COWLES FOUNDATION DISCUSSION PAPER NO. 30

The Estimation of Relationships in which the Dependent Variable is Subject to Inertia

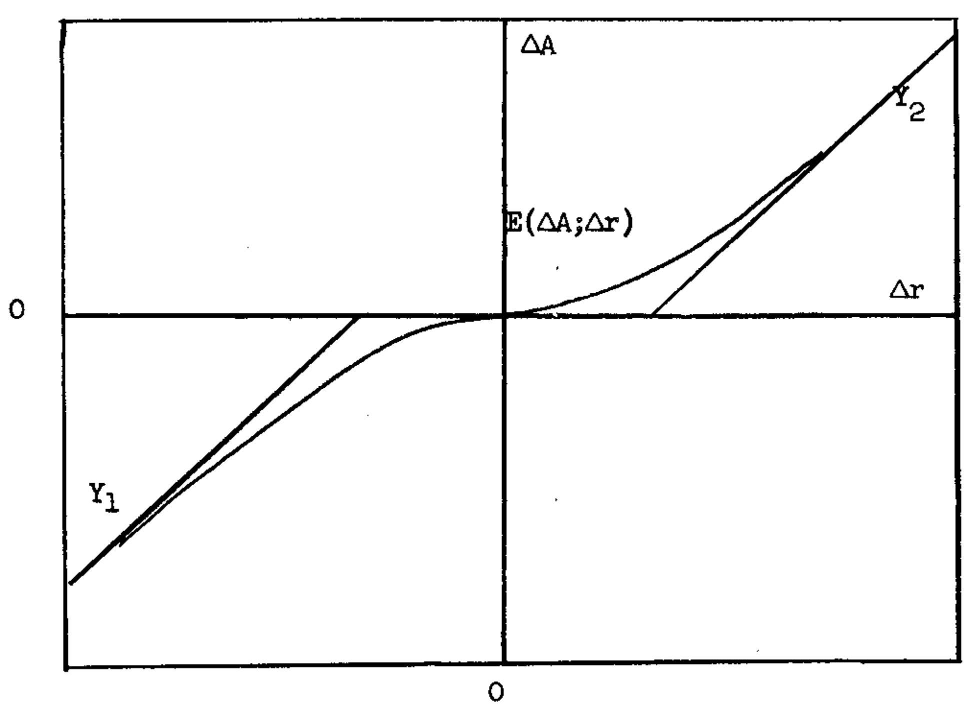
Richard Rosett

May 24, 1957

The Estimation of Relationships in which the Dependent Variable is Subject to Inertia

There are many relationships in economics in which the dependent variable is subject to inertia. The most usual cases are those involving a transaction cost. For example, if one were to examine the effects of changes in yield on changes in the holdings of a particular type of asset by a certain class of investors, it might be found that small changes in yield would have no effect because of transaction costs. Figure 1 represents such a relationship. ΔA represents changes in asset holdings and Δr represents changes in yield.

Figure 1



It is possible to obtain a procedure for estimating such a relationship by modifying the prob t-regression model first presented by Tobin.*

---

* James Tobin, "Estimation of Relationships for Limited Dependent Variables." CFDP No. 3.

Let W be an independent variable subject to inertia, and let $X_k, \ldots, X_m$ be a set of independent variables which are related to W. If the effects of the independent variables are assumed to be linear and additive, the relationship can be specified as follows:

$$(1) \qquad \begin{aligned} Y_1 &= \beta_o' + \sum_{i-1}^{m} \beta_i X_i \\ Y_2 &= \beta_o'' + \sum_{i=1}^{m} \beta_i X_i \end{aligned}$$

If deviations (u) from the true relationship are taken to be random and normally distributed, then W is determined by

$$(2) \qquad \begin{aligned} W &= Y_1 - u & &(Y_1 - u < L) \\ W &= L & &(Y_1 - u > L \text{ and } Y_2 - u < L) \\ W &= Y_2 - u & &(Y_2 - u > L) \end{aligned}$$

$P(x)$ is the value of the unit-normal distribution function, $Q(x) = 1 - P(x)$, and $Z(x)$ is the value of the unit-normal density function.

$$\Pr(W > x > L \mid X_1, \ldots, X_m, L)$$

$$= \Pr(u < Y_2 - x) = P(\frac{Y_2 - x}{\sigma})$$

(3) $$\Pr(W = L \mid X_1, \ldots, X_m, L) = \Pr(Y_1 - u > 1 \text{ and } Y_2 - u < L)$$

$$= \Pr(Y_1 - L > u > Y_2 - L)$$

$$= Q(\frac{Y_2 - L}{\sigma}) - Q(\frac{Y_1 - L}{\sigma})$$

$$\Pr(L > x > W \mid X_1, \ldots, X_m, L) = \Pr(u > Y_1 - x)$$

$$= Q(\frac{Y_1 - x}{\sigma})$$

The distribution function of W is

$$F(x; X_1, \ldots, X_m, L) = Q(\frac{Y_1 - x}{\sigma}) \qquad (x < L)$$

(4) $$F(L; X_1, \ldots, X_m, L) = Q(\frac{Y_2 - L}{\sigma}) - Q(\frac{Y_1 - L}{\sigma})$$

$$F(x; X_1, \ldots, X_m, L) = Q(\frac{Y_2 - x}{\sigma}) \qquad (x > L)$$

The corresponding density function is

(5)

$$f(x; X_1, \ldots, X_m, L) = \frac{1}{\sigma} Z(\frac{Y_1 - x}{\sigma}) \qquad (x < L)$$

$$f(x; X_1, \ldots, X_m, L) = \frac{1}{\sigma} Z(\frac{Y_2 - x}{\sigma}) \qquad (x > L)$$

For ease of exposition and computation, it is convenient to normalize on σ. Let $(a_o', a_o'', a_1, \ldots, a_m, a)$ be estimates of $(\frac{\beta_o'}{\sigma}, \frac{\beta_o''}{\sigma}, \frac{\beta_1}{\sigma}, \ldots, \frac{\beta_m}{\sigma}, \frac{1}{\sigma})$,

(6)
$$I_1 = a\ Y_1 = a_o' + \sum_{i=1}^{m} a_i X_i$$
$$I_2 = a\ Y_2 = a_o'' + \sum_{i=1}^{m} a_i X_i$$

Each observation in a sample consists of a set of values for the independent variables and the dependent variable. The value of L may be different from observation to observation, but it must be known. Since the distribution function has a mass point, the likelihood function $\varphi(a_o', a_o'', a_1, \ldots, a_m, a)$ will be a mixture of densities and probabilities. Assume that a sample contains n observations, and that for p of these observations $W < L$, for q observations $W = L$ and for r observations $W > L$. Then

(7)
$$\varphi(a_o', a_o'', a_1, \ldots, a_m, a)$$
$$= \prod_{j=1}^{p} a\ Z(I_{1j} - a\ W_j) \prod_{k=1}^{q} [Q(I_{2k} - W_k) - Q(I_{1k} - W_k)] \cdot$$
$$\prod_{\ell=1}^{r} a\ Z(I_{2\ell} - W_\ell)$$

The natural logarithm of φ is

$$\varphi^*(a_o', a_o'', a_1, \ldots, a_m, a) =$$

$$-\frac{1}{2} \sum_{j=1}^{p} (I_{1j} - a\ W_j)^2 + \sum_{k=1}^{q} \ell n\ [Q(I_{2k} - W_k) - Q(I_{1k} - W_k)]$$

$$- \frac{1}{2} \sum_{\ell=1}^{r} (I_{2\ell} - a\, W_\ell)^2 + (p + r)\, \ell n\, a - \frac{p + r}{2}\, \ell n\, 2\pi$$

Let $X_o = 1$ for all observations. Taking the derivatives of φ^* with respect to all parameter estimator gives the following system of $m + 3$ equations

(8)

$$\varphi^*_{a'_o} = - \sum_{j=1}^{p} (I_{1j} - a\, W_j)\, X_{oj} + \sum_{k=1}^{q} \frac{Z(I_{1k} - a\, W_k)}{Q(I_{2k} - W_k) - Q(I_{1k} - W_k)}\, X_{ok}$$

$$\varphi^*_{a''_o} = - \sum_{k=1}^{q} \frac{Z(I_{2k} - W_K)}{Q(I_{2k} - W_k) - Q(I_{1k} - W_k)}\, X_{ok} - \sum_{\ell=1}^{r} (I_{2\ell} - a\, W_\ell)\, X_{o\ell}$$

$$\varphi^*_{a_i} = - \sum_{j=1}^{p} (I_{ij} - a\, W_j)\, X_{ij} + \sum_{k=1}^{q} \frac{Z(I_{1k} - W_k) - Z(I_{2k} - W_k)}{Q(I_{2k} - W_k) - Q(I_{1k} - W_k)}\, X_{ik}$$

$$- \sum_{\ell=1}^{r} (I_{1\ell} - a\, W_\ell)\, X_{i\ell} \qquad (i = 1,2,\ldots,m)$$

$$\varphi^*_{a} = \sum_{j=1}^{p} (I_{1j} - a\, W_j)\, W_j - \sum_{k=1}^{q} \frac{Z(I_{1k} - a\, W_k) - Z(I_{2k} - a\, W_k)}{Q(I_{2k} - a\, W_k) - Q(I_{1k} - W_k)}\, W_k$$

$$+ \sum_{\ell=1}^{r} (I_{2\ell} - a\, W_\ell)\, W_\ell + \frac{p + r}{a}$$

As is the case of the probit-regression model, these equations are non linear, and the suggested method of solution is exactly the same as that described in CFDP No. 3.

Let A be the vector of parameter estimates, M the matrix of second derivative of φ^* and V the vector of first derivatives. An initial trial value of A is obtained, $A^{(o)}$. An approximation to A is obtained by iteration; each iteration consists of solving the equation

$$(9) \qquad M(A^{(i)} - A^{(i-1)}) = -V$$

Letting $u_1 = I_{1k} - a W_k$ and $u_2 = I_{2k} - a W_k$, the second derivatives of φ^* are

$$\varphi^*_{a_o'a_o''} = \sum_{k=1}^{q} \frac{Z(u_1)\, Z(u_2)}{[Q(u_2) - Q(u_1)]^2} X_{ok}^2$$

$$\varphi^*_{a_o'a_i} = -\sum_{j=1}^{p} X_{ij}X_{oj} - \sum_{k=1}^{q} \frac{u_1\, Z(u_1)\, [Q(u_2) - Q(u_1) + Z(u_1)\, [Z(u_1) - Z(u_2)]}{[Q(u_2) - Q(u_1)]^2} X_{ok}X_{ik}$$

$$\varphi^*_{a_o''a_i} = \sum_{k=1}^{q} \frac{u_2\, Z(u_2)\, [Q(u_2) - Q(u_1)] + Z(u_2)\, [Z(u_1) - Z(u_2)]}{[Q(u_2) - Q(u_1)]^2} X_{ok}X_{ij} - \sum_{\ell=1}^{r} X_{o\ell}X_{i\ell}$$

$$\varphi^*_{a_ia_t} = -\sum_{j=1}^{p} X_{ij}X_{tj} - \sum_{k=1}^{q} \frac{[u_1\, Z(u_1) - u_2\, Z(u_2)]\, [Q(u_2) - Q(u_1)] + [Z(u_1) - Z(u_2)]^2}{[Q(u_2) - Q(u_1)]^2} X_{ik}X_{tk}$$

$$\varphi^*_{a_o'a} = \sum_{j=1}^{p} X_{oj}W_j + \sum_{k=1}^{q} \frac{u_1 Z(u_1) [Q(u_2) - Q(u_1) + Z(u_1)] [Z(u_1) - Z(u_2)]}{[Q(u_2) - Q(u_1)]^2} X_{ok}W_k$$

$$\varphi^*_{a_o''a} = - \sum_{k=1}^{q} \frac{u_2 Z(u_2) + Z(u_2) [Z(u_1) - Z(u_2)]}{[Q(u_2) - Q(u_1)]^2} X_{ok}W_k + \sum_{\ell=1}^{r} X_{o\ell}W_\ell$$

$$\varphi^*_{a_ia} = - \sum_{j=1}^{p} X_{ij}W_j + \sum_{k=1}^{q} \frac{[u_1 Z(u_1) - u_2 Z(u_2)] [Q(u_2) - Q(u_1) + [Z(u_1) - Z(u_2)]^2}{[Q(u_2) - Q(u_1)]^2} X_{ij}W_k$$

$$- \sum_{\ell=1}^{r} X_{i\ell}W_\ell$$

$$\varphi^*_{aa} = \sum_{j=1}^{q} W_j^2 - \sum_{k=1}^{r} \frac{[u_1 Z(u_1) - u_2 Z(u_2)] [Q(u_2) - Q(u_1)] + [Z(u_1) - Z(u_2)]^2}{[Q(u_2) - Q(u_1)]^2} W_k^2$$

$$+ \sum_{\ell=1}^{r} W_\ell^2 - \frac{p + r}{a^2}$$

The computation of the first and second derivatives can be accomplished through a trivial modification of a program which has been written for obtaining estimates of the probit-regression model. A Linear regression could be used as an initial trial value, A^o . Doubtless, better methods for obtaining initial trial values are available, but this problem has not yet been examined. The significance testsare exactly those discribed in CFDP No. 3. The negative inverse of the matrix of second derivatives is the matrix of estimated variances and covariances of the parameter estimates, and likelihood ratio tests can be applied to the parameter estimates.

The locus of expected values of W is not the relationship which is estimated, but can be computed as follows:

$$E(W;\ X_1,\ldots,X_m,\ L) = \int_{-\infty}^{L} \frac{x}{\sigma}\ Z\left(\frac{Y_1 - x}{\sigma}\right)\ dx$$

$$+ L\left[Q\left(\frac{Y_2 - L}{\sigma}\right) - Q\left(\frac{Y_1 - L}{\sigma}\right)\right] + \int_{L}^{\infty} \frac{x}{\sigma}\ Z\left(\frac{Y_1 - x}{\sigma}\right)\ dx$$

$$= Y_1\ Q\left(\frac{Y_1 - L}{\sigma}\right) - \sigma\ Z\left(\frac{Y_1 - L}{\sigma}\right) + L\ \left[Q\left(\frac{Y_2 - L}{\sigma}\right) - Q\left(\frac{Y_1 - L}{\sigma}\right)\right]$$

$$+ Y_2\ P\left(\frac{Y_2 - L}{\sigma}\right) + \sigma\ Z\left(\frac{Y_2 - L}{\sigma}\right)$$

The locus of expected values of W is represented in Figure 1.