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A Simplex Method
for the
Portfolio Selection Problem

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Philip Wolfe [1] has shown that a slight variant of the simplex method can be used to solve the problem

\[ \text{minimize } f(x) = x'Cx \]
\[ x \geq 0 \]
\[ Ax = b \]

where \( x \) is an \( n \) by \( 1 \) matrix
\( x' \) is its transpose
\( C \) is an \( n \) by \( n \) positive semi-definite matrix and
\( A \) is an \( m \) by \( n \) matrix.

To solve the above problem by Wolfe's technique the simplex procedure must be modified so that certain pairs of variables are not allowed in the basis simultaneously.

In the portfolio selection problem [2] \( f(x) \) is minimized subject to the constraints

\[ Ax = b \]
\[ x \geq 0 \]

and

\[ \mu x \geq E \]

where \( \mu \) is an \( 1 \) by \( n \) matrix and \( E \) is a scalar. \( E \) is not fixed in value; rather the problem is to find \( \min f \) for all possible values of \( E \).
This note shows that the simplex method, with Wolfe's amendment, can be used to solve the whole portfolio selection problem. Since one by-product of the portfolio selection computation is the point \( x \) which minimizes \( f(x) \) subject only to \( Ax = b \) and \( x \geq 0 \), the procedure presented in this paper also provides an alternate method of solving this problem.

We define

\[
\eta_j = \frac{\partial f(x) + \lambda A x - \lambda_k \mu x}{\partial x_j}
\]

\[
= \sum_k c_{jk} x_k + \sum_1 \lambda_i a_{ij} \lambda_i x_j
\]

Where

- \( c_{jk} \) is the \((j,k)\)th element of \( C \)
- \( a_{ij} \) is the \((ij)\)th element of \( A \)
- \( \mu_j \) is the \( j \)th element of \( \mu \)

and \( \lambda \) is a \( 1 \) by \( m \) matrix of "Lagrangian Multipliers" whose \( i \)th component is \( \lambda_i \).

The critical line method of quadratic programming [3] uses the fact that there is a piecewise linear set of points \((x)\) which give minimum \( f(x) \) for each value of \( E \). Associated with any linear segment of this set is a set of variables \( \mathcal{J} = (j_1, \ldots, j_j) \). Along the linear segment we have

\[
\begin{align*}
\eta_j &\geq 0 \\
x_j &\geq 0 \\
\eta_j &= 0 \quad \text{for} \quad j \in \mathcal{J} \\
x_j &= 0 \quad \text{for} \quad j \notin \mathcal{J}
\end{align*}
\]
We also have

\[
\begin{pmatrix}
CA' - I \\
A0 0
\end{pmatrix}
\begin{bmatrix}
x \\
\lambda \\
\eta
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
b
\end{bmatrix} + 
\begin{bmatrix}
\mu \\
0
\end{bmatrix}
\lambda_E.
\]

In the critical line procedure \( \lambda_E \) is reduced until some \( \eta_j \notin \mathcal{J} \) or \( x_j \in \mathcal{J} \) reaches zero. If an \( \eta_{j_0} \) reaches zero first \( j_0 \) "goes into" the set \( \mathcal{J} \), so that along the next critical line

\[
\eta_{j_0} = 0
\]

as well as

\[
\eta_j = 0 \text{ for } j \text{ in the old } \mathcal{J}.
\]

Conversely if an \( x_{j_0} \) reaches zero first, \( j_0 \) "goes out" of \( \mathcal{J} \) for the next critical line. The computing procedure continues until \( \lambda_E = 0 \) is reached. At this point \( f(x) \) is minimized subject only to \( Ax = b, x \geq 0 \).

Consider the "amended" linear programming problem:

minimize \( \lambda_E \)

subject to

\[
\begin{pmatrix}
CA' - I \\
A0 0
\end{pmatrix}
\begin{bmatrix}
x \\
\lambda \\
\eta
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
\mu \\
b
\end{bmatrix}
\lambda_E = 
\begin{bmatrix}
0 \\
b
\end{bmatrix}
\]
where: 1) \( \eta_j \) and \( x_j \) can never be in a basis simultaneously; 2) \( \lambda = (\lambda_1 \ldots \lambda_m) \) is not constrained to be non-negative; and 3) \( \lambda_1, \ldots, \lambda_m \) are always in the basis. A basis to this problem contains \( m + n \) variables.

As long as \( \lambda_E > 0 \) the basis must consist of

\[
\begin{align*}
\lambda_E \\
\lambda_1, \ldots, \lambda_m
\end{align*}
\]

and

\( n - 1 \) of the \( x_j \) and \( \eta_j \).

We will assume that there are no degenerate bases. This is a convenient but unessential assumption; see [4] concerning degeneracy in linear programming and [3] concerning degeneracy in the critical line method. By construction, if \( x_j \) is in the basis, then \( \eta_j \) is not; and vice versa. Since there are \( n - 1 \) of the \( x_j \) or \( \eta_j \) in the basis, there can be only one \( j_o \) for which both its \( x_{j_o} \) and \( \eta_{j_o} \) are not in the basis. According to the amended procedure one of these must go into the basis.

As will be discussed below, the introduction of one of these into the basis will increase \( \lambda_E \); the introduction of the other will decrease \( \lambda_E \).

The variable \( (x_{j_o} \text{ or } \eta_{j_o}) \) which decreases \( \lambda_E \) is introduced into the basis by increasing its value until some other variable \( (x_j \text{ or } \eta_j \text{ with } j \neq j_o) \) goes to zero. This again leaves us with one \( j_1 \) with \( x_{j_1} \text{ and } \eta_{j_1} \) out of the basis. The procedure is repeated until \( \lambda_E = 0 \) is reached.
Comparison of the amended simplex computation and the critical line procedure shows that if they are started out together they will continue together.* $x$ and $\eta$ vary with $\lambda_E$ along a critical line by exactly the same formula that relates them to $\lambda_E$ when the new variable is introduced into the basis. The sequence of $x$'s of $\eta$'s which go to zero, and their counterparts which become non-zero, are exactly the same.

The proof that the critical line method works, therefore, is a proof that the amended simplex method produces the desired results. It is this equivalence between the two procedures that implies that when $x_{jo}$ and $\eta_{jo}$ are both outside the simplex basis, one will increase $\lambda_E$ while the other will decrease it.

* See [3] for starting the portfolio problem.
References


