COWLES FOUNDATION DISCUSSION PAPER NO. 27

A Simplex Method for the Portfolio Selection Problem

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March 27, 1957

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Philip Wolfe [1] has shown that a slight variant of the simplex method can be used to solve the problem

$$\text{minimize} \quad f(x) = x'Cx$$

$$x \geq 0$$

$$Ax = b$$

where x is a n by 1 matrix
x' is its transpose
C is a n by n positive semi-definite matrix and
A is a m by n matrix.

To solve the above problem by Wolfe's technique the simplex procedure must be modified so that certain pairs of variables are not allowed in the basis simultaneously.

In the portfolio selection problem [2] $f(x)$ is minimized subject to the constraints

$$Ax = b$$

$$x \geq 0$$

and

$$\mu x \geq E$$

where μ is a 1 by n matrix and E is a scalar. E is not fixed in value; rather the problem is to find min f for all possible values of E.

This note shows that the simplex method, with Wolfe's amendment, can be used to solve the whole portfolio selection problem. Since one by-product of the portfolio selection computation is the point x which minimizes $f(x)$ subject only to $Ax = b$ and $x \geq 0$, the procedure presented in this paper also provides an alternate method of solving this problem.

We define
$$\eta_j = \frac{\partial f(x) + \lambda A x - \lambda_E \mu x}{\partial x_j}$$
$$= \sum_k c_{jk} x_k + \sum_i \lambda_i a_{ij} \; \lambda_E \mu_j$$

Where c_{jk} is the (j,k)th element of C

a_{ij} is the (ij)th element of A

μ_j is the jth element of μ

and λ is a 1 by m matrix of "Lagrangian Multipliers" whose ith component is λ_i.

The critical line method of quadratic programming [3] uses the fact that there is a piecewise linear set of points (x) which give minimum $f(x)$ for each value of E. Associated with any linear segment of this set is a set of variables $\mathcal{J} = (j_1, \ldots, j_J)$. Along the linear segment we have

$$\left.\begin{aligned} \eta_j &\geq 0 \\ x_j &\geq 0 \end{aligned}\right\} \text{ for all } j$$

$$\eta_j = 0 \quad \text{for } j \in \mathcal{J}$$

$$x_j = 0 \quad \text{for } j \notin \mathcal{J}$$

We also have

$$\begin{pmatrix} CA' & -I \\ A0 & 0 \end{pmatrix} \begin{bmatrix} x \\ \lambda \\ \eta \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix} + \begin{bmatrix} \mu \\ 0 \end{bmatrix} \lambda_E .$$

In the critical line procedure λ_E is reduced until some $\eta_j \notin \mathcal{J}$ or $X_j \in \mathcal{J}$ reaches zero. If an η_{jo} reaches zero first j_o "goes into" the set $\mathcal{J}$, so that along the next critical line

$$\eta_{jo} = 0$$

as well as

$$\eta_j = 0 \text{ for } j \text{ in the old } \mathcal{J}.$$

Conversely if an x_{jo} reaches zero first, j_o "goes out" of $\mathcal{J}$ for the next critical line. The computing procedure continues until $\lambda_E = 0$ is reached. At this point $f(x)$ is minimized subject only to $Ax = b, x \geq 0$.

Consider the "amended" linear programming problem:

minimize λ_E

subject to

$$\begin{pmatrix} C & A' & -I \\ A & 0 & 0 \end{pmatrix} \begin{bmatrix} x \\ \lambda \\ \eta \end{bmatrix} - \begin{bmatrix} \mu \\ 0 \end{bmatrix} \lambda_E = \begin{bmatrix} 0 \\ b \end{bmatrix}$$

where: 1) η_j and x_j can never be in a basis simultaneously; 2) $\lambda = (\lambda_1 \ldots \lambda_m)$ is not constrained to be non-negative; and 3) $\lambda_1, \ldots, \lambda_m$ are always in the basis. A basis to this problem contains $m + n$ variables. As long as $\lambda_E > 0$ the basis must consist of

$$\lambda_E$$
$$\lambda_1, \ldots, \lambda_m$$

and

$$n - 1 \text{ of the } x_j \text{ and } \eta_j .$$

[We will assume that there are no degenerate bases. This is a convenient but unessential assumption; see [4] concerning degeneracy in linear programming and [3] concerning degeneracy in the critical line method.] By construction, if x_j is in the basis, then η_j is not; and vice versa. Since there are $n - 1$ of the x_j or η_j in the basis, there can be only one j_o for which both its x_{jo} and η_{jo} are not in the basis. According to the amended procedure one of these must go into the basis.

As will be discussed below, the introduction of one of these into the basis will increase λ_E ; the introduction of the other will decrease λ_E . The variable (x_{jo} or η_{jo}) which decreases λ_E is introduced into the basis by increasing its value until some other variable (x_j or η_j with $j \neq j_o$) goes to zero. This again leaves us with one j_1 with x_{j1} and η_{j1} out of the basis. The procedure is repeated until $\lambda_E = 0$ is reached.

Comparison of the amended simplex computation and the critical line procedure shows that if they are started out together they will continue together.* x and η vary with λ_E along a critical line by exactly the same formula that relates them to λ_E when the new variable is introduced into the basis. The sequence of x's of η's which go to zero, and their counterparts which become non zero, are exactly the same.

The proof that the critical line method works, therefore, is a proof that the amended simplex method produces the desired results. It is this equivalence between the two procedures that implies that when x_{jo} and η_{jo} are both outside the simplex basis, one will increase λ_E while the other will decrease it.

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* See [3] for starting the portfolio problem.

References

[1] Philip Wolfe, "A Simplex Method for Quadratic Programming."

[2] Harry Markowitz, "Portfolio Selection," Cowles Commission Papers, The University of Chicago, 1952.

[3] Harry Markowitz, "The Optimization of a Quadratic Function Subject to Linear Constraints," Naval Res. Logistics Q. 3, 111-133 (1956).

[4] G.B. Dantzig, A. Orden, P. Wolfe, The generalized simplex method for minimizing a linear form under linear inequality constraints, Pac. J. Math. 5, 183-195 (1955).