COWLES FOUNDATION DISCUSSION PAPER NO. 21

A new Approach to the Aggregation Problem.

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December 22, 1956

I.

In previous investigations on the aggregation of economic relations, one has generally stayed within the borders of the so-called consistency approach.* This means that one has accepted as the most inportant goal

* Cf. H. Theil "Linear Aggreation of Economic Relations". Amsterdam 1954 p. 5.

or one of the most important goals in the construction of macromodels that a micromodel and a macromodel dealing with "the same piece of reality" shall be consistent with each other. The research done seems to indicate that to find a macromodel consistent with a given micromodel of some size will generally take a lot of work, and that the parameters of such a macromodel will have to be rather complicated functions of the parameters of the micromodel.

In this article we will be concerned with a macromodel which is not consistent with a micromodel which is supposed to be true; but whose parameters are simple functions of the parameters of the micromodel. Our interest in this macromodel has two sources, one which belongs to the field of theoretical economics, and another which belongs to the field of computational technics.

In macroeconomics we often find used what has been called the analogy approach to the aggregation problem. The research worker believes that some microeconomic relations are a good description of that piece of reality with which he is dealing. He constructs a macromodel, the relations of which he makes analogous to the assumed microrelations, and he makes assumptions about the parameters of the macromodel on the basis of what he believes about the parameters of the microrelations. If, for example, he believes in the descriptive value of individual demand relations, then he uses in his model a macro demand relation. If he believes that the micro demand relations are such that higher prices mean lower demand, then he will assume that the demand for a group of goods is a decreasing function of an index of the relevant prices. And if he believes that a certain event gives some micro demand functions a constant positive shift while no other changes in the microparameter values take place, then he will suppose that this event gives his macro demand function a constant positive shift while no other change take place in his macrorelations.

It may be worth while to investigate the soundness of this analogy approach. Research on the aggregation problem along the lines of the consistency approach may perhaps seem to be the most appropriate way of contributing to such an investigation. Such research is however, as mentioned above, likely to imply a lot of work if one deals with micromodels of some size. And it seems difficult if at all possible, in this way to make a satisfactory analysis of cases involving n microvariables.

Another approach is to specify a micromodel, postulate a macromodel and state that the macroparameters are such simple functions of the microparameters as implicitly assumed by many economists. We will then generally get a macromodel which is wrong (if the micromodel is assumed to be true). By investigating what determines "the degree of wrongness" in the macromodel we may perhaps obtain knowledge which makes it possible to describe some types of economic situations in which the analogy approach is a sound one, and other types in which it is dangerous. - We will in this article try to give a very modest contribution to such knowledge. We concern ourselves with a micromodel which consists of demand equations and supply equations. All equations are linear and static. The macrovariables are Laspeyre indexes of the microvariables. The macromodel is not derived from the micromodel but constructed on the assumption that the macrorelations are simple functions of the microrelations.

The second reason for being interested in macromodels of the type exemplified in the second section of this article, is due to the fact that we may be faced with computational problems of the following type: "We have given a set of equations, called the micromodel, in a set of variables called the microvariables. We have also given a set of aggregates of the microvariables. We want to find the implications of structural changes on the values of the aggregates. The information about structural changes is given in the form of changes in the values of the parameters of the micromodel." A straightforward way to handle this problem is to use the micromodel to find the values of the microvariables associated with every relevant structure, and then to find the values of the aggregates by using the definitions of these quantities. But such a procedure will require a lot of work if the micromodel is of some size. It may therefore be worth while to try to find methods which involves less work. As a part of a search for such methods it seems reasonable to investigate what can be achieved by using macromodels which are constructed on the basis of an assumption that macrorelations are simple functions of microrelations.

II.

We have n commodities. Their quantities are X_i, $(i = 1,\ldots n)$, and their prices are P_i. The quantity aggregrate x and the price aggregate p are defined by

$$(1) \qquad x = \frac{\sum_{i=1}^{n} P_i^o X_i}{\sum_{i=1}^{n} P_i^o X_i^o}, \qquad p = \frac{\sum_{i=1}^{n} X_i^o P_i}{\sum_{i=1}^{n} X_i^o P_i^o}$$

where P_i^o and X_i^o are the values of P_i and X_i in a known state of the economy.

It will prove convenient in our analysis to use the following transformations of the variables X_i and P_i

$$(2)\qquad x_i = \frac{P_i^o X_i}{\sum_{i=1}^{n} P_i^o X_i^o} \qquad\qquad p_i = \frac{X_i^o P_i}{\sum_{i=1}^{n} P_i^o X_i^o} .$$

It is easily verified that

$$(3)\qquad x = \sum_{i=1}^{n} x_i \qquad\qquad p = \sum_{i=1}^{n} p_i .$$

The variables x_i and p_i satisfy a set of demand relations

$$(4)\qquad x_i = b_i + \sum_{j=1}^{n} a_{ij} p_j \qquad i = 1,\ldots,n\ldots,$$

and a set of supply equations

$$(5)\qquad x_i = d_i + \sum_{j=1}^{n} c_{ij} p_j \qquad i = 1,\ldots,n.$$

The set of equations (4) and (5) will be call the micromodel, a_{ij}, b_i, c_{ij} and d_i will be called the microparameters, and x_i and p_i will be called the microvariables.

A structure is a combination of the micromodel and a set of values of the microparameters. Structures will be denoted by letters or numbers in quotation marks. If e denotes any of the microvariables or microparameters then e^y denotes the value of e associated with the structure "y". The structure "0" is identical with the aformentioned known state of the economy where $X_i = X_i^o$ and $P_i = P_i^o$. For this structure we know explicitly the values both of the microparameters and the microvariables. For the other relevant structures we know explicitly the values of the microparameters, but we do not know explicitly the values of the microvariables. - We will suppose that every relevant set of microparameter values are such that in connection with (4) and (5) it will imply a unique set of non-negative microvariable values.

It can be verified that

$$(6) \qquad a^o_{ij} = e^o_{ij} \frac{P^o_i X^o_i}{P^o_j X^o_j} , \qquad c^o_{ij} = s^o_{ij} \frac{P^o_i X^o_i}{P^o_j X^o_j} ,$$

where e^o_{ij} the demand elasticity of commodity i with respect to the price of commodity j in the structure "O", and where s^o_{ij} is the corresponding supply elasticity.

It is desired to find values (or estimates of the values) of x and p associated with relevant structures. Our task in the present section is to describe one method by which this can be done.

We will think of the market mechanism as consisting of a demand relation

$$(7) \qquad x = a\,p + b$$

and a supply relation

$$(8) \qquad x = c\,p + d .$$

The set of equations (7) and (8) will be called the macromodel, a, b, c and d will be called the macroparameters and x and p will be called the macrovariables. The aformentioned interpretation of e^y will be extended to the case where e is any of the macrovariables or macroparameters.

The information about the different structures is given by sets of values of the macroparameters. We must therefore, in order to connect the macromodel with our information, establish relationships between the macroparameters and the microparameters. This can be done in different ways.

One of them consists of deriving the macro demand equation by eleminations within a set equations consisting of (3), (4) and $(n - 1)$ of the n equations (5); and to derive the macro supply equation in an analogue way.*

* This is an example of an approach recommended by K. May. in his article "Technological Change and Aggregation", Econometrica vol. 15.

We will however proceed differently.

Let "y" and "z" be any pair of relevant structures. We will, as a part of the definition of the macroparameters, accept the following relations:

$$a^z - a^y = \sum_{j=1}^{n} p_j^o \sum_{i=1}^{n} (a_{ij}^z - a_{ij}^y)$$

$$b^z - b^y = \sum_{i=1}^{n} (b_i^z - b_i^y)$$

$$(9) \qquad c^z - c^y = \sum_{j=1}^{n} p_j^o \sum_{i=1}^{n} (c_{ij}^z - c_{ij}^y)$$

$$d^z - d^y = \sum_{i=1}^{n} (d_i^z - d_i^y)$$

It is obvious that when (9) is accepted, then the definition of the macroparameter values associated with one relevant structure, no matter which, is sufficient to complete the definition of the macroparameters.

Let "k" and "t" be a given pair of relevant structures. We will for a moment assume that the "k"-values of the macroparameters shall be defined in such a way that the macromodel implies correct statements about the "k"-values and the "t"-values of x and p. This assumption is equivalent to the acceptance of the set of equations.

$$x^k = a^k p^k + b^k$$

$$x^k = c^k p^k + d^k$$

$$(10) \qquad x^t = [a^k + (a^t - a^k)]\, p^t + [b^k + (b^t - b^k)]$$

$$x^t = [c^k + (c^t - c^k)]\, p^t + [d^k + (d^t - d^k)]$$

The only quantities in (10) which are not defined without the acceptance of (10), are the four quantities a^k, b^k, c^k and d^k. The four equations (10) are therefore (if the equations are independent and consistent) a definition of these quantities, and (9) and (10) are a complete definition of the macroparameters, We will, in order to indicate the role which structure "t" has played, denote the quantities defined by (10) by $a^k(t)$, $b^k(t)$, $c^k(t)$ and $d^k(t)$.

It can be shown that the definition of the macroparameters given by (9) and (10) may, and generally will, give a macromodel which is inconsistent with the values of x and p associated with an abritarily chosen structure. We may decribe the stiuation by saying that different pair of structures will, when relations (9) are accepted, call for different ways of making complete the definition of the macroparameters.

We will now derive expressions for $a^k(t)$ and $c^k(t)$ which will later on prove to be useful. We will, in doing this, let Δe denote $(e^t - e^k)$, where e may be any of the parameters or variables of the micromodel or the macromodel. We first use (4) to get

$$(11) \qquad \begin{aligned} \left(\sum_{i=1}^{n} \Delta x_i \right) &= \left(\sum_{i=1}^{n} \Delta b_i \right) + \left(\sum_{j=1}^{n} \Delta p_j \sum_{i=1}^{n} a_{ij}^k \right) \\ &+ \left(\sum_{j=1}^{n} p_j^k \sum_{i=1}^{n} \Delta a_{ij} \right) + \left(\sum_{j=1}^{n} \Delta p_j \sum_{i=1}^{n} \Delta a_{ij} \right) \end{aligned}$$

Using (7) we get

$$\Delta x = \Delta b + a^k(t) \cdot \Delta p + p^k \cdot \Delta a + \Delta p \cdot \Delta a. \tag{12}$$

The right hand side of (11) is equal to the right hand side of (12). Using this relationship and replacing Δa and Δb with functions of the microparameters given by (9), we get.*

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* Two terms in (13) will always cancell out, and another pair of terms will cancell out if "k" is the structure "0". The definitions (9) were deliberately chosen (among the set of definitions appropriate according to the remarks about our goals in section I) in such a way that these cancellationstake place.

$$\begin{aligned} &\left(\sum_{i=1}^{n} \Delta b_i\right) + a^k(t) \cdot \Delta p + \left(p^k \sum_{j=1}^{n} p_j^o \sum_{i=1}^{n} \Delta a_{ij}\right) \\ &\quad + \left(\Delta p \cdot \sum_{j=1}^{n} p_j^o \sum_{i=1}^{n} \Delta a_{ij}\right) = \left(\sum_{i=1}^{n} \Delta b_i\right) + \left(\sum_{j=1}^{n} \Delta p_j \sum_{i=1}^{n} a_{ij}^k\right) \\ &\quad + \left(\sum_{j=1}^{n} p_j^k \sum_{i=1}^{n} \Delta a_{ij}\right) + \left(\sum_{j=1}^{n} \Delta p_j \sum_{i=1}^{n} \Delta a_{ij}\right) \end{aligned} \tag{13}$$

Solving (13) with respect to $a^k(t)$ we get

$$\begin{aligned} a^k(t) &= \left(\sum_{j=1}^{n} \frac{\Delta p_j}{\Delta p} \sum_{i=1}^{n} a_{ij}^k\right) \\ &\quad + \left[\left(\sum_{j=1}^{n} \frac{\Delta p_j}{\Delta p} - \frac{p_j^o}{p^o}\right) + \frac{p^k}{\Delta p} \cdot \sum_{j=1}^{n} \left(\frac{p_j^k}{p^k} - \frac{p_j^o}{p^o}\right)\right] \cdot \sum_{i=1}^{n} \Delta a_{ij} . \end{aligned} \tag{14}$$

We can in an analogue way, from the supply equations of the micromodel and the macromodel derive the equation

$$c^k(t) = \left(\sum_{j=1}^{n} \frac{\Delta p_j}{\Delta p} \sum_{i=1}^{n} c^k_{ij}\right)$$

$$(15) \qquad + \left[\left(\sum_{j=1}^{n} \frac{\Delta p_j}{\Delta p} - \frac{p^o_j}{p^o}\right) + \frac{p^k}{\Delta p}\left(\sum_{j=1}^{n} \frac{p^k_j}{p^k} \cdot \frac{p^o_j}{p^o}\right)\right] \cdot \sum_{i=1}^{n} \Delta c_{ij} \ .$$

We will in the remaining part of this section suppose that the following conditions are satisfied for every set of values of the microparameters with which we will have to deal.

$$(16) \qquad c_{ii} - a_{ii} > 0 \qquad\qquad i = 1,\ldots,n$$

$$(17) \qquad c_{ij} - a_{ij} \leqq 0 \qquad\qquad i = 1,\ldots,n, \ j = 1\ldots)i(\ldots,n$$

$$(18) \qquad \sum_{j=1}^{n} (c_{ij} - a_{ij}) > 0 \qquad\qquad i = 1,\ldots,n \ .$$

Condition (16) will in all normal cases be satisfied. Conditions (17) and (18) will be satisfied in many cases, but not in all, so it is a relevant loss of generality of to accept them. The complete listing of types of cases which will contradict (17) is an easy but tedious task. A necessary condition for the positiveness of $(c_{ij} - a_{ij})$ is that commodity nr. i is complementary to commodity nr. j either in demand or in supply. The equations (6) may be useful when judging condition (18). A necessary condition for a case that contradicts (18) is that $(\sum_{j=1}^{n} a_{ij}) \geqq 0$ or $(\sum_{j=1}^{n} c_{ij}) \leqq 0$. Speaking loosely one may say that

$(\sum_{j=1}^{n} a_{ij}) \geqq 0$ means that the demand for commodity nr. j is not a decreasing function of "the price level" and $(\sum_{j=1}^{n} c_{ij}) \leqq 0$ means that the supply of commodity nr. i is not an increasing function of "the price level."

The following theorems can be proved.*

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* Outline of the proofs are given in Appendix A.

Theorem I:

If the micromodel (4) and (5) and the conditions (16), (17) and (18) are satisfied, and if some or all of the values of the parameters of the demand equations (4) increase (decrease) and if no other changes in the values of the microparameters take place, then all the resulting changes in the microvariables p_j $(j = 1...n)$, will be positive (negative).

Theorem II:

If the micromodel (4) and (5) and the conditions (16), (17) and (18) are satisfied, and if some or all of the values of the parameters of the supply equations (5) increase (decrease), and if no other changes in the values of the microparameters take place, then all the resulting changes in the microvariables p_j $(j = 1...n)$, will be negative (positive).

We will now describe a method for estimating how much the macrovariable values associated with one given structure differs from the macrovariable values associated with another given structure. Estimates of such differences will in connection with the known "0"-values of x and p enable us to fulfill the goal of finding estimates of relevant macrovariable values.

We will first be concerned with a case where a structure "t" can be obtained from a structure "k" by some positive and some negative changes in the values of the parameters b_i, while all other microparameters are unchanged. We will consider "t" as a result of two subsequent structural changes. The first of these consists of the positive changes in some b_i and takes us from "k" to a structure we will call "s". The second structural change consists of the negative changes in some b_i and takes us from "s" to "t". We can derive the following equations from the macromodel:

$$(19) \qquad \begin{aligned} (x^s - x^k) &= a^k(s).(p^s - p^k) + (b^s - b^k) \\ (x^s - x^k) &= c^k(s).(p^s - p^k) \end{aligned}$$

where $a^k(s)$ and $c^k(s)$ are those "k"-values of a and b which make the macromodel correct with respect to the structures "k" and "s". Solving (19) with respect to $(x^s - x^k)$ and $(p^s - p^k)$ we get

$$(20) \qquad \begin{aligned} (x^s - x^k) &= \frac{c^k(s)}{c^k(s) - a^k(s)}(b^s - b^k) \\ (p^s - p^k) &= \frac{1}{c^k(s) - a^k(s)}(b^s - b^k) . \end{aligned}$$

We will let $\underset{j}{\text{Max}}\,(\sum_{i=1}^{n} a_{ij}^k)$ and $\underset{j}{\text{Min}}\,(\sum_{i=1}^{n} a_{ij}^k)$ denote the maximum and the minium of $\sum_{i=1}^{n} a_{ij}^k$ under variation in j. $\underset{j}{\text{Max}}\,(\sum_{i=1}^{n} c_{ij}^k)$ and $\underset{j}{\text{Min}}\,(\sum_{i=1}^{n} c_{ij}^k)$ will be used in an analogue way. Theorem I implies that all the differences $(p^s - p^k)$ are non-negative. This together with (14) gives

$$(21) \qquad \underset{j}{\text{Min}}\,(\sum_{i=1}^{n} a_{ij}^k) \leq a^k(s) \leq \underset{j}{\text{Max}}\,(\sum_{i=1}^{n} a_{ij}^k) .$$

We have not yet said anything about how we shall define a "k"-value of a which we may use in calculations of an estimate of $(x^s - x^k)$ and $(p^s - p^k)$.

We will want such a value to be easy to calculate and to be close to $a^k(s)$. It may in view of (21) be an idea to use $\hat{a}^k$ defined by:

$$(22) \qquad \hat{a}^k = 1/2\ [\ \underset{j}{\text{Min}}\ (\sum_{i=1}^{n} a_{ij}^k) + \underset{j}{\text{Max}}\ (\sum_{i=1}^{n} a_{ij}^k)],$$

To accept such a definition is also consistent with the idea that we, (cf. section I), as a part of an investigation of "the analogy approach", should define macroparameters which are related to the microparameters in a simple way.

It is obvious from (21) and (22) that $\hat{a}^k$ and $a^k(s)$ are related to each other in the following way:

$$(23) \qquad |\ a^k(s) - \hat{a}^k\ | \leqq 1/2\ [\underset{j}{\text{Max}}\ (\sum_{i=1}^{n} a_{ij}^k) - \underset{j}{\text{Min}}\ (\sum_{i=1}^{n} a_{ij}^k)].$$

The same type of considerations which may lead to the definition of $\hat{a}^k$ may lead to a definition of a "k"-value of c by

$$(24) \qquad \hat{c}^k = 1/2\ [\underset{j}{\text{Min}}\ (\sum_{i=1}^{n} c_{ij}^k) + \underset{j}{\text{Max}}\ (\sum_{i=1}^{n} c_{ij}^k)]$$

$c^k(s)$ and $\hat{c}^k$ will satisfy the relations.

$$(25) \qquad \begin{aligned} &|\ c^k(s) - \hat{c}^k\ | \leqq 1/2\ [\underset{j}{\text{Max}}\ (\sum_{i=1}^{n} c_{ij}^k) - \underset{j}{\text{Min}}\ (\sum_{i=1}^{n} c_{ij}^k)]. \\ &\underset{j}{\text{Min}}\ (\sum_{i=1}^{n} c_{ij}^k) \leqq c^k(s) \leqq \underset{j}{\text{Max}}\ (\sum_{i=1}^{n} c_{ij}^k) \end{aligned}$$

Now we need some new notation. Let "y" and "z" be any pair of structures. We will denote our estimates of $(x^z - x^y)$ and $(p^z - p^y)$ by $(x^z - x^y)^{Est}$ and $(p^z - p^y)^{Est}$. The expressions $[(x^z - x^y) - (x^z - x^y)^{Est}]$ and $[(p^z - p^y) - (p^z - p^y)^{Est}]$ will be called estimation errors and denoted by

$\epsilon_x^{z,y}$ and $\epsilon_p^{z,y}$.

Our estimates $(x^s - x^k)^{Est}$ and $(p^s - p^k)^{Est}$ will be defined by

(26)
$$(x^s - x^k)^{Est} = \frac{\hat{c}^k}{\hat{c}^k - \hat{a}^k}\,(b^s - b^k)$$
$$(p^s - p^k)^{Est} = \frac{1}{\hat{c}^k - \hat{a}^k}\,(b^s - b^k)$$

These estimates derive any justification they have, from the relations (20) - (25). From (20),(21),(25) and (26) we may find upper and lower bounds for the estimation errors. In the normal case where $\underset{j}{\text{Max}}\,(\sum_{i=1}^{n} a^k_{ij}) < 0$ and $\underset{j}{\text{Min}}\,(\sum_{i=1}^{n} c^k_{ij}) > 0$, we have

(27)
$$\epsilon_x^{s,k} \leqq \frac{\underset{j}{\text{Max}}\,(\sum_{i=1}^{n} c^k_{ij})(b^s - b^k)}{\underset{j}{\text{Max}}\,(\sum_{i=1}^{n} c^k_{ij}) - \underset{j}{\text{Max}}\,(\sum_{i=1}^{n} a^k_{ij})} - (x^s - x^k)^{Est}$$

$$\epsilon_x^{s,k} \geqq \frac{\underset{j}{\text{Min}}\,(\sum_{i=1}^{n} c^k_{ij})\,(b^s - b^k)}{\underset{j}{\text{Min}}\,(\sum_{i=1}^{n} c^k_{ij}) - \underset{j}{\text{Min}}\,(\sum_{i=1}^{n} a^k_{ij})} - (x^s - x^k)^{Est}$$

$$\epsilon_p^{s,k} \leqq \frac{(b^s - b^k)}{\underset{j}{\text{Min}}\,(\sum_{i=1}^{n} c^k_{ij}) - \underset{j}{\text{Max}}\,(\sum_{j}^{n} a^k_{ij})} - (p^s - p^k)^{Est}$$

$$\epsilon_p^{s,k} \geqq \frac{(b^s - b^k)}{\underset{j}{\text{Max}}\,(\sum_{i=1}^{n} c^k_{ij}) - \underset{j}{\text{Min}}\,(\sum_{i=1}^{n} a^k_{ij})} - (p^s - p^k)^{Est}$$

We can find estimates for $(x^t - x^s)$ and $(p^t - p^s)$ in exactly the same way as we found estimates for $(x^t - x^s)$ and $(p^t - p^s)$ and we can also find upper and lower bound for the estimation errors $\epsilon_x^{t,s}$ and $\epsilon_p^{t,s}$. Estimates for $(x^t - x^k)$ and $(p^t - p^k)$ are defined by

$$(28) \qquad \begin{aligned} (x^t - x^k)^{Est} &= (x^s - x^k)^{Est} + (x^t - x^s)^{Est} \\ (p^t - p^k)^{Est} &= (p^s - p^k)^{Est} + (p^t - p^s)^{Est} . \end{aligned}$$

Let now e be any quantity. We will let $U(e)$ denote an upper bound for e and $L(e)$ denote a lower bound for e .

We have

$$(29) \qquad \begin{aligned} L(\epsilon_x^{s,k}) + L(\epsilon_x^{t,s}) \leqq \epsilon_x^{t,k} \leqq U(\epsilon_x^{s,k}) + U(\epsilon_x^{t,s}) \\ L(\epsilon_p^{s,k}) + L(\epsilon_p^{t,s}) \leqq \epsilon_p^{t,k} \leqq U(\epsilon_p^{s,k}) + U(\epsilon_p^{t,s}) . \end{aligned}$$

These relations establish upper and lower bounds for $\epsilon_x^{t,k}$ and $\epsilon_p^{t,k}$. It is easily verified that $(x^t - x^k)^{Est}$ and $(p^t - p^k)^{Est}$ could be calculated more directly by the equations

$$(30) \qquad \begin{aligned} (x^t - x^k)^{Est} &= \frac{\hat{c}^k}{\hat{c}^k - \hat{a}^k} (b^t - b^k) \\ (p^t - p^k)^{Est} &= \frac{1}{\hat{c}^k - \hat{a}^k} (b^t - b^k) . \end{aligned}$$

But it seems difficult to find convenient expressions for upper and lower bounds for the estimation errors without introducing the structure "s" .

We will now turn to a case where one structure can be obtained from another structure by some positive and some negative changes in the parameters a_{ij}, while all other microparameters are unchanged. This case can be handled in a way analogous to that used in the case treated above.

Let us also in this case use "k", "s" and "t" as names on the structures to be considered. We look upon the structure "t" as a result of two subsequent structural changes. The first one consists of positive changes in the parameters a_{ij} and takes us from the structure "k" to a structure "s". The second change consists of negative changes in a_{ij} and takes us from "s" to "t". We derive a set of equations

$$(31)\qquad \begin{aligned} (x^s - x^k) &= \frac{c^k(s)}{c^k(s) - a^k(s) - (a^s - a^k)}\, p^k\,(a^s - a^k) \\ (p^s - p^k) &= \frac{1}{c^k(s) - a^k(s) - (a^s - a^k)}\, p^k\,(a^s - a^k). \end{aligned}$$

These equations will play the same role in the present case as equations (20) did in connection with the case previously analysed.

Corresponding to the second of the equations (25), we have

$$(32)\qquad \operatorname*{Min}_{j}\,\Big(\sum_{i=1}^{n} c_{ij}^{k}\Big) \leq c^k(s) \leq \operatorname*{Max}_{j}\,\Big(\sum_{i=1}^{n} c_{ij}^{k}\Big)$$

but the relation corresponding to (21) will be more complicated. From theorem I, and relation (14) we get

$$(33)\qquad \begin{aligned} a^k(s) &> \operatorname*{Min}_{j}\,\Big(\sum_{i=1}^{n} a_{ij}^{k}\Big) - (a^s - a^k) + L(G) \\ a^k(s) &< \operatorname*{Max}_{j}\,\Big(\sum_{i=1}^{n} a_{ij}^{s}\Big) + U(G). \end{aligned}$$

where $$G = \frac{p^k}{p^s - p^k} \sum_{j=1}^{n} \left(\frac{p_j^k}{p^k} - \frac{p_j^o}{p^o} \right) \cdot \sum_{i=1}^{n} (a_{ij}^s - a_{ij}^k)$$

Convenient expressions for the upper and the lower bound of G are difficult to give when "k" is completely unspecified, but may be obtained (in more or less complicated ways) for all cases of interest to us. If "k" is identical with "0" then G is equal to zero.

One way of estimating $(x^s - x^k)$ and $(p^s - p^k)$ is to use the estimates defined by the equations

$$(34) \quad \begin{aligned} (x^s - x^k)^{Est} &= \frac{\hat{c}^k}{\hat{c}^k - \hat{a}^k - (a^s - a^k)} (p^k)^{Est} (a^s - a^k) \\ (p^s - p^k)^{Est} &= \frac{1}{\hat{c}^k - \hat{a}^k - (a^s - a^k)} (p^k)^{Est} (a^s - a^k) \end{aligned}$$

where $\hat{a}^k$ and $\hat{c}^k$ are defined by (22) and (24) and where $(p^k)^{Est}$ is an estimate of p^k which is found by starting with structure "0" and using the method outlined in this article. Upper and lower bounds for the estimation errors $\epsilon_x^{s,k}$ and $\epsilon_p^{s,k}$ may be derived from equations (31) - (34) and knowledge of an upper and lower bound for $[p^k - (p^k)^{Est}]$.

The methods used in estimating $(x^s - x^k)$ and $(p^s - p^k)$ will also be used* in estimating $(x^t - x^s)$ and $(p^t - p^s)$.

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* This statement implies that we use a "s"-value of a which pr. definition is equal to $1/2\,[\mathrm{Min}_j\,(\sum_{i=1}^{n} a_{ij}^s) + \mathrm{Max}_j\,(\sum_{i=1}^{n} a_{ij}^s)]$. It should be noticed that this "s"-value of a is not necessarily equal to $\hat{a}^k + \sum_{j=1}^{n} p_j^o \sum_{i=1}^{n} (a_{ij}^s - a_{ij}^k)$

The situation may be described by saying that the macromodel we use in estimating $(x^t - x^s)$ and $(p^t - p^s)$ is likely to be different from the macromodel we use in estimating $(x^s - x^k)$ and $(p^s - p^k)$.

Any structural change which only alters the micro demand parameters, can be looked upon as a combination of two subsequent changes, one which alters only the parameters a_{ij} and one which alters only the parameters b_i . The methods outlined above therefore enable us to deal with any structural change restricted to the demand equations - . Changes which alters only the parameters of the supply equations can be investigated by the methods we have used in dealing with changes which alters only the parameters of the demand equations. Any structural change may be looked upon as a combination of one change which alters only the demand parameters and another change which alters only the supply parameters. Any structural change can therefore be handled by the methods outlined above.

III.

In this section we will make some remarks about the quality of estimates obtained by using the methods described in section II.

We will first introduce a couple of new symbols. $E_x^{z,y}$ will denote the largest of the following two quantities: 1) Our upper bound for $\epsilon_x^{z,y}$, and 2) the numerical value of our lower bound for $\epsilon_x^{z,y}$. $E_p^{z,y}$ will be given an analogous meaning.

A possible measure for the quality of the estimate $(x^z - x^y)^{Est}$ is the quotient of $|x^z - x^y|$ and $|\epsilon_x^{z,y}|$. We don't know any of these quantities. But as a substitute for $|x^z - x^y|$ we may use $|(x^z - x^y)^{Est}|$. And we know that $|\epsilon_x^{z,y}|$ is smaller than $E_x^{z,y}$. We will therefore as a "quanlity coefficient" for $(x^z - x^y)^{Est}$ use

$$\frac{|(x^z - x^y)^{Est}|}{E_x^{z,y}}$$

. And we will use

$$\frac{|(p^z - p^y)^{Est}|}{E_x^{z,y}}$$ as a quality coefficient for $(p^z - p^y)^{Est}$.

We will interpret the meaning of these coefficients by saying that higher values of them in one case than another, means that the quality of the estimates is better in the first case than in the second. The degree of reasonableness of such an interpretation will of course depend on the purposes of the estimates.

We will first deal with the case where "z" can be obtained from "y" by changes only in the parameters b_i. Relation (30) states that $|(x^z - x^y)^{Est}|$ and $|(p^z - p^y)^{Est}|$ both will be proportional to $|b^z - b^y|$, which pr. definition is equal to $|\sum_i^n (b_i^z - b_i^y)|$. From 27 -(30) it may be verified that $E_x^{z,y}$ and $E_p^{z,y}$ both are proportional to $(\sum_i^n |b_i^z - b_i^y|)$. This means that the quality coefficients remain unchanged if all the differences $(b_i^z - b_i^y)$ are multiplied with the same number. Stated somewhat loosely all this means that to use the estimation methods when the changes in the parameters b_i are large, is as good (or as bad) as when the changes are small.

Another observation which may be made is that the method will, "other things being equal", give higher values of the quality coefficients the greater the value of

$$\frac{|\sum_{i=1}^n b_i|}{\sum_{i=1}^n |b_i|}$$ is.

Stated loosely this means that we get better estimates the more one sign (plus or minus) dominates in the set of changes $(b_i^z - b_i^y)$.

It may also be of interest that the values of the coefficients are independent of whether the quantities x^y and p^y are known or not. Let us also notice that the "y"-values of a_{ij} and c_{ij} play an important role in determining the quality of our estimates. We may observe that the quantities $E_x^{z,y}$ and $E_p^{z,y}$ are zero (and the values of the quality coefficients infinitely large) when both $\sum_{i=1}^{n} a_{ij}^y$ and $\sum_{i=1}^{n} c_{ij}^y$ are independent of j .

Let us finally notice that the "y"-values of b_i influence the quality coefficients only through the differences $(b_i^z - b_i^y)$ and that the "y"-values of d_i have no influence at all.

Let us now turn to the case where "z" can be obtained from "y" by changes only in the parameters a_{ij} . In some aspects this case is different from and in other aspects it is like the case discussed above.

Suppose for a moment that all the quantities $(a_{ij}^z - a_{ij}^y)$ are multiplied by the same positive number larger than one. What will then happen to the quality coefficients?

This question is difficult to answer completely, but it seems justified from an investigation of relations (31)-(34) to say that they will in most cases increase.

A conclusion analogous to one made for the case where the parameters b_i changes, it that our method will "other things being equal" give higher values of the quality coefficients the more one sign (plus or minus) dominates in the set of changes $(a_{ij}^z - a_{ij}^y)$. This conclusion can be derived in the same way as its analogue.

It may be of interest to notice [cf. (34) and the terms $L(G)$ and $U(G)$ in (33)] that the value of the quality coefficients will be greater the more precise is our knowledge of the values of p^y and p_i^y.

The "y"-values of a_{ij} and c_{ij} play also in this case an important role in the determination of the quantities $E_x^{z,y}$ and $E_p^{z,y}$. But that restrictions on these parameter values alone are not sufficient to make $E_x^{z,y}$ and $E_p^{z,y}$ equal to zero is clearly indicated by relation (33). - We will finally notice that the "y"-values of b_i and d_i have no influence on the quality coefficients except for the role they may play on the determination of $(p^y)^{Est}$ and the establishment of lower and upper bounds for the quantity G.

A discussion of the consequences of changes in the parameters d_i will give conclusions analogous to those reached in the above discussion of consequences of changes in b_i, and a discussion of consequences of changes in c_{ij} will give conclusions analogous to those reached in the above discussion of consequences of changes in a_{ij}.

IV.

In section I it was declared that we would try to make a contribution to knowledge which might enable us to judge the analogy approach. Have we succeeded in doing this?

We think that the conclusions of section III may justify a positive answer to this question. But it is important to keep in mind that $|\epsilon_x^{z,y}|$ and $|\epsilon_p^{z,y}|$ may be zero or small when $E_x^{z,y}$ and $E_p^{z,y}$ are large. One implication of this is that the analogy approach may work well in cases where the values of our quality coefficients are small. Another implication is that conclusions about how changes in the values of the quality coefficients are related to given changes in the set of relevant structures, are not a rigorous basis for drawing conclusions about how the appropriateness of the analogy approach is related to the set of relevant structures.

But in spite of this it seems reasonable, as long as better knowledge is lacking, to take into consideration the conclusions reached in section III, when we want to judge the use of the analogy approach in particular cases. These conclusions may at any rate enable us to recognize cases where the methods safely can be used.

In section I we also considered the possibility of finding methods which could be used to save work when we were faced with a certain type of computational problem. It seems however unlikely that the method outlined in section II will be much used for this purpose. Developments in computing machines decrease the value of methods which save computational work, but often give only crude approximations to the desired results. If the main idea of our method nevertheless should be used, it is likely to be in a different form; since there seem to be several rather simple modifications which in most cases will diminish estimation errors and known bounds for the numerical values of the estimation errors.

We will finally point out the conclusions reached in section III will be of relevance also if we deal with some generalized versions of the models of this article. The parameters b_i with which we have been dealing may be looked upon as defined by a set of equations

$$(36) \qquad b_i = b_{io} + b_{i1} Z_1 + \ldots + b_{ih} Z_h + \epsilon_i \quad (i = 1, \ldots, n)$$

where b_{ij}, $(j = 0, 1...k)$, are microparameters of a more "basic" type than b_i, Z_j, $(j = 1,...,k)$, are a set of exogenious variables, and ϵ_i is a stochastic variable. The conclusions in section III regarding the quality of our estimates of x and p when the parameters b_i are changing, give a basis for conclusions regarding the use of a macromodel for dealing with changes in exogenious variables, and microparameters associated with such variables, when variables of this type occur in linear demand and supply equations. The conclusions in section III also give a starting point for a discussion of the consequences of changes in the distribution of additive stochastic elements occuring in such equations.

Appendix A

Outline of Proofs for Theorems I and II.

We will in this appendix be working with the micromodel:

$$(4) \qquad x_i = b_i + \sum_{j=1}^{n} a_{ij}\, p_j \qquad i = 1\ldots n$$

$$(5) \qquad x_i = d_j + \sum_{j=1}^{n} c_{ij}\, p_j \qquad i = 1\ldots n.$$

The parameters of this model will always be assumed to satisfy the following conditions

$$(16) \qquad c_{ii} - a_{ii} > 0 \qquad i = 1\ldots n$$

$$(17) \qquad c_{ij} - a_{ij} \leqq 0 \qquad i = 1\ldots n \;\; j = 1\ldots)i(\ldots n$$

$$(18) \qquad \sum_{j=1}^{n} (c_{ij} - a_{ij}) > 0 \qquad i = 1\ldots n.$$

We will give an outline of the proofs of these theorems:

Theorem I:

If the micromodel (4) and (5) and the conditions (16),(17) and (18) are satisfied, and if some or all of the values of the parameters of the demand equations (4) increase (decrease) and if no other changes in the values of the microparameters take place, then all resulting changes in the values of the microvariables p_j will be positive (negative).

Theorem II:

If the micromodel (4) and (5) and the conditions (16),(17) and (18) are satisfied, and if some or all of the values of the parameters of the supply equations (5) increase (decrease) and if no other changes in the values of the microparameters take place, then all resulting changes in the values of the microvariables p_j will be negative (positive).

Let s and k be given positive numbers less than or equal to n. Suppose there occurs a change Δb_k in the value of b_k while all other parameters are unchanged. Let Δp_j $(j = 1 \ldots n)$ denote the price changes caused by Δb_k.

The following statement is true:

Lemma I: If the relation $|\Delta p_s| \geqq |\Delta p_t|$ is satisfied for all values of t such that $\Delta p_s \cdot \Delta p_t > 0$, then $s = k$.

For, suppose that lemma I is not true. Then it will be possible to find a certain s, $s \neq k$, such that Δp_s is numerically larger then any other price change of the same sign. Using the demand and supply equations for commodity s, we get

$$(37) \qquad 0 = \sum_{j=1}^{n} (c_{sj} - a_{sj}) \frac{\Delta p_j}{\Delta p_s} .$$

The righthand side of (37) may be regarded as a weighted sum of the expressions $(c_{sj} - a_{sj})$; $j = 1, \ldots n$, where the weight for the positive element $(c_{ss} - a_{ss})$ is one, and where the weight for the other elements which all are non-positive, are less than or equal to one. Such a sum must, cf. (18), be positive. This contradiction of (37) proves lemma I.

We can further prove:

Lemma II: All non-zero changes in the prices will be of the same sign.

Suppose that lemma II was not true. Then there exist a case where there are both a group of positive price changes and a group of negative price changes. The group not containing Δp_k must, since it is finite, contain a member which is numerically at least as great as any other number of the group. But lemma I denies the existence of such a member. Our assumption that lemma II is not true, must therefore be wrong.

From the two lemmas is it easily found that Δp_k is numerically larger than any other price change. We can further prove.

Lemma III: The relation $\Delta b_k \; \Delta p_j \geqq 0$ is satisfied for all values of j.

Using the demand equation and the supply equation for the k'th commodity we get

$$(38) \qquad \Delta b_k = \sum_{j=1}^{n} (c_{kj} - a_{kj}) \, \Delta p_j \quad .$$

From conditions (16) - (18) and the fact that Δp_k is the numerically greatest price change, it may be verified that $(c_{kk} - a_{kk}) \, p_k$ is numerically greater than the sum of all other terms on the right hand side of (38). This in connection with (16) and (38) implies that Δb_k and Δp_k are of the same sign. Lemma III must be therefore be true, since lemma II states that all non-zero price changes are of the same sign.

Let us now turn to the case where there is a change Δa_{kt} in the parameter a_{kt} while all other parameters remain unchanged. We will also in this case denote the resulting price changes by Δp_j . The following statement can be proven:

Lemma IV The relation $\Delta a_{kt} \, \Delta p_j \geqq 0$ is satisfied for all values of j.

Lemma IV can be proven by a reasoning analogous to that one which is used in proving lemma III.

Theorem I can now be proven. We look upon each of the changes dealt with in theorem I as the result of a sequence of changes. Each of the members of such a sequence is a change in one and only one parameters of equations (4).

It is easily verified that theorem I is true if lemmas III and IV are true.

Theorem II may be proven by a reasoning analogous to that one outlined above.