H^i_B are necessary and sufficient for the existence of the (non-observable distribution F^i?

This question is studied in our Section 2 (Choice among n objects), for the case when the set A is finite, and hence the offered subsets B are finite. Let R be the set of all permutations of $A = (1, \ldots, n)$; denote its generic element by $r = (r_1, \ldots, r_n)$, and let the subset $R_{j,B}$ consist of all those permutations under which $r_j \geq r_k$ for all k in B. Denote by $P(j,B)$ the probability that the individual, when offered B, chooses its element j. The quantities $P(j,B)$ can be, in principle, estimated from observations. If the distribution $F(r)$ of permutations (rankings) exists, then, for any B and any j in B,

$$P(j,B) = \Sigma F(r) ,$$

the summation extending over the set $R_{j,B}$.

Section 1 deals with a more special case: here the offered subsets B are pairs. Assume that the individual i, when forced to choose from the pair (x,y) chooses x with probability

$P^i(x,y)$ or $P(x,y)$ for brevity]. When $P(x,y) > 1/2$ we say that he prefers x to y stochastically. This replaces the concept of absolute preference. If the relation of stochastic preference is transitive, i.e., if $P(x,y) \geq 1/2$ and $P(y,z) \geq 1/2$ imply $P(x,z) \geq 1/2$, a simple ordering on A is established. Then, under Debreu's restrictions on A there will exist a real valued function w on A, unique up to a monotone transformation and such that $w(x) \geq w(y)$ whenever $P(x,y) \geq 1/2$. We may call w a "weak utility function." An assumption that is strictly stronger than the transitivity of stochastic preference is the existence of a "strong utility function" u on A , unique up to a linear transformation and such that $P(x,y)$ is a non-decreasing function of $u(x) - u(y)$: for all x,y, $P(x,y) = \varphi[u(x) - u(y)]$ where φ is a distribution function whose mean and median are zero.

The transitivity of stochastic preferences and the existence of a strong utility function are shown to be the extreme links in a chain of conditions on the probabilities $P(x,y)$, each condition strictly stronger than the preceding one. This is the subject matter of Section 1.

The two ways of weakening absolute consistency into stochastic consistency--by assuming the existence of a distribution F of tastes (as in Section 2) or by assuming the existence of a weak (w) or a strong (u) utility function (as in Section 1)--are independent. F can exist without w (let alone u); and u (and hence w) can exist without F (cf 2.43). However, both u and F exist for any finite set of alternatives $A = (1, \dots, n)$ if there exist: 1) a random vector $U = (U_1, \dots, U_n)$ such that, for any j in $B \subseteq A$, the quantity $P(j,B)$ defined above is equal to $\Pr(U_j \geq U_k$, all k in B) , and 2) a constant vector $c = (c_1, \ldots, c_n)$ such that $c - U$ is distributed symmetrically in its variables. Then $F(r) = \Pr(U_{r_1} \geq U_{r_2} \geq \ldots \geq U_{r_n})$; and $u(j) = c_j$. The random variable U_j may be called the "current" utility of j , while the parameter u(j) is its "permanent" utility. If U is normal with equal variances and equal covariances, the means are permanent utilities.

In essence though without formal elaboration, the strong utility function u(j) —with "sensations" playing the role of our utilities—was used already by Fechner [4]. He applied to the results of his psychophysical trials a rough test of the existence of the univariate distribution ψ (defined above), assuming ψ to be normal. Thurstone [5] introduced, in essence, the symmetric distribution of the random vector U and assumed it to be normal (symmetry implying that all variances be equal and all covariances be equal); Mosteller [6] proposed statistical tests for Thurstone's hypothesis, permitting however the symmetry condition to drop.

While Fechner and Thurstone (and their numerous successors) deal with the psychology of perceptions and of attitudes, respectively, an extension of Fechner's approach or of some still weaker stochastic assumptions on choices between two alternatives (as in Section 1) to decisions and to economics was undertaken during the last decade by Hans Reichenbach [7], Clyde Coombs [8], Stephan Vail [9], Leo Tornquist [10], Ward Edwards [11], Duncan Luce [12], Andreas Papandreou (in collaboration with Leonid Hurwicz and others) [14]; Kenneth O. May [15]; Donald Davidson (in collaboration with one of the present authors) [16]; and others. The application to economics of some stochastic concepts relevant to the choice between n alternatives (as in our Section 2) was suggested by Nicholas Georgescu-Roegen [17], David Rosenblatt [18], Duncan Luce [13], and J. Marschak [19].

Our Section 3 deals with statistical tests. Each of the conditions of the preceding two sections defines a "region of consistency" in the space of certain probabilities. After observing a limited number of choices made by the individual from various feasible subsets of alternatives, one has to test the hypothesis that the relevant probabilities lie in that region. In psychophysical experiments, the replication of a trial with the same feasible subset for a large number of times is limited by learning and fatigue, unless one pools the trials on several individuals and assumes the latter to have identical stochastic properties. When experimenting with human choices (and attitudes) the replication possibilities are even more severely limited, because of the effects of memory: the subject may feel "committed" to a recent choice and is likely to repeat it. The statistician may even have to consider the extreme case, of each feasible subset being offered to the individual only <u>once</u>. This is analogous to the case when a coin-minting machine is assumed to have a certain unknown probability distribution γ of the chance variable P, the probability that a coin falls heads. One is permitted to toss several coins, each only a limited number of times (or perhaps indeed only once) in order to make inferences about γ .

We had the benefit of discussions with A. Calderon, Wassily Hoeffding, L. Hurwicz, and A. Sharma.

1. Choice between two objects.

1.1 Definition. Let A be a set of elements (alternatives) (a,b,...

For a given person at a given time we assume that for each pair (a,b) $a \neq b$ of elements from A, there is a certain probability P(a,b) that, if the person is forced to choose between a and b, he chooses a. For brevity we denote P(a,b) by ab. We define $aa = \frac{1}{2}$. Then $ab + ba \equiv 1$. The set of numbers {ab,...} thus defined might or might not have the following properties:

U (Utility Condition): There exists a real valued function u on A such that for each a,b,c,d in A:

$$u(a) - u(b) \geqq u(c) - u(d) \text{ if and only if } ab \geqq cd$$

S_s (Strong Condition on Sextuples): Any six elements $a_1, a_2, a_3, b_1, b_2, b_3$ in A which satisfy $a_1 a_2 \geqq b_2 b_3$ and $a_2 a_3 \geqq b_1 b_2$ also satisfy $a_1 a_3 \geqq b_1 b_3$.

Q (Condition on Quadruples): Any four elements a,b,c,d in A which satisfy $ab \geqq cd$ also satisfy $ac \geqq bd$.

S_w (Weak Condition on Sextuples): Any six elements, $a_1, a_2, a_3, b_1, b_2, b_3$ in A which satisfy $a_1 a_2 \geqq b_1 b_2$ and $a_2 a_3 \geqq b_2 b_3$ also satisfy $a_1 a_3 \geqq b_1 b_3$.

T_s (Strong Transitivity, or Strong Condition on Triples): Any three elements a,b,c in A which satisfy $ab \geqq \frac{1}{2}$ and $bc \geqq \frac{1}{2}$ also satisfy

$ac \geqq \max [ab, bc]$.

T_w (Weak Transitivity, or Weak Condition on Triples): Any three elements a,b,c in A which satisfy $ab \geqq \frac{1}{2}$ and $bc \geqq \frac{1}{2}$ also satisfy $ac \geqq \frac{1}{2}$

1.2. Remarks.

1.21) U can also be stated thus: There exists a real valued function u on A and a monotone function f of the probabilities ab such that $f(ab) = u(a)-u(b)$; or, alternatively: There exists a monotone function ψ such that $ab = \psi(u(a)-u(b))$. The function ψ can be regarded as a distribution function, with $\psi(z) + \psi(-z) = 1$; hence $\psi(0) = \frac{1}{2}$, i.e., the median is zero. If the first moment exists then $E(z) = 0$; if a density function ψ' exists then ψ' is an even function.

1.22) One might consider extending S_w to the condition that, for any integer n, $a_1a_2 \geqq b_1b_2$, $a_2a_3 \geqq b_2b_3, \ldots, a_{n-1}a_n \geqq b_{n-1}b_n$ implies that $a_1a_n \geqq b_1b_n$. However it is easy to see by successive applications of S_w that this is equivalent to S_w. On the other hand consider the following extension of S_s, n_s: Let n be any integer and let r be any permutation of the integers $1,2,\ldots,n-1$; then $a_1a_2 \geqq$

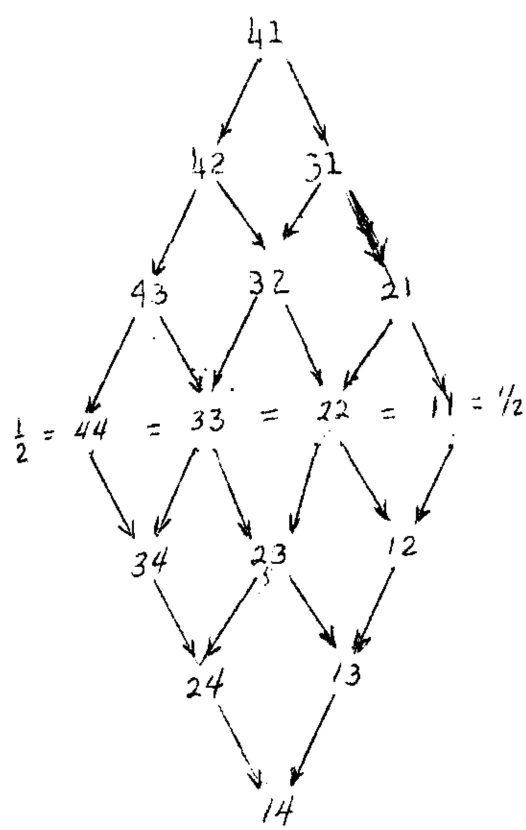
$b_{r(1)}b_{r(1)+1}$, $a_2a_3 \geqq b_{r(2)}b_{r(2)+1}, \ldots, a_{n-1}a_n \geqq b_{r(n-1)}b_{r(n-1)+1}$

imply that $a_1a_n \geqq b_1b_n$. We have not yet examined the condition n_s. Although it clearly implies S_s it seems likely that it is not implied by S_s; and although U clearly implies n_s it is not clear whether or not n_s implies U.

1.23) T_w implies that a transitive order relation "$\geqq$" exists among the elements of A: $a \geqq b$ if and only if $ab \geqq \frac{1}{2}$. Under certain mild restrictions [1], this is equivalent to the existence of a real valued "weak utility function" w on A, such that $w(a) \geqq w(b)$ if and only if $ab \geqq 1/2$.

1.24) T_s is equivalent to the following property; call it T_s^* : if $ab \geqq \frac{1}{2}$ then $ax \geqq bx$ for every x in A. Proof. If T_s holds and $ab \geqq \frac{1}{2}$ then for any x such that $bx \geqq \frac{1}{2}$, $ax \geqq bx$. If $by < \frac{1}{2}$ and $ya \geqq \frac{1}{2}$ then $yb \geqq \max[ya, ab]$, so $ay \geqq by$. If $by < \frac{1}{2}$ and $ya < \frac{1}{2}$ then $ay > \frac{1}{2} > by$. This T_s implies T_s^*. Conversely if T_s^* holds and $ab \geqq \frac{1}{2}$, $bc \geqq \frac{1}{2}$ then $ax \geqq bx$ and $by \geqq cy$. Letting $x = c$ we get $ac \geqq bc$ and letting $y = a$ we get $ba \geqq ca$ and hence $ac \geqq ab$. Thus T_s^* implies T_s.

1.25) In the presence of T_s the order relation "$\geqq$" imposed by T_w on the set A (see Remark 1.23) now implies that <u>ab</u> is an increasing function of <u>a</u> and a decreasing function of <u>b</u>. Thus the probabilities can be put into the form of a lattice; e.g. if A consists of four elements we shall have



where the arrowheads indicate " $\geqq$ ". The directions of the arrow between 42 and 31 or between 43 and 31, e.g., are left undetermined by T_s.

1.3 Theorem 1:

$$U \not\Longleftarrow\!\!\Longrightarrow S_s \not\Longleftarrow\!\!\Longrightarrow Q \not\Longleftarrow\!\!\Longrightarrow S_w \not\Longleftarrow\!\!\Longrightarrow T_s \not\Longleftarrow\!\!\Longrightarrow T_w, \text{ (where } A \not\Longleftarrow\!\!\Longrightarrow B$$

means that A implies B but B does not imply A).

Proof (i) U implies S_s. If $a_1a_2 \geqq b_2b_3$ and $a_2a_3 \geqq b_1b_2$ then, by U, $u(a_1) - u(a_2) \geqq u(b_2) - u(b_3)$, $u(a_2) - u(a_3) \geqq u(b_1) - u(b_2)$. Adding the last two inequalities we get $u(a_1) - u(a_3) \geqq u(b_1) - u(b_3)$, which by U implies that $a_1a_3 \geqq b_1b_3$.

(ii) S_s implies Q. Let $ab \geqq cd$; since $bc \geqq bc$, we get by S_s (letting $a_1 = a$, $b_1 = a_2 = b$, $a_3 = b_2 = c$, $b_3 = d$) $ac \geqq bd$.

(iii) Q implies S_w. If $a_1a_2 \geqq b_1b_2$ and $a_2a_3 \geqq b_2b_3$ then, by Q, $a_1b_1 \geqq a_2b_2$, $a_2b_2 \geqq a_3b_3$. Hence $a_1b_1 \geqq a_3b_3$, and by Q, $a_1a_3 \geqq b_1b_3$

(iv) S_w implies T_s. Let $ab \geqq \frac{1}{2} = bb$. Since $bc \geqq bc$, S_w implies that $ac \geqq bc$. Similarly $bc \geqq \frac{1}{2} = bb$, and $ab \geqq ab$ gives, by S_w, $ac \geqq ab$.

(v) T_s implies T_w. (obvious)

(vi) T_w does not imply T_s. Let $A = (a,b,c)$ and let $ab > ac > 1/2$ and $bc > \frac{1}{2}$. Then T_w is satisfied but not T_s.

(vii) <u>T_s does not imply S_w.</u> Let $A = (a,b,c,d)$ and take $\frac{1}{2} < ab < bc < cd < bd < ac < ad$. Then T_s is satisfied but not S_w; for $ac > bd$ and $bc > ab$ $(ba > cb)$ would give by S_w, $bc > cd$. In the notation of the ordering given in Remark 1.25 above, this example takes the form $\frac{1}{2} < 43 < 32 < 21 < 31 < 42 < 41$. Another example would be $\frac{1}{2} < 21 < 32 < 43 < 42 < 31 < 41$; cf. Remark 1.42 below

(viii) <u>S_w does not imply Q.</u> With $A = (1 < 2 < 3 < 4)$ we may use either $\frac{1}{2} < 32 < 43 < 21 < 31 < 42 < 41$ or $\frac{1}{2} < 43 < 21 < 32 < 31 < 42 < 41$. Each of these satisfies S_w but not Q; since $31 < 42$, Q requires $43 > 21$. The verification that S_w is satisfied can be expedited by noting that most of the relations of S_w are implied by T_s (which is easy to verify directly) and by making use of the symmetries; cf Remark 1.43 below.

(ix) <u>Q does not imply S_s.</u> It is possible to give several cases involving five elements which have the property Q but not S_s; e.g. $\frac{1}{2} < 21 < 54 < 32 < 43 < 53 < 31 < 42 < 41 < 52 < 51$.

Here Q is easily verifiable while from $54 \geqq 21$ and $43 \geqq 32$, S_s would require $53 \geqq 31$. Another set of examples is obtained by taking $\frac{1}{2} < 32 < 43 < 21 < 54 < 53 < 31 < 41 < 52 < 51$, with $43 < 42 < 53$. Cf. Remark 1.44 below.

(x) <u>S_s does not imply U.</u> The following system with A consisting of nine elements satisfies S_s but not U.

$\frac{1}{2} < 76 < 21 < 98 < 32 < 31 < 65 < 43 < 87 < 54 < 75 < 86 < 42 <$

$< 97 < 41 < 96 < 64 < 53 < 85 < 74 < 52 <$

$< 51 < 95 < 63 < 73 < 84 < 62 < 72 <$

$< 61 < 94 < 71 < 83 < 93 < 82 < 81 < 92 <$

$< 91.$

To see that U is not satisfied note that 21 > 76, 32 > 98, 43 > 65, 54 > 87 which, by U, would imply that 51 > 95. To see that S_s is satisfied, start with a slightly different system, namely one satisfying U with u(1) = 0, u(2) = 101., u(3) = 302., u(4) = 612., u(5) = 1017., u(6) = 1326., u(7) = 1426., u(8) = 1826., u(9) = 2026. This gives rise to a set of probabilities which differs from the given set only in that the position of 51 and 95 are reversed (and of course the images 15 and 59 are also reversed). Since S_s is automatically satisfied in the new system, the only possible way in which it can be violated in the given system is in sextuples where one of these occurs on each side of the inequality; but, for these, S_s is readily verified directly. Cf. Remark 1.45 below

1.4 Remarks

1.41) If A consists of only three elements then T_s implies U. If A has four or fewer elements, then Q implies U. If A has five or fewer elements then it appears that S_s implies U; the proof of this last statement is however not quite complete.

1.42) The examples in (vii) were arrived at by the following considerations. We first restate S_w thus: "The three relations

$a_1a_2 \geq b_1b_2$, $a_2a_3 \geq b_2b_3$, $a_3a_1 \geq b_3b_1$ imply that all three of the $\geq$ are equalities." Now any two (unordered) triples of elements (a_1,a_2,a_3), (b_1,b_2,b_3) generate the following six triples of relations (using $\geq$ throughout without loss of generality):

(1) $a_1a_2 \geq b_1b_2$, $a_2a_3 \geq b_2b_3$, $a_3a_1 \geq b_3b_1$

(2) $a_1a_2 \geq b_2b_3$, $a_2a_3 \geq b_3b_1$, $a_3a_1 \geq b_1b_2$

(3) $a_1a_2 \geq b_3b_1$, $a_2a_3 \geq b_1b_2$, $a_3a_1 \geq b_2b_3$

(4) $a_1a_2 \geq b_3b_2$, $a_2a_3 \geq b_2b_1$, $a_3a_1 \geq b_1b_3$

(5) $a_1a_2 \geq b_2b_1$, $a_2a_3 \geq b_1b_3$, $a_3a_1 \geq b_3b_2$

(6) $a_1a_2 \geq b_1b_3$, $a_2a_3 \geq b_3b_2$, $a_3a_1 \geq b_2b_1$

If A consists of four elements then if the two sets of triples have zero, one or three elements in common then T_s does imply in (1)-(6) the equality holds in each case. For example with triples (x,a,x) and (y,a,z) the relations (1) become $xa \geq ya$, hence $xy \geq \frac{1}{2}$, $x \geq y$ (in the notation of 1.23); $ax \geq az$, hence $z \geq x$; $xx \geq zy$; hence $z = y = x$ and equality holds throughout (1). Similarly with the relations (2)-(6). If the two triples of elements have two elements in common as in (a,x,b), (a,y,b), then by similar argument, T_s^* will make the relations (1), (4), (5), (6) into equalities. There remains the relations (2); relations (3), equivalent with (2) under interchange of letters, we need not consider. We have by (2): $ax \geq yb$, $xb \geq ba$, $ba \geq ay$. By T_s, if $a \leq b$, these relations imply

$\frac{1}{2} \geq ab \geq ya \geq yb \geq ax$ and hence the sequence $x \geq b \geq a \geq y$; if $a \geq b$, they imply the reverse sequence. Therefore we can, without loss of generality set $x = 4$, $b = 3$, $a = 2$, $y = 1$. The triple of relations now becomes $31 \geq 42$; $43 \geq 32 \geq 21$; or, the reverse $31 \leq 42$; $43 \leq 32 \leq 21$. Using the lattice condition of 1.25 (it suffices to consider its upper part) we see that the only ways to obtain T_s without S_w are 41,31; 42,43; 32; 21 and 41,42; 31,21; 32; 43, where the commas and semi-colons stand for ≥, but at least one of the semi-colons signifies >.

1.43) To construct the examples in (viii) one uses the symmetries and proceeds very much like in Remark 1.42 above.

1.44) The construction of the examples in (ix) is based on reasoning similar to that in Remark 1.42, employing the symmetries to simplify the calculations.

1.45) The construction of the example in (x) is motivated as follows. We assume that the numbers xy have been modified (cf Remark 1.46 below) so that $xy = \frac{1}{2} + u(x) - u(y)$. Suppose that the objects have already been arranged in a lattice (Remark 1.25) e.g. $9 > 8 > . . > 1$. The direction of the missing arrows* is determined by the difference of the u's, i.e. let $u(i+1)-u(i) = \varepsilon_i$. Then e.g. $98 = \varepsilon_8 + 1/2$, $87 = \varepsilon_7 + \frac{1}{2}$ and $98 > 87$ if and only if $\varepsilon_8 > \varepsilon_7$. Moreover the sum of a consecutive run of ε's is expressible as one of the numbers xy; e.g. $\varepsilon_5 + \varepsilon_6 = u(6)-u(5)+u(7)-u(6) = u(7)-u(5) = 75 - 1/2$. Now if we take $\varepsilon_1 > \varepsilon_6$, $\varepsilon_2 > \varepsilon_8$, $\varepsilon_3 > \varepsilon_5$, $\varepsilon_4 > \varepsilon_7$, it follows by adding that $51 > 95$. But from only these assumptions on the ε's, S_s

* This, incidentally, furnishes a workable algorithm for testing whether a given set of numbers satisfies U. The relative size of the ε's are determined easily from the order and the order may then be tested for consistency. This is not however the explicit, simple criterion sought in 1.46.

does not imply 51 > 95. It remains only to pick such ε's, form the u's and compute the ordering. The ε's are then adjusted so as to make 51 and 95 adjacent in the ordering; this makes the verification of S_s simpler. The ε's used for the example given were $\varepsilon_1 = 101$, $\varepsilon_2 = 201$, $\varepsilon_3 = 310$, $\varepsilon_4 = 415$, $\varepsilon_5 = 309$, $\varepsilon_6 = 100$, $\varepsilon_7 = 400$, $\varepsilon_8 = 200$

1.46) The following question, which arises in an obvious way from the foregoing, has not yet been answered. Let (k_{ij}) be an anti-symmetric matrix with no off-diagonal elements equal. Suppose the numbers k_{ij} (which correspond to the probabilities minus $\frac{1}{2}$) are marked as points on the real line in the usual way. Call this an <u>arrangement</u> of the points. Now move these points on the line in any way which leaves the order unchanged. Call such an arrangement equivalent to the original one. We would like to know for which of the arrangements there exists an equivalent arrangement k'_{ij} with the property that there is an increasing function u such that $k'_{ij} = u(i)-u(j)$. An alternative formulation is to ask that there exist increasing functions h and u such that $k_{ij} = h(u(i)-u(j))$. Clearly a necessary condition is that k_{ij} be an increasing function of <u>i</u> and decreasing in <u>j</u>. Conditions analogous to S_s, Q and S_w are also clearly necessary, but what is desired is a reasonably simple criterion characterizing those arrangements which admit the above representation.

If A can be represented by a real interval, say [0,1], the matrix (k_{ij}) is replaced by a function $f(x,y)$ on the unit square; and the question analogous to the one just discussed is answered by the theorems in the next section (1.5).

(For our purposes, read $v \equiv u$, $b \equiv a$, D the unit square in what follows.)

1.5 Theorems.

Let $f(x,y)$ be a continuously differentiable function on a two dimensional domain D and whose range is a set of real numbers Γ. Let $\frac{\partial f}{\partial y} \neq 0$ in D. A curve $y = \eta(x)$ on which $f(x,y)$ is constant is called a contour line of f; clearly along a contour line the slope

$\frac{d\eta}{dx} = \frac{\frac{\partial f(x,y)}{\partial x}}{-\frac{\partial f(x,y)}{\partial y}}$. Then we have

Theorem 2. If there exists a function g on Γ and differentiable functions $u(x), v(y)$ defined over the projection of D on the X and Y axes respectively such that $g(f(x,y)) = u(x)-v(y)$ then at each point $\frac{d\eta}{dx} = \frac{u'(x)}{u'(y)}$. Conversely if at each point $\frac{d\eta}{dx} = \frac{a(x)}{b(y)}$ then there are functions g, u, v such that $g(f(x,y)) = u(x)-v(y)$.

Proof: If $g(f(x,y)) = u(x)-v(y)$ then, along a contour $f(x,y) = s$, $g(s) = u(x)-v(\eta(x))$, so $u'(x) = v'(y)\frac{d\eta}{dx}$. Conversely if along each contour $f(x,y) = s$, $\frac{d\eta}{dx} = \frac{a(x)}{b(y)}$, let $u'(x) = a(x)$, $v'(y) = b(y)$; then along that contour $v'(y)dy = u'(x)dx$ or $u(x)-v(y) = C(s)$, say. Hence $u(x)-v(y) = C(f(x,y))$.

1.51) Corollary 1. If, in addition, f is monotone increasing in x and decreasing in y then: in the first part of the theorem the monotonicity of g implies that of u and v and vice-versa; and in the second part, the g, u, v will all be monotone.

1.52) Illustration. The theorem can be used to determine whether a given specific function has the desired representation and, if so, the explicit form of it. For example let $f(x,y) = \frac{x-y}{1+x+y}$. We ask: Do there exist monotone functions g and u such that $g(f(x,y)) = u(x)-u(y)$? This does not appear to be immediately obvious. However we form $\frac{d\eta}{dx} = \frac{f_1}{-f_2} = \frac{a(x)}{a(y)}$ where $a(x) = \frac{1}{1+2x}$ so that the desired representation is possible. We can take $u(x) = \log(1+2x)$. To find g, let $f(x,y) = \frac{x-y}{1+x+y} = s$; i.e., $y = \frac{x-s-sx}{s+1}$. Then $g(s) = u(x)-u(y(x)) = \log\frac{1+s}{1-s}$.

An alternative representation is to let $h = g^{-1}$ so that $f(x,y) = \frac{x-y}{1+x+y}$ is equal to $h(u(x)-u(y))$, where $h(z) = \frac{e^z-1}{e^z+1} = \tanh\frac{z}{2}$ and $u(x) = \log(1+2x)$.

1.53 Corollary 2. Suppose that A is representable by an interval I and that $P(x,y)$ (the probability that x is chosen over y) is continuously differentiable on the domain $D = A \times A$ with $\frac{\partial P}{\partial y} \neq 0$ on D. Then the condition S_W is sufficient to guarantee that the condition U is satisfied.

Proof: Let $f(x,y) = P(x,y) - \frac{1}{2}$. Let a,b,c be three arbitrary numbers in I. Let $(a+\Delta x, b+\Delta y)$ lie on the contour through (a,b) so that $f(a,b) = f(a+\Delta x, b+\Delta y)$. Let $(b+\Delta y, c+\Delta z)$ lie on the contour through (b,c) so that $f(b,c) = f(b+\Delta y, c+\Delta z)$. From the last two equations S_W implies that $f(a,c) = f(a+\Delta x, c+\Delta z)$. Thus when we let $\Delta x \to 0$, we have $\Delta y \to 0$, $\Delta z \to 0$, $\frac{dy}{dx}$ is the slope of the contour at (a,b), $\frac{dz}{dy}$ the slope of the contour at (b,c) and $\frac{dz}{dx}$ the slope of the contour at (a,c). But $\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = \frac{f_1(b,c)f_1(a,b)}{f_2(b,c)f_2(a,b)}$. Since $P(x,y)+P(y,x) = 1$, we have $f(y,x) = -f(x,y)$ and it follows that $f_2(y,x) = -f_1(x,y)$. With b fixed let $\psi(x) = \frac{f_1(x,b)}{f_2(x,b)}$. Then $\frac{dz}{dx} = \frac{\psi(a)}{\psi(c)}$, which, since <u>a</u> and <u>c</u> are arbitrary, is the sufficient condition of the theorem.

1.54 <u>Remark</u>. Since Q implies S_w it follows, under the conditions of Corollary 2 above, that Q implies U. However we give the following proof of this because of its simplicity. Along a contour $f(a,b) = f(a+\Delta x, b+\Delta y)$. By Q, $f(a,a+\Delta x) = f(b,b+\Delta y)$. Then by the mean value theorem, $f_2(a,a) = f_2(b,b)\frac{dy}{dx}$, or $\frac{dy}{dx} = \frac{f_2(a,a)}{f_2(b,b)} = \frac{\psi(a)}{\psi(b)}$.

1.55 <u>Reference</u>. With regard to the material in this section cf. Debreu [2], where, however, a different definition of the continuity of the set A is used.

§2. <u>Choice among n objects</u>.

2.1) <u>Three alternatives</u>. Suppose that A consists of three alternatives a,b,c. Let x,y,z denote distinct generic elements of A If we are given the probabilities, xy, that x is chosen when x and y are offered, then one might ask whether there exists a probability distribution on the six permutations of x,y,z such that

$$xy = xyz + xzy + zxy. \tag{1}$$

Here xyz denotes the probability* that the person's ranking of the triple (x,y,z) is such that x comes first, y second and z third; and equation (1) expresses the fact that the probability of choosing x over y should be the sum of the probabilities of those rankings in

* In section 2.3 below we shall use P(xyz) to denote this probabilit in order to distinguish between the permutation itself and its proba bility.

which x is ahead of y. This will be possible if the six equations (1) have solutions xyz,... which form a probability distribution, i.e., are non-negative and add up to one. It can be shown that this is possible if and only if

(2) $1 \leq ab + bc + ca \leq 2$

or, equivalently if and only if

(3) $xy + yz + zx \geq 1.$

This may also be written

(4) $xy + yz \geq xz.$

However, instead of proceeding to show this directly we introduce the probabilities x(x,y,z) of choosing x from among the set (x,y,z); these presumably will be observable, as well as xy which will henceforth be denoted by x(x,y).

Suppose the observable probabilities of first choices, x(x,y) and x(x,y,z) are known and are in fact the result of an underlying probability distribution on the six permutations. Then, in addition to (1), the equations

(5) $x(x,y,z) = xyz + xzy$

will also be satisfied.

The solution to the system (1), (5) is

(6) $xyz = y(y,z) - y(x,y,z).$

These solutions sum up to 1, and hence the condition

(7) $x(x,y) \geq x(x,y,z)$

is necessary and sufficient for the existence of the desired probability distribution. <u>Thus, if we specify all the probabilities of</u>

<u>first choices x(x,y), x(x,y,z), the probability distribution on the permutations which generates them exists if and only if (7) holds; when it exists it is unique</u>.

It is easily verified that (7) implies (3), while if (3) is satisfied it is possible to specify three non-negative numbers $x(x,y,z) \leqq \min(xy, xz)$ which will add up to 1; they will satisfy (7). To see this let $\gamma = \min(ab,ac) + \min(bc,ba) + \min(ca,cb)$. Without loss of generality say ab is minimal among the six numbers xy. Then ba is maximal. Hence $\gamma = ab + bc + \min(ca,cb)$. But $ab + bc + ca \geqq 1$ by (3), while $ab + bc + cb = ab+1 \geqq 1$. Hence $\gamma \geqq 1$. This proves the existence of the desired numbers x(x,y,z). Furthermore if the inequality holds in (3) and all $xy > 0$, then $\gamma > 1$, and the numbers x(x,y,z) may be chosen in more than one way; by (6) this shows that the solution xyz is not unique. Thus we have proved that in the case in which only all the probabilities of first choices out of two, x(x,y), are specified, then an underlying probability distribution on the permutations which generate them will exist if and only if (3) [or equivalently (2) or (4)] holds; if the inequality sign holds in (3) and if all $x(x,y) > 0$ then the solution is not unique.

2.2) <u>Four Alternatives</u>. If A consists of four alternatives a, b, c, d and we are given the probabilities of first choice w(w,x), w(w,x,y), w(w,x,y,z), where w,x,y,z, are distinct generic elements of A, then we ask if there exists a probability distribution on the 24 permutations abcd, abdc, ... which generate the given probabilities of first choice; i.e. (letting wxyz denote the probability of that permutation)

(1) $w(w,x) = wxyz + wxzy + wyxz + \ldots + zywx$

(2) $w(w,x,y) = wxyz + wyxz + \ldots + zwyx$

(3) $w(w,x,y,z) = wxyz + wyxz + \ldots + wzyx,$

where the sum in (1) is over all permutations in which w precedes x; in (2) where w precedes x and y, and in (3) where w precedes x,y and z. Before proceeding to solve (1), (2), (3), we note that

(4) $w(w,x,y) - w(w,x,y,z) = zwxy + zwyx$

and that

(5) $w(w,x) - [w(w,x,y)+w(w,x,z)] + w(w,x,y,z) = yzwx + zywx,$

which gives the necessary conditions

(6) $w(w,x,y) \geqq w(w,x,y,z)$ and

(7) $w(w,x) - [w(w,x,y) + w(w,x,z)] + w(w,x,y,z) \geqq 0.$

We shall show that the conditions (6) and (7) are indeed also sufficient for the existence of the desired probability distribution although the solution is not now unique. The system (4), (5) consists of twenty four equations; each permutation occurs exactly once in the set (4) and exactly once in the set (5). Thus the system (4), (5) breaks into six sets, with four equations in each set; each set involves exactly four permutations, none of which appear in the other five sets: viz.

$$wxyz + wxzy = x(x,y,z) - x(w,x,y,z)$$

$$wxyz + xwyz = y(y,z) - [y(w,y,z)+y(x,y,z)] + y(w,x,y,z)$$

$$xwyz + xwzy = w(w,y,z) - w(w,x,y,z)$$

$$xwzy + wxzy = z(y,z) - [z(w,y,z)+z(x,y,z)] + z(w,x,y,z)$$

which is of the form

$$p_1 + p_2 = \alpha$$

$$p_2 + p_3 = \beta$$

$$p_3 + p_4 = \gamma$$

$$p_4 + p_1 = \delta = a - \beta + \gamma ,$$

where α, β, γ, δ are non-negative by (6), (7). This has non-negative solutions p_1, p_2, p_3, p_4 since, if $\beta \leqq \gamma$ we may take $p_2 = 0$, while if $\beta > \gamma$ we may take $p_4 = 0$. Thus for each of the six ways of choosing a pair of objects for the first two (or, equivalently, the last two) one gets four permutations (i.e., wxyz, wxzy, xwyz, xwzy); of the probabilities to be assigned to these permutations, one may be chosen freely in a suitable region while the other three are then determined.

We have thus established the existence of non-negative solutions of (4), (5). By substitution into the right hand sides of (1),(2), (3) it may be verified that any solution of the system (4),(5) also satisfies (1),(2),(3). By adding, e.g. (3) over the four values of w it is seen that the sum of the desired probabilities is indeed unity.

2.21 Remark. Although the necessary condition for the three element case (condition 7 of section 2.1) does not appear explicitly in the above, it is of course implied by the conditions (6) and (7), as may be seen by adding (6) and (7) or from the fact that the existence of the probability distribution on the permutations of four elements must imply the existence of the marginal probabilities of the permutations of a specified three; i.e., xyz of the three element case would be wxyz + xwyz + xywz + xyzw of the four element case.

2.3 n Alternatives. Suppose that A consists of n elements a,b,... and suppose that, for each subset B of A a probability distribution for the first choice out of B is given; i.e., non-negative numbers x(B) such that $\sum_{x\varepsilon B} x(B) = 1$. If there exists a probability distribution on the set of the n! permutations of the n elements of A, (uv...z) such that x(B) is equal to the sum of the probabilities P(uv...z) taken over those permutations in which x precedes the other elements of B, then we say that the given probabilities have the property D (distribution). The generalization of 2.1 (7) or 2.2 (6):

(1) If $C \subseteq B \subseteq A$ then $x(C) \geq x(B)$ for each $x \varepsilon C$

is again a necessary condition for D. This follows from the fact that x(C) - x(B) must be equal to the sum of the probabilities of those permutations of B in which at least one element of B-C precedes x while x precedes the elements of C-x. (cf. 2.1 (6) and 2.2 (4)).

Moreover we also have the identity generalizing 2.2 (5), viz. if $(u,v,x_1,x_2,\ldots,x_m)$ is a subset $B \subseteq A$ then

(2)
$$u(u,v) - [u(u,v,x_1)+u(u,v,x_2)+\ldots+u(u,v,x_m)] +$$
$$[u(u,v,x_1,x_2)+u(u,v,x_1,x_3)+\ldots+u(u,v,x_{m-1},x_m)]$$
$$- \ldots \pm u(u,v,x_1,\ldots,x_m) = \Sigma P(x_{i_1}\ldots x_{i_m}\ uv),$$

the sum on the right being taken over all permutations of B which end in uv. To verify (2) one may count the number of times, with sign, that any particular permutation is included on the left side. A term of the type ...u...v..., where $k \geq 1$ elements besides v follow u, will have the coefficient

$$1 - \binom{k}{1} + \binom{k}{2} - \ldots \pm \binom{k}{k} = (1-1)^k = 0$$

and so the only terms remaining will be of the specified type.

Thus we have as necessary conditions (1) and that the left side of (2) shall be non-negative; whether these conditions are also sufficient for D when $n \geq 5$ we can only conjecture.

2.4 Remark. We have not yet discussed the logical relations between the conditions formulated in the present §3 and those formulated in §2. §3 has dealt with conditions that are to be satisfied by the probabilities of first choices from all subsets of A, in order to establish the condition D, i.e., the existence of a probability distribution on the permutations of the elements of A. §2, on the other hand, dealt with a sequence of conditions of increasing strength (T_w, T_s, S_w, Q, S_s, U) which are to be satisfied by the probabilities of first choices from all pairs of elements of A, in order to establish the existence of a weak utility function (by T_w) or a strong utility function (by U) or of certain intermediate types of stochastic consistency of choices.

We can show that D does not imply T_w (and hence does not imply U); and that U does not imply D.

2.41. The following two examples, suggested by P. R. Halmos and C. Winsten, respectively, show that D does not imply T_w.

(i) Let A, B, C be three dice, loaded so as to turn up with the following probabilities

Face No	1	2	3	4	5	6
Die: A	0	0	.5	.5	0	0
B	0	.6	0	0	0	.4
C	.4	0	0	0	.6	0

Let X, Y, Z be (independent chance variables) the number of spots turning up on A, B, C respectively. Then $P(X > Y) = .6 > \frac{1}{2}$, $P(Y > Z) = .64 > \frac{1}{2}$, and $P(Z > X) = .6 > \frac{1}{2}$.

ii) Let $0 < \alpha < \frac{1}{6}$ then the following six numbers are positive and add up to unity: $abc = bca = cab = \frac{1}{6} + \alpha$; $cba = bac = acb = \frac{1}{6} - \alpha$. Here $ab = bc = ca > \frac{1}{2}$.

2.42. On the other hand T_w does not imply the condition (3) of 2.1 and hence certainly does not imply D. This is shown by the example: $A = (a,b,c)$ with $ab = .1$, $bc = .2$, $ac = .4$. Here T_w is satisfied but since $ab + bc + ca = .7$, the condition (3) of 2.1 is violated.

2.43. We shall now show that T_s implies the condition (3) of §2.1. For, if 2.1 (3) does not hold, then with a suitable labelling of the elements we have $a(a,b) + b(b,c) + c(a,c) < 1$. Then $a(a,b) + b(b,c) < 1$; $a(a,b) < c(b,c)$ (or in the notation of §1, $ab < cb$). It then follows from T_s (see 1.25) that $c(a,c) > \frac{1}{2}$. Similarly $a(a,b) > \frac{1}{2}$, $b(b,c) > \frac{1}{2}$ so that $a(a,b) + b(b,c) + c(a,c) > 3/2$, a contradiction. Hence T_s implies 2.1 (3).

This shows that if only all the probabilities of first choices out of pairs are observed (or specified) then (for three objects) T_s does imply that an underlying distribution can exist which generates the given probabilities. On the other hand if we are given all the probabilities of first choices out of all subsets, the satisfaction of T_s (which is a condition only on the probabilities of first choices out of pairs), or even indeed U, is not sufficient to guarantee D, as may be seen from the example:

$a(a,b) = 1 - b(a,b) = .7$; $b(b,c) = 1 - c(b,c) = .7$; $c(a,c) = 1 - a(a,c) = .1$; $a(a,b,c) = b(a,b,c) = .3$, $c(a,b,c) = .4$.

Here U is satisfied since we may take $u(a) = .7$, $u(b) = .5$, $u(c) = .3$ and $x(x,y) = \frac{1}{2} + u(x)-u(y)$, but since $c(a,b,c) > c(a,c)$ the condition (7) of §2.1 is violated and hence D is not satisfied.

§3. Statistical Tests.

We intend in a later paper to develop statistical tests for all the conditions U, S_s, Q, S_w, T_s, T_w, D. Here we present two methods of testing T_w.

During the course of the experiment the subject will be asked to choose one from each of the sets (a,b), (b,c), (a,c). Let p_1 be the probability of choosing a from (a,b), p_2 the probability of choosing b from (b,c), p_3 the probability of choosing c from (a,c); then $p = (p_1,p_2,p_3)$ is a vector in $[0,1]^3$. Assume that the subject has a certain probability distribution $\mathcal{P}$ of choosing points in $[0,1]^3$. Let $f(p) = p_1p_2p_3 + (1-p_1)(1-p_2)(1-p_3) = \frac{1}{4} + \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1$, where $p_i = \frac{1}{2} + \alpha_i$. The region of $[0,1]^3$ where T_w is not satisfied (call it R) is characterized by the fact that all three α_i's are of the same sign. It is desired to estimate $\mathcal{P}(R) = \theta$.

If $p \epsilon R$ then $f(p) \geqq \frac{1}{4}$. If $p \not\epsilon R$ then $f(p) \leqq \frac{1}{2}$, for if $\alpha_1 = -\alpha$, $\alpha_2 = \beta$, $\alpha_3 = \gamma$ (α, β, γ non-negative) $f(p) = \frac{1}{4} + \beta\gamma - \alpha(\beta+\gamma) \leqq \frac{1}{4} + \beta\gamma \leqq \frac{1}{2}$ and the other possibilities are covered by symmetry.

When the experiment has been performed, let $X_1 = 1$ if a is chosen from (a,b) and 0 if b is chosen; let $X_2 = 1$ if b is chosen from (b,c) and 0 if <u>c</u> is chosen and let $X_3 = 1$ if <u>c</u> is chosen from (a,c) and 0 if <u>a</u> is chosen. Let $Z = X_1X_2X_3 + (1-X_1)(1-X_2)(1-X_3)$. For a given vector p, $P(Z=1) = f(p)$. Now

$P(Z=1) = P(Z=1|p\varepsilon R)\mathcal{P}(p\varepsilon R) + P(Z=1|p\notin R)\mathcal{P}(p\notin R)$. Hence

$P(Z=1) \leqq \theta + \frac{1}{2}(1-\theta) = \frac{1}{2}(1+\theta)$ and $P(Z=1) \geqq \frac{\theta}{4} + 0 = \frac{\theta}{4}$. Thus Z is a binomial chance variable with mean $\mu = P(Z=1)$ where

$$(1) \ldots \frac{\theta}{4} \leqq \mu \leqq \frac{1}{2} + \frac{\theta}{2} .$$

Let $Z_1, Z_2, \ldots, Z_n$ be independent replicates of Z. (We assume that a random choice of triples (a,b,c) leads to a random choice from $[0,1]^3$ with the distribution $\mathcal{P}$ on it.) Let $\overline{Z} = \frac{1}{n}\sum_{i=1}^{n} Z_i$. Our first test is based on the following. 1) Let $0 \leq \theta_o < \frac{1}{2}$. If $\theta \leqq \theta_o$ then $\mu_i \leqq \frac{1}{2} + \frac{\theta_o}{2}$. Let V_o be a binomial random variable with $P(V_o=1) = \frac{1}{2} + \frac{\theta_o}{2}$. Then $P(Z_i=1) \leqq P(V_o=1)$. Let $a_o \geq \frac{1}{2} + \frac{\theta_o}{2}$. Then $P(\overline{Z} \geqq a_o) \leqq P(\overline{V}_o \geqq a_o)$. Now if a confidence level α_o is specified one can choose n sufficiently large so that $P(\overline{V}_o \geqq a_o) \leq \alpha_o$. Hence:

$$\text{if } \theta \leqq \theta_o,\ P(\overline{Z} \geqq a_o) \leqq \alpha_o.$$

On the other hand let $0 < \theta_1 \leq 1$: 2) if $\theta \geq \theta_1$ then $\mu_i \geqq \frac{\theta_1}{4}$. Let V_1 be a binomial chance variable with $P(V_1=1) = \frac{\theta_1}{4}$. Then $P(Z_i=0) \leqq P(V_1 = 0)$. Let $a_1 \leq \frac{\theta_1}{4}$. Then $P(\overline{Z} \leq a_1) \leqq P(\overline{V}_1 \leqq a_1)$. Again with a specified confidence level α_1, we can choose n large enough so that $P(\overline{V}_1 \leqq a_1) \leq \alpha_1$. Hence:

$$\text{if } \theta \geqq \theta_1,\ P(\overline{Z} \leqq a_1) \leqq \alpha_1.$$

The test then consists of the following. If $\overline{Z} \geqq a_o$ we assert $\theta > \theta_o$; if $\overline{Z} \leqq a_1$ we assert $\theta < \theta_1$; if $a_1 < \overline{Z} < a_o$ we assert nothing. The consequences of this are shown in the following table of probabilities of occurance

True State:→	$\theta \leqq \theta_o$	$\theta \geqq \theta_1$
Assertion:↓		
$\theta > \theta_o$	$<\alpha_o$	-
$\theta < \theta_1$	-	$<\alpha_1$
Nothing	-	-

where the dashes indicate probabilities that we have not estimated. This procedure protects us against extreme errors of classification but leaves open the possibility of coming to no decision.

If n is large enough so that the normal approximation to the binomial may be used, then the constants involved in the test are determined by the equations

$$\int_{\frac{\sqrt{n}\,(2a_o-1-\theta_o)}{\sqrt{(1+\theta_o)(1-\theta_o)}}}^{\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,dy = \alpha_o \quad \text{and}$$

$$\int_{\frac{(\theta_1-4a_1)\sqrt{n}}{\sqrt{\theta_1(4-\theta_1)}}}^{\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,dy = \alpha_1$$

Our second method is to find a confidence interval for θ. Let $0 < \delta < 1$ and $0 < \alpha < 1$ be given. Then it follows that for n sufficiently large $P(-\delta < \overline{Z} - \mu < \delta) > 1-\alpha$; i.e.
$P(\overline{Z}-\mu < \delta$ and $\overline{Z}-\mu > -\delta) > 1-\alpha$, or $P(\overline{Z} < \delta + \mu$ and $\overline{Z} > \mu-\delta) > 1-\alpha$; from (1) it follows that $P(\overline{Z} < \delta + \frac{1}{2} + \frac{\theta}{2}$ and $\overline{Z} > \frac{\theta}{4} - \delta) > 1-\alpha$. Thus $P(2\overline{Z} - 2\delta-1 < \theta < 4\overline{Z} + 4\delta) > 1-\alpha$.

While it is true that the length of the confidence interval $(2\overline{Z} - 2\delta - 1, 4\overline{Z} + 4\delta)$ is $2\overline{Z} + 6\delta + 1 > 1$, it is not centered at $\frac{1}{2}$ so that if $\overline{Z}$ is near zero or unity the effective length of the interval may be quite small; e.g. if $\overline{Z} = 0$ the conclusion is $-2\delta-1 < \theta < 4\delta$ which has an effective length of only 4δ; while if $\overline{Z} = 1$ the conclusion is $1-2\delta < \theta < 4 + 4\delta$ which has an effective length of 2δ. Since for a prescribed significance level $1-\alpha$ we do not know δ explicitly we can use the estimate $\delta = k(\alpha)\sigma = k(\alpha)\sqrt{\frac{pq}{n}} \leq \frac{k(\alpha)}{2\sqrt{n}}$.
Thus $P(2\overline{Z} - \frac{k(\alpha)}{\sqrt{n}} -1 < \theta < 4\overline{Z} + \frac{2k(\alpha)}{\sqrt{n}}) > 1-\alpha$.
If the normal approximation for the binomial is used $k(\alpha)$ is determined from the equation

$$\int_{-k(\alpha)}^{k(\alpha)} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy = 1 - \alpha.$$

The confidence interval may also be used for making decisions, e.g., let $0 < \theta_2 \leqq \theta_3 < 1$. If the confidence interval is contained in the interval $(0,\theta_2)$ assert that $\theta \leqq \theta_2$; if the confidence interval is contained in the interval $(\theta_3,1)$ assert that $\theta \geq \theta_3$: otherwise assert nothing. In other words if $\overline{Z} < \frac{\theta_2}{4} - \frac{k(\alpha)}{2\sqrt{n}}$ assert that $\theta \leq \theta_2$; if $\overline{Z} > \frac{1}{2}(\theta_3 + 1 + \frac{k(\alpha)}{\sqrt{n}})$ assert that $\theta \geq \theta_3$; if $\frac{\theta_2}{4} - \frac{k(\alpha)}{2\sqrt{n}} \leq \overline{Z} \leq \frac{1}{2}(1+\theta_3+\frac{k(\alpha)}{\sqrt{n}})$ assert nothing. We then get the following table of probabilities

True State:→	$0 \leq \theta < \theta_2$	$\theta_2 \leq \theta \leq \theta_3$	$\theta_3 < \theta \leq 1$
Assertion: ↓			
$\theta \leq \theta_2$	-	$<\alpha$	$<\alpha$
$\theta \geq \theta_3$	$<\alpha$	$<\alpha$	-
Nothing	-	-	-

FOOTNOTE

(1) Research sponsored by the Office of Naval Research.

References

[1] Debreu, Gerard, "Representation of a Preference Ordering by a Numerical Function," in Decision Processes, Thrall, Davis, and Coombs, eds., New York, 1954.

[2] ______, "Stochastic Choice and Cardinal Utility," Cowles Foundation Discussion Paper No. 39, 1957 (mimeogr.).

[3] Wald, Abraham, "The Approximate Determination of Indifference Surfaces by Means of Engel Curves," Econometrica, 1940, pp. 144-175.

[4] Fechner, Gustav Theodor, Elemente der Psychophysik, 1859. (See especially pp. 70-103 of the 1889 edition.)

[5] Thurstone, Louis L., "A Law of Comparative Judgment," Psychological Review, 1927, pp. 273-286.

[6] Mosteller, Frederick, "Remarks on the Méthod of Paired Comparison, I," Psychometrika, 1951, pp. 1-9.

[7] Reichenbach, Hans, an unpublished RAND paper dated August 1949.

[8] Coombs, Clyde H., "Inconsistency of Preferences as a Measure of Psychological Distance," (Mimeogr.) University of Michigan, 1956.

[9] Vail, Stephan, "A Stochastic Model for Utilities," Seminar on the application of Mathematics to the Social Sciences," University of Michigan, 1953 (dittoed).

[10] Tornqvist, Leo, "A Model for Stochastic Decision Making," Cowles Commission Discussion Paper, Economics 2100, 1954 (dittoed).

[11] Edwards, Ward. Articles on Probability Preferences--American Journal of Psychology, 1953, 66, pp. 345-364; 1954, 67, pp. 56-67; 1954, 67, 68-95.

[12] Luce, Duncan, "A Probabilistic Theory of Utility," 1957 (dittoed).

[13] ____, "A Theory of Individual Choice Behavior," 1957 (mimeogr.). The present authors feel, however, that Arrow's axiom of irrelevant alternatives is logically relevant to our condition (1) in 2.3, rather than to Luce's "Axiom 1" of whose intuitive plausibility we are not convinced.

[14] Papandreou, Andreas G., with the collaboration of O. H. Sauerlander, O. H. Brownlee, L. Hurwicz, and W. Franklyn, "A Test of a Stochastic Theory of Choice," University of California Publications in Economics, Vol. 16, No. 1, pp. 1-18.

[15] May, Kenneth O., "Intransitivity, Utility and the Aggregation of Preference Patterns." Econometrica Vol. 22 (1954) pp. 1-13.

[16] Davidson, Donald and Marschak, Jacob, "Experimental Tests of Stochastic Decision Theory," Cowles Foundation Discussion Paper No. 22, 1957 (mimeogr.). To be published in Symposium on Measurement, John Wiley and Son, New York.

[17] Georgescu-Roegen, Nicholas, "The Pure Theory of Consumer's Behavior," Quarterly Journal of Economics, 50, pp. 545-593, 1936.

[18] Rosenblatt, David, "On Some Stochastic Process Formulations of Individual Preference and Consumer Behavior" (abstract), Econometrica, 1956, pp. 347-348.

[19] Marschak, J., "Norms and Habits of Decision Making under Certainty," Mathematical Models of Human Behavior, Proceedings of a Symposium, Dunlap and Associates, Inc., Stamford, Conn., 1951.