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On the Two Bin Inventory Policy:

An Application of the Arrow-Harris-Marschak Model*

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1. Introduction

The inventory problem as formulated by Arrow, Harris and Marschak applies with slight modifications to the following problem in management science: a firm or department sells a commodity of which stock is taken at the beginning of each "period." The demand for this commodity is a random variable identically and independently distributed in different periods. Unfilled demand is backlogged to the beginning of the next period. The firm follows an s, S or "two-bin" inventory policy: whenever at the beginning of a period stock is below the reordering point $\underline{s}$ an order is placed, and filled immediately, which brings stock back to a fixed level $\underline{S}$.

With each order is associated a constant cost and a carrying cost which is proportional to the initial stock for the period under consideration. If no sales are lost, total revenue and any proportional costs of orders are fixed except for interest changes which may be absorbed in the carrying cost. However, if sales are lost or a backlog is incurred then a penalty arises reflecting the loss in profit and goodwill to the firm or the probability of this happening. The problem is to determine those values of s and S which will minimize the loss, namely the total expected discounted cost including penalities.

A specific instance of this problem occurs in the storing of machinery repair parts for a manufacturing firm in Chicago. These repair parts are used in the firm's own plant and are stored in a central warehouse. In this paper we analyze the distributions of demand of several of these parts and investigate the applicability of the Arrow-Harris-Marschak inventory model. It turns out that the observed distributions are approximately negative exponential. For this distribution the integral equation of the Arrow-Harris-Marschak model may be solved in closed form and simple expressions may be obtained for the unknowns s and S. These calculations, which may be of some general interest, are carried out in the latter part of the paper.

2. Distribution of Demand for Some Machine Repair Parts.

In this section we shall analyze some data on the demand for machine repairs parts at the central warehouse of the manufacturing firm referred to in the introduction. We shall test the data for stationarity and independence to determine if the Arrow-Harris-Marschak model might be applied. We state the rationale for approximating the distribution of demand by the negative exponential function and ask how well this distrubtion fits the observed demand data.

2.1 Demands at the part using level are generated by failure of parts installed in operating machinery. As is well known, if the conditional probability of failure of a part is independent of its age, the distribution of length of time between two successive failures is negative exponential and that of failures per unit time is Poisson, assuming that replacements are made instantly. Davis has examined a large number of sets of time-to-failure data and concludes that "the exponential theory of failure may be regarded as a useful approximation of certain classes of failure distribution" [2, p. 123]. Thus one might expect, as a first approximation, that part usage is distributed according to the Poisson law. But in this case there are many part using plants, each with many machines, which use these parts. If the Poisson parameter for these is itself distributed negative exponentially, then the resulting distribution of part usage is geometric [4, pp. 124-125]. While this distribution is discrete it may be approximated by the continuous negative exponential distribution. These considerations, of course, relate to the usage of repair parts in the part using plants. The fact that they send orders to the central warehouse only when their stocks fall to certain levels might cause the distribution of total orders placed by all plants--central warehouse demand--to differ substantially from that of total part usage in these same plants.

2.2 Departures from stationarity might be of at least two different types, trend and seasonal variation. Sales of the company's final output have been increasing and with more machines of a particular type being used or existing machines being used more intensively it might be expected that more repair parts would be needed as time progresses. Also there is a tendency for sales to be greater in the last two quarters of the year than in the first two so that one might expect that repair part demand would vary from quarter to quarter. We thus wish to test for the existence of trend and seasonal variation.

One way to do so is to use a two way analysis of variance, classifying our observations by year and quarter. Since we can't assume that the observations have come from a normal population we have used Friedman's Ranked Analysis of Variance [3], a standard non-parametric technique.

In Table I, below, the values of his X_r^2 statistic for each of the six parts for the year and quarter comparisons as well as the approximate significance level are given (in the limit X_r^2 is distributed as X^2). In each case the test were based upon four quarterly observations for each of seven years.

Table I

Rank Analysis of Variance

Part	X_r^{2*}	P
A, year	7.205	0.30
quarter	3.643	0.31
B, year	7.688	0.26
quarter	5.229	0.16
C, year	7.714	0.26
quarter	2.486	0.48
D, year	7.607	0.27
quarter	0.728	0.87
E, year	10.955	0.09
quarter	1.586	0.66
F, year	5.571	0.47
quarter	8.657	0.03

* For the year comparsion X_r^2 has six degrees of freedom, for the quarter comparsion three.

In only one of the twelve cases is the test statistic significant at the 5 per cent level, and if the test were independent, the probability of at least one "significant result" if there really is no trend or seasonal variation is about 0.46. It does not appear from these data that there is a substantial departure from stationarity.

If the distributions of the observations at different times were really statistically independent we would expect to find that the observed demands are uncorrelated. To test for correlation of the observations we have again used a non-parametric test, Spearman's rank correlation coefficient ρ . On the null hypothesis of independence this is distributed approximately as Student's t [4, p. 401]. Values of ρ for current demand and demand lagged once and for current demand and demand lagged twice have been computed. The results are shown in Table II. None of the ρ's is significant

Table II

- Spearman's ρ -

Part	ρ	P*
A, one lag	0.171	0.38
two lags	-0.237	0.24
B, one lag	-0.0979	0.66
two lags	0.240	0.22
C, one lag	0.123	0.56
two lags	-0.219	0.28
D, one lag	0.0673	0.72
two lags	0.164	0.44
E, one lag	0.235	0.24
two lags	0.352	0.08
F, one lag	-0.260	0.19
two lags	-0.0649	0.76

* For Parts A and B, the one lag "t" has 26 degrees of freedom, while the two lag "t" has 25.

at the 5 per cent level. Thus the data do not appear to refute the hypothesis of independence.

2.3 When we turn to testing our hypothesis about the form of the distribution of repair part demand it is difficult to form a judgement about the goodness of fit. We would really like to know how close the actual distribution is to the negative exponential in terms of the difference in discounted expected loss associated with the order policies which are optimal for these two distributions. Conventional goodness of fit tests, such as the chi-square test, may be poorly adapted to this purpose. An observed distribution of repair part demand might differ significantly, as judged by the chi-square criterion, from what we would expect if the parent distribution were really negative exponential even though the "true" optimal order policy differs but little, as measured by discounted expected loss, from the one that is optimal for the negative exponential distribution. But the construction of a statistical test appropriate for this purpose is quite beyond the scope of this paper. Thus, while we shall apply the chi-square test, it should be interpreted only as a means of describing certain aspects of the observed demands.

The negative exponential distribution was fitted to all six repair part distributions and the chi-square test applied to each of the fitted distributions. The results of these are reproduced in Table III, while cumulative observed and fitted negative exponential distribution functions for each part are plotted in the accompanying diagrams. There is considerable variation in the "goodness of fit." One sees that the negative exponential fits very well in two of the six cases, parts A and F, while it fits very badly for two others, parts C and E. For the remaining two the test statistic is "significant" at

Table III

Chi-square Goodness of Fit Tests

Part A*

Demand	Observed	Expected
Under 25	6	5.63
25-49	4	4.39
50-99	7	5.88
100-199	5	5.38
Over 199	2	2.72

$X^2(3) = 0.488$, $P = 0.92$

Part B*

Demand	Observed	Expected
Under 25	4	6.17
25-49	3	4.68
50-99	11	5.98
100-199	4	5.04
Over 199	2	2.13

$X^2(3) = 5.793$, $P = 0.12$

Part C

Demand	Observed	Expected
Under 50	3	7.50
50-99	4	5.54
100-149	6	4.04
150-249	13	5.10
Over 249	2	5.81

$X^2(3) = 18.801$, $P = 0.0003$

Part D

Demand	Observed	Expected
Under 10	6	5.58
10-19	2	4.68
20-39	4	6.63
40-59	12	6.75
Over 59	4	4.35

$X^2(3) = 6.720$, $P = 0.08$

Part E

Demand	Observed	Expected
Under 10	13	7.12
10-19	2	5.55
20-39	2	7.07
40-59	9	5.86
Over 59	2	2.40

$X^2(3) = 12.497$, $P = 0.006$

Part F

Demand	Observed	Expected
Under 25	5	5.13
25-49	4	4.27
50-99	7	6.30
100-199	7	6.92
Over 199	5	5.39

$X^2(3) = 0.128$, $P = 0.99$

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* Tests for Parts A and B were based upon demand for 24 quarters only since the reported demands for 5 quarters are suspected of being subject to serious error.

about the ten percent level. From the charts it appears that, expect for part C, the fit is good in the upper tail. In view of the remarks in the above paragraph it is difficult to judge the closeness of fit for our purposes. It should be noted, however, that this distribution appears to give the best simple approximation to the observed distributions.*

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* In all six cases it would appear that the Poisson distribution fits very badly. For each of them there is a much greater variation in demand than one would expect if the distribution were Poisson. In only one case, part C, was it possible to obtain a better fit with the more general Gamma distribution, of which the negative exponential is a member.

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3. Solution of the Inventory Equation for Exponentially Distributed Demand

3.1 We may define the commodity unit as equal to the average demand per period. This casts the distribution function into the particularly simple form $dF = e^{-x}dx$.

Notation

$x + s$	stock
$\ell(y)$	the expected loss during a period for which initial stock is y
α	discount rate
$F(x)$	cumulative distribution function of demand per period
c	the (average) carrying cost per unit commodity per period
a	the penalty per unit of shortage.

Now

$$(1) \qquad \ell(y) = cy + a\int_y^\infty (z-y)e^{-z}dz = cy + ae^{-y}$$

The argument of the Arrow-Harris-Marschak paper yields an integral equation [1, p. 264, equation 5.6] for our loss function $\lambda(x)$.

(2) $$\lambda(x) = \ell(x + s) + \alpha\, \lambda(0) \quad [1 - F(x)] + \alpha \int_0^x \lambda(x-t)\,dF(t) \qquad x > 0$$

Its derivation will not be repeated here. By continuity

$$\lambda(0) = \ell(s) + \alpha\, \lambda(0)$$

(3) $$= \frac{\ell(s)}{1-\alpha}$$

Substituting (1) in (2)

(4) $$\lambda(x) = c(x + s) + ae^{-(x + s)} + \alpha\, \lambda(0)e^{-x} + \alpha \int_0^x \lambda(x - t)e^{-t}dt \ .$$

Putting

(5) $$\int_0^x \lambda(t)e^{t}dt = u(x)$$

we may rewrite (4) as a differential equation

(6) $$u'(x) - \alpha u(x) = ae^{-s} + c(x + s)e^{x} + \alpha\, \lambda(0)$$

with

$$u(0) = 0 .$$

Its solution is

$$u(x) = e^{x}\left[\frac{c}{1-\alpha}(s + x) - \frac{c}{(1-\alpha)^2}\right] + e^{\alpha x}\left[\frac{ae^{-s}}{\alpha} + \lambda(0) - \frac{cs}{1-\alpha} + \frac{c}{(1-\alpha)^2}\right] - \frac{ae^{-s}}{\alpha} - \lambda(0)$$

from which, inverting (5)

$$\lambda(x) = e^{-x}u + x = \frac{c}{1-\alpha}(s + 1 - x) - \frac{c}{(1-\alpha)^2}$$

$$+ \alpha e^{(\alpha-1)x}\left[\frac{ae^{-s}}{\alpha} + \lambda(0) - \frac{cs}{1-\alpha} + \frac{c}{(1-\alpha)^2}\right]$$

Determining $\lambda(0)$ from (3) and (1) and substituting we finally have

$$\lambda(x) = \frac{c(x+s)}{1-\alpha} - \frac{\alpha c}{(1-\alpha)^2} + e^{(\alpha-1)x}\left[\frac{ae^{-s}}{1-\alpha} + \frac{\alpha c}{(1-\alpha)^2}\right] \tag{7}$$

Write

$$S - s = \sigma \tag{8}$$

To determine s and S we have the conditions

$$\lambda'(\sigma) = 0 \tag{9}$$

$$\lambda(0) - \lambda(\sigma) = K \tag{10}$$

Now

$$\lambda'(x) = \frac{c}{1-\alpha} + (\alpha-1)e^{(\alpha-1)x}\left[\frac{ae^{-s}}{1-\alpha} + \frac{\alpha c}{(1-\alpha)^2}\right]$$

(9) becomes therefore*

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* The original paper [1,p.269] contains a similar formula for the case that $\alpha = 1$, i.e., that no discounting takes place.

$$\sigma = \frac{1}{1-\alpha} \log \left[\alpha + (1-\alpha)\frac{a}{c}e^{-s}\right] \tag{11}$$

for brevity write:

$$y = \alpha + (1-\alpha)\frac{a}{c}e^{-s} \tag{12}$$

Condition (10) gives use to

$$\frac{\alpha c}{(1-\alpha)^2} + \frac{ae^{-s}}{(1-\alpha)} - \frac{c\sigma}{(1-\alpha)} - e^{(\alpha-1)\sigma}\left[\frac{ae^{-s}}{(1-\alpha)} + \frac{\alpha c}{(1-\alpha)^2}\right] = K,$$

or

$$(13) \qquad [\alpha + (1-\alpha)\frac{a}{c}e^{-s}] - (1-\alpha)\,\sigma - e^{(\alpha-1)\sigma}\left[(1-\alpha)\frac{a}{c}e^{-s} + \alpha\right] = \frac{(1-\alpha)^2}{c}K$$

Substituting y, where $\sigma = \frac{1}{(1-\alpha)} \log y$,

$$(14) \qquad y = \log y + 1 + \frac{(1-\alpha)^2 K}{c}$$

The values s and S may now be found as follows: first determine y by iteration of (14) or through an approximation formula (below).

s, σ, and S are then given by

$$(15) \qquad \begin{cases} s = \log \dfrac{a(1-\alpha)}{c(y-\alpha)} \\ \sigma = \dfrac{1}{1-\alpha} \log y \\ S = s + \sigma \end{cases}$$

3.2 The following example applies to one of the parts whose distribution was analyzed in section 2.

$K = \$20.00$

$c = \$\ 15$ per 100 parts per quarter

$\alpha = 0.975$ per quarter

$$a = \begin{cases} \$150 \\ \qquad\qquad \text{per 100 parts} \\ \$1500 \end{cases}$$

then

$y = 1.0408$

a	s (100 parts)	σ (100 parts)	S (100 parts)
150	0.73	1.60	2.33
1500	3.04	1.60	4.64

3.3 The following approximation formula for y is useful and leads to interesting estimates of s and σ. This formula is valid whenever the term $\frac{K}{c}(1-\alpha)^2$ on the right hand side of (14) is of small order compared to 1. In view of the fact that c is the carrying cost for as much as the mean demand per period, and that α is the discount factor for a period usually much less than a year, this is apt to be true in all practical cases.

For small values of the last term in (14) y is approximately 1. Writing $y = 1 + z$ we may use the Taylor approximation $\log y \doteq z - \frac{z^2}{2}$ which when inserted in (14) yields $1 + z \doteq z - \frac{z^2}{2} + 1 + \frac{K}{c}(1-\alpha)^2$ or

$$(17) \qquad y \doteq 1 + (1-\alpha)\sqrt{\frac{2K}{c}} .$$

Upon substitution in (16) we obtain

$$\sigma \doteq \frac{1}{1-\alpha} \log \left[1 + (1-\alpha)\sqrt{\frac{2K}{c}} \right]$$

$$\doteq \frac{1}{1-\alpha} \left[(1-\alpha)\sqrt{\frac{2K}{c}} - (1-\alpha)^2 \frac{2K}{c} \right]$$

$$(18) \qquad \sigma \doteq \sqrt{\frac{2K}{c}} - (1-\alpha) \frac{2K}{c}$$

Thus to a first approximation, the order size σ is independent of the penalty $\underline{a}$ and of the reordering point s. Without the second (small) term on the right hand side, (18) represents the familiar lot size rule [5, p. 32 sqq.] minimizing the average cost per item in a lot of σ, $\frac{K}{\sigma} + \frac{\sigma}{2} c$. (Since one item is sold per day on the average $\frac{\sigma}{2}$ represents the mean storage time).

When substituted in (15), (17) yields

$$(19) \qquad s \doteq \log \frac{a}{c} \quad \frac{1}{1 + \sqrt{\frac{2K}{c}}} \doteq \log \frac{a}{c} - \log \ (1 + \sigma)$$

This may be compared with the one-period formula* for an optimal initial stock

* obtained by minimizing $c \int_0^x (x - z) \, dF(z) + a \int_x^\infty (z - x) \, dF(z)$

$$\frac{F(x)}{1 - F(x)} = \frac{a}{c} \; .$$

x, representing an average initial stock, may be equated to $s + \theta \sigma$ where $0 < \theta < 1$. With $F(x) = 1 - e^{-x}$ we obtain

$$(20) \qquad s \doteq \log \ (1 + \frac{a}{c}) - \theta \sigma$$

3.4 The cost of an inventory policy

$$\lambda(- s) = \lambda(0) = \frac{\ell(s)}{1-\alpha}$$

will now be considered as a function of the average demand m per period. Previously $m = 1$. Let now the units of demand be m times the old units. Then

$$a = a_o \cdot m$$

$$c = c_o \cdot m$$

From the approximation formula (19) we obtain

$$s \doteq \log \frac{a_o}{c_o} - \log \left(1 + \sqrt{\frac{2K}{a_o m}}\right)$$

and so

$$\frac{\ell(s)}{1-\alpha} \doteq [c_o s + a_o e^{-s}] \frac{m}{1-\alpha}$$

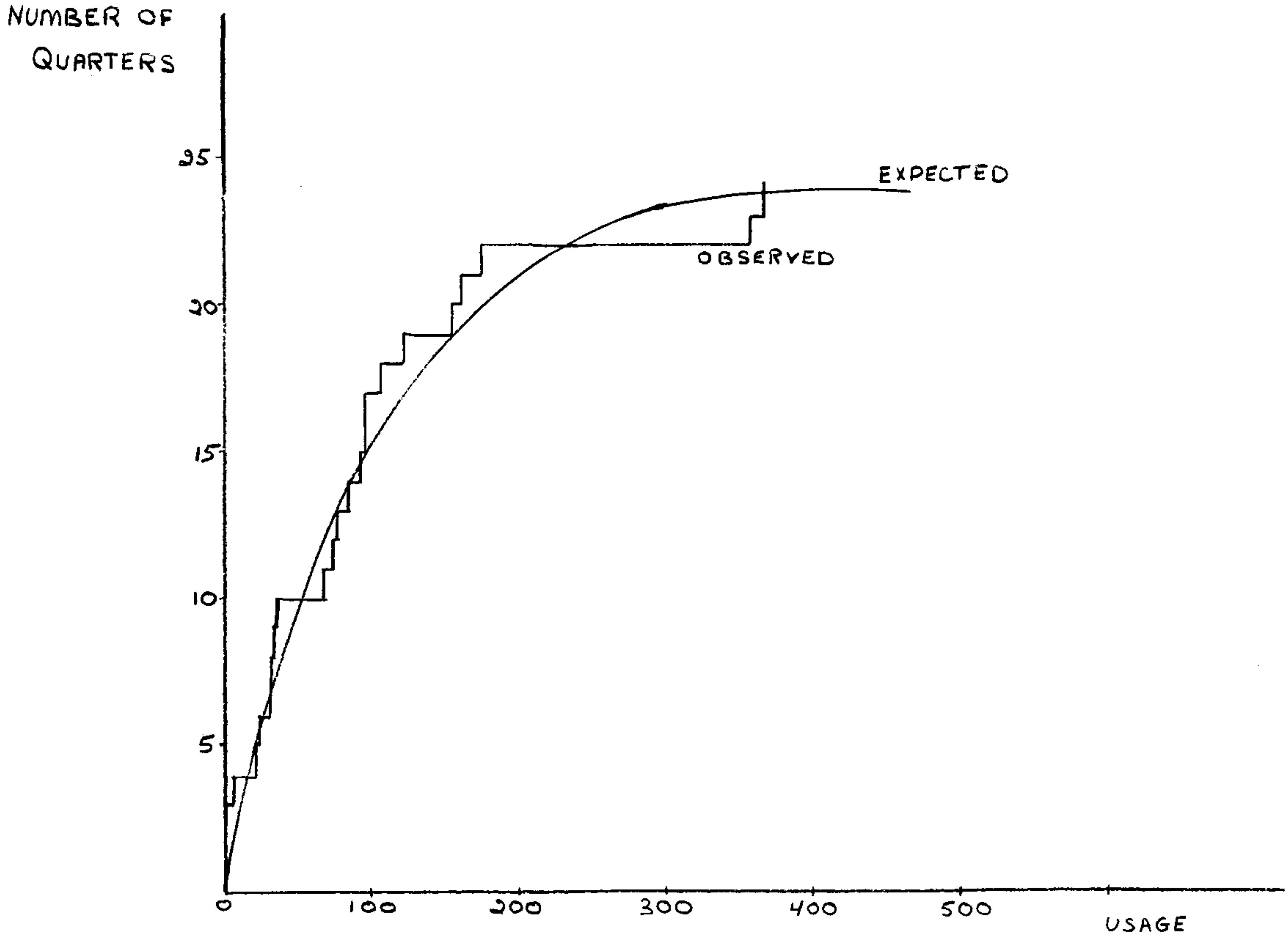
$$= [c_o \log \frac{a_o}{c_o} - c_o \log \left(1 + \sqrt{\frac{2K}{c_o m}}\right)] \frac{m}{1-\alpha}$$

$$+ \frac{c_o m}{1-\alpha} \left(1 + \sqrt{\frac{2K}{c_o m}}\right)$$

For large m the Taylor approximation of the log may be used yielding

$$\lambda(-s) = \frac{\ell(s)}{1-\alpha} \doteq \frac{c_o}{1-\alpha} \left(1 + \log \frac{a_o}{c_o}\right) m + \frac{K}{1-\alpha}$$

This is the cost that would be incurred if an order were placed at the beginning of each period so as to make stocks equal to $(1 + \log \frac{a_o}{c_o})\, m$. To a first degree of approximation the cost of an inventory policy is thus a linear function of the size of demand, the positive constant term giving rise to economies of scale.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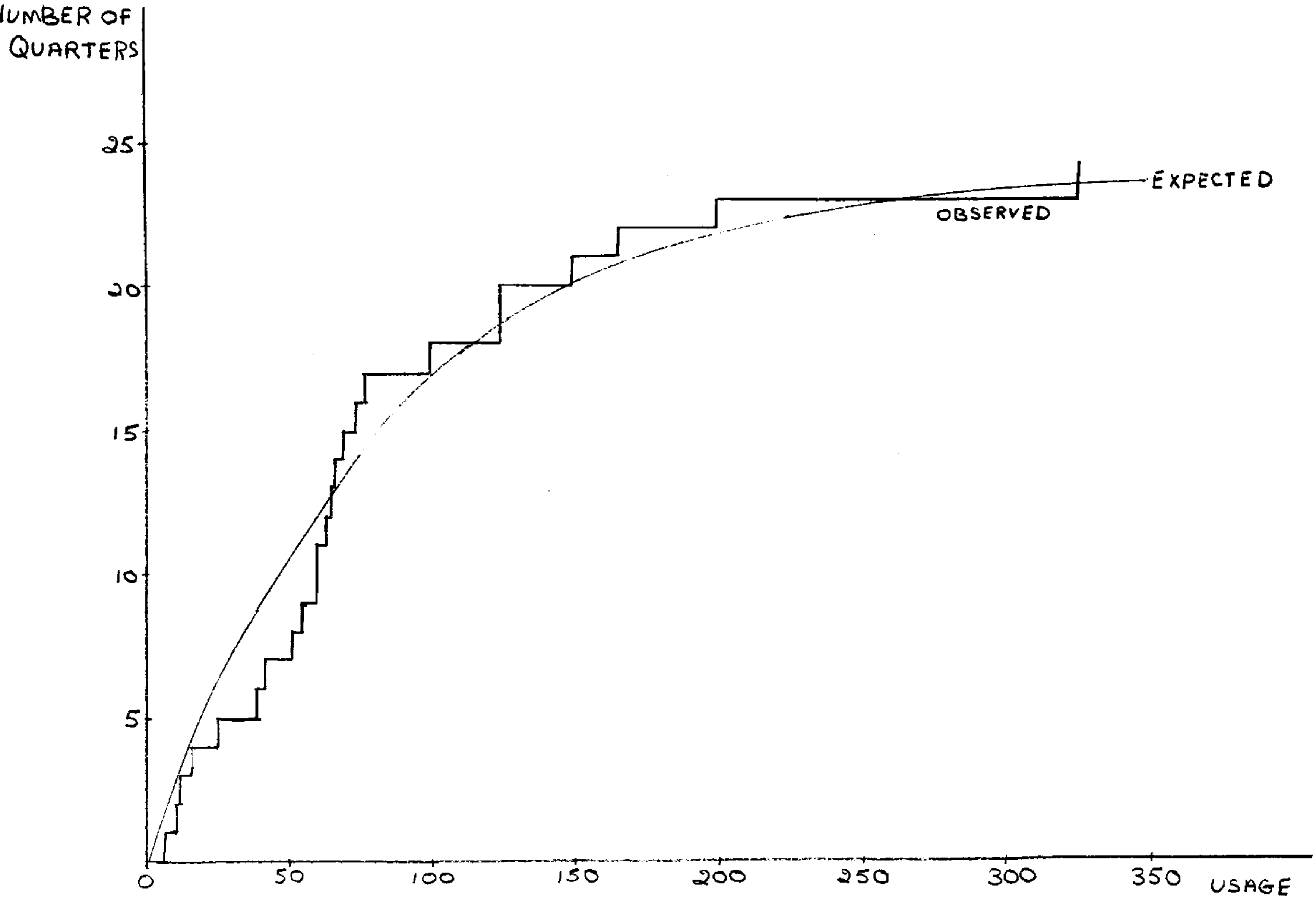
15
10
5
0
100
200
300
400
500
USAGE
EXPECTED
OBSERVED

PART A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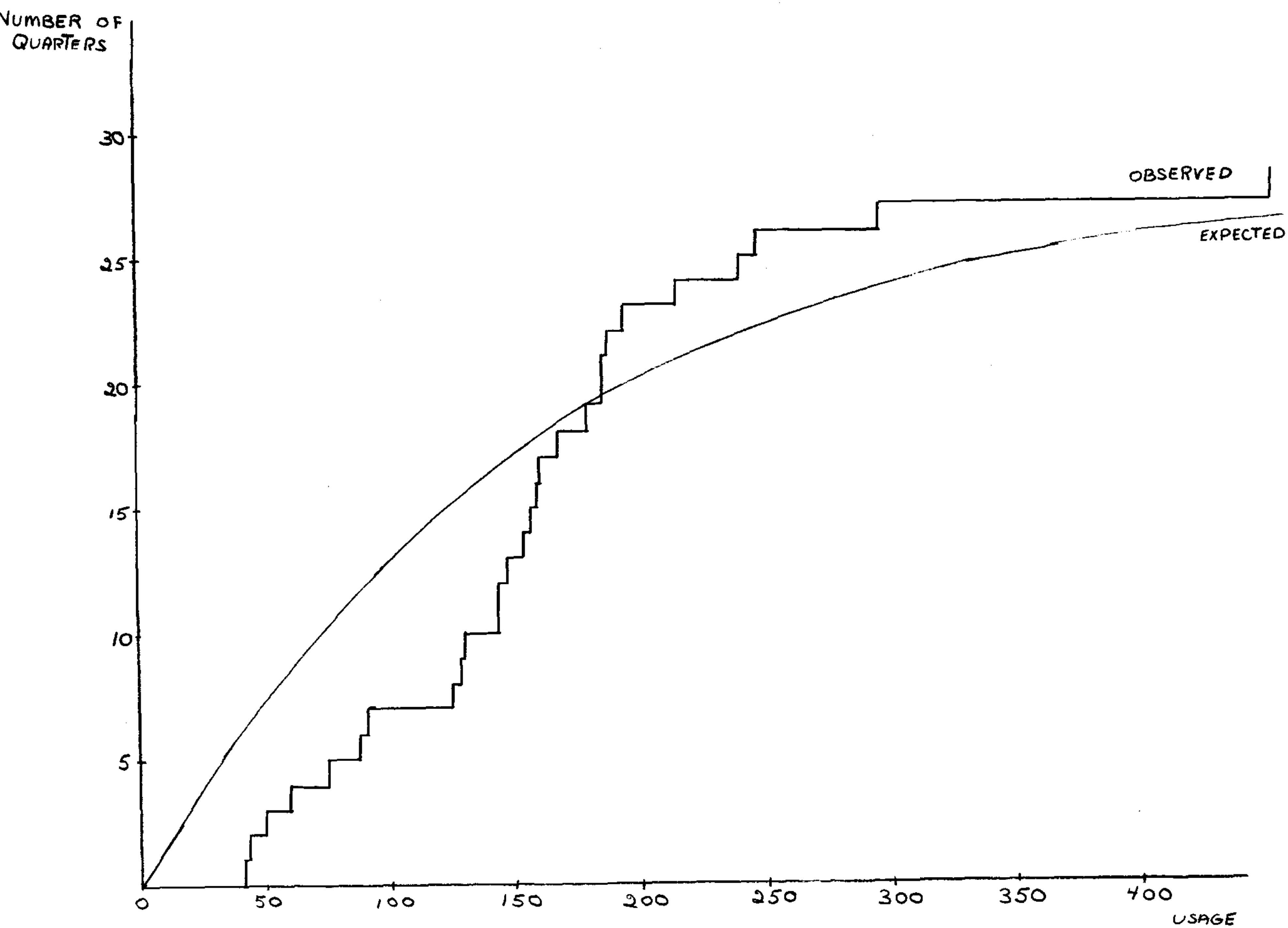
USAGE
EXPECTED
OBSERVED

PART B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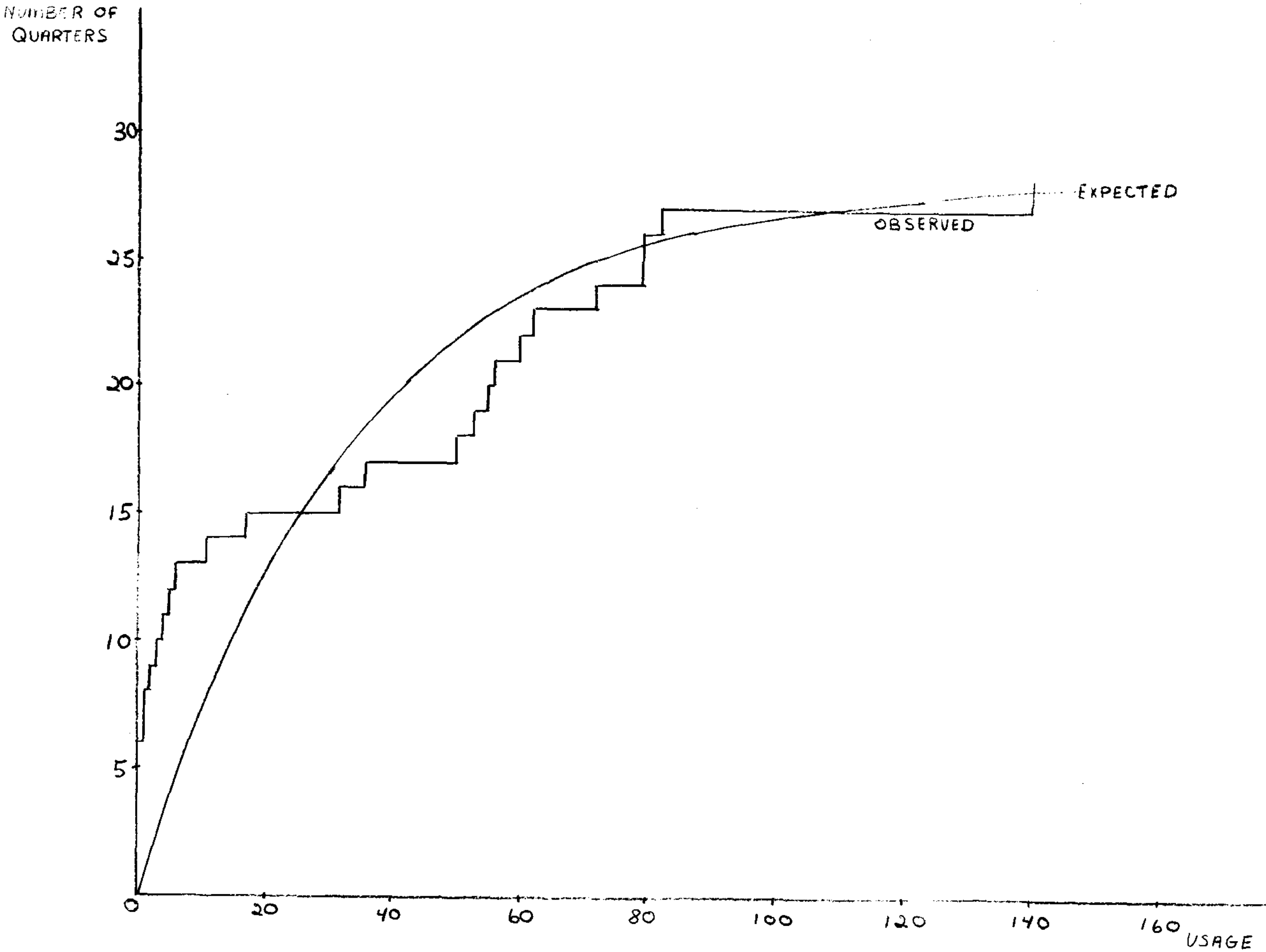
OBSERVED
EXPECTED

PART C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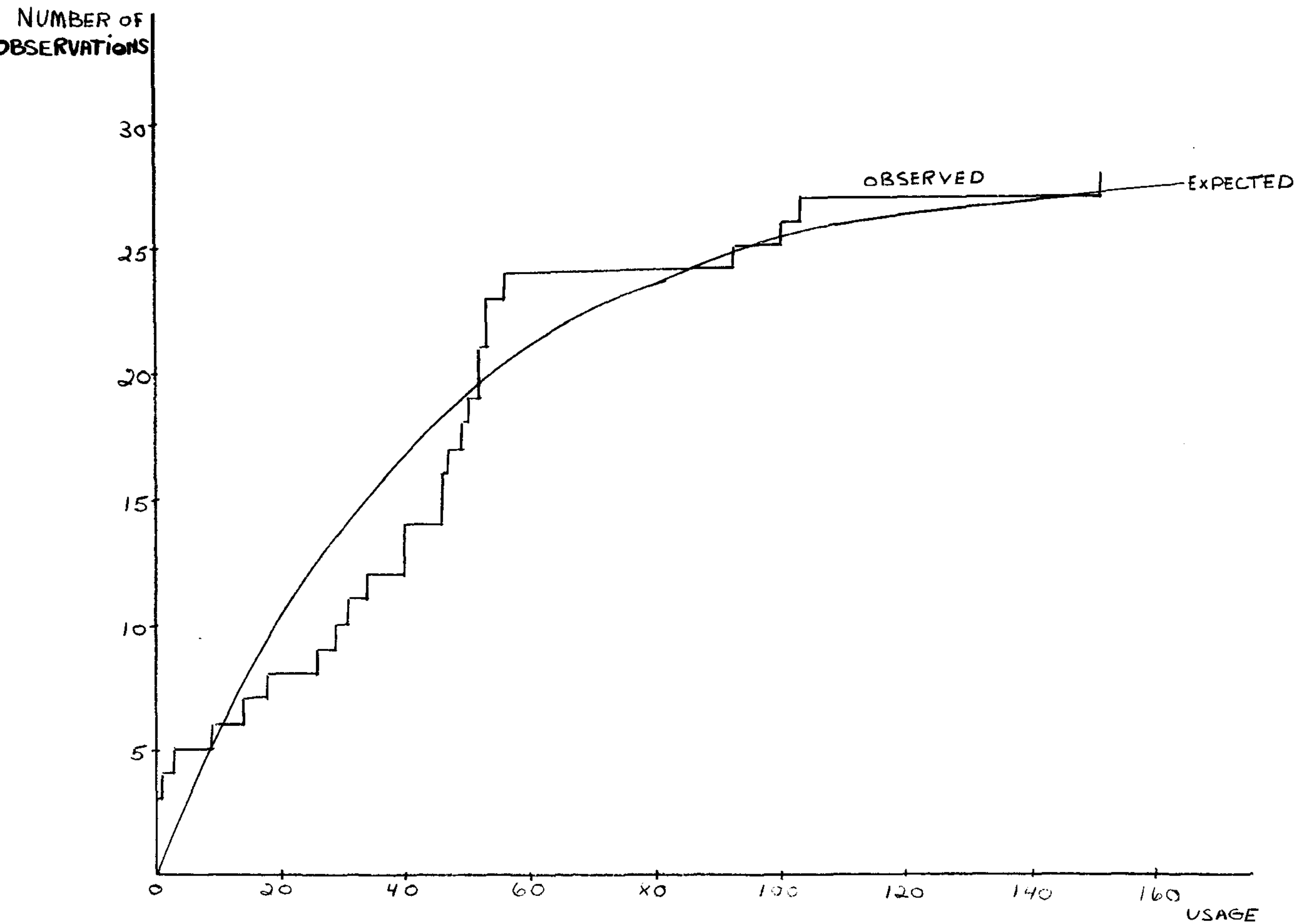
EXPECTED
OBSERVED

PART D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USAGE
OBSERVED
EXPECTED

PART E

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