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The Theory of Backwardation for a Stationary Economy with Random Stocks.*

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1. Introduction

In a commodity market with forward trading, two features must be taken into account by any theory: At any time there is not just one but a whole series of prices, whose mutual relationship must be explained. Prices are not determined by the equilibrium of current supply and demand, because stocks of the commodity, if storable (as is usually the case), may be carried forward into future periods. But what determines the amounts of commodity that will be carried forward? The conclusion seems inevitable that in some way <u>beliefs</u> about the future must enter into the decision of those individuals that are responsible for the carryover, namely those who hold unhedged stock or have bought forward on net balance. These beliefs or expectations, to be sure may be vague and may differ widely between people. If nothing more could be said about them, prices would be random, and all explanation had to stop here. To quote Hawtrey: "When the future movement of prices becomes very uncertain there are wide differences of opinion among dealers, and both bulls and bears hope for big gains. The market seeks from hour to hour the price which will just divide the bulls and the bears at the point at which buyers and sellers balance". [Mr. Kaldor on the Forward Market, Rev. Ec. Studies VII (1939/40) p. 204]. On the other hand any attempt to find a common element in expectations is up against a "fatal objection to the introduction of any aggregates or averages of expectations into economic reasoning, not merely that owing to differences in expectations of different individuals there is no one expected price, but there are gaps in the series where there is no expectation at all...."

This statement is unexceptionable if one interprets expectations to mean "single valued" expectations, as they were indeed considered in the paper by Kalecki referred by Hawtrey. It might seem that to assume that traders hold probabilistic expectations is to make matters worse. Certainly not every trader in the real world has something approximating a subjective probability distribution of prices and future dates in his mind when engaging in a transaction. But perhaps the problem of expectations is so perplexing only because one starts out with a complex real world, with all its imperfections and irrational people. In other contexts it has been recognized that to explain something a theory need not, and probably cannot, be of photographic accuracy. The problem of expectations ceases to be an obstacle to further analysis if we look at it in the context of a stationary economy. The reason is simply, that in a stationary economy it takes no brains to form an opinion about the probability distribution of prices at future dates, it can be inferred from the empirical frequency distributions of prices in the past. No rational person can afford to disregard these observations from the past except two classes of people: Those who on no account would be speculators in a commodity market, say because of their poverty. And those who although under an occupational necessity to buy or sell that commodity find it a more comfortable existence to hedge themselves completely. (That means that at any time the amount of stock owned plus forward purchases is exactly balanced by the amount of stock sold forward. Then if it can be supposed that forward and spot prices move always in the same direction and by the same amounts (approximately) it follows that they can neither gain nor lose by movements in prices). Everybody else is essentially a speculator. And a speculator must have some opinion about prices in the future. In a stationary economy every rational speculator will have the same opinion. Yet because of different attitudes toward the same "objective" risk there will be trading among them. Hawtrey's assertion that if all held the same expectation, there would be no speculation [ibid. p. 204] does not apply, when expectations refer to a probability distribution, as we shall see in detail later on.

Expectations of all concerned have thus been reduced to a "constant" -- that is, as long as conditional expectations about the future are impossible. For a crop commodity this would be the case during the time after an annual harvest when no forecasts of next year's crop size can yet be made. To be rigrous we must also rule out either any connections with other commodities, substitutes or complements, or we must assume that neither are conditional forecasts of other crops available at that time. Apart from other constants of the stationary economy, what do the commodity prices at such times depend on? Obviously only the stocks of the commodity itself and those of related commodities. If for the sake of simplicity we rule out interdependence, the commodity price is just a function of the total commodity stock.

What is this function? How do the constants of the economy enter into it? These constants include for instance the demand functions of consumers for this commodity, its storage cost, the acreage devoted to this crop, the yield (or if the acreage is constant the crop size) distribution. This is the problem to be considered in this paper.

2. The Model

The following assumptions are believed to simplify the problem as suggested in the introduction while retaining its most significant features:

1. Discreteness. The market is assumed to convene only once every year, after the annual harvest, at which time all contracts are concluded.
2. Stationarity. More precisely
   - 2.1 Independent and identical crop size distributions which are not conditional on any facts known one year before
   - 2.2 A stationary economy, in particular constant (time independent) demand and cost functions
   - 2.3 No anticipation or deferment of demand for consumption
3. Uniformity of the commodity traded forward, a single location for delivery.

These assumptions will be spelled out in greater detail in the following discussion of the types of demand.

1. Supply

Regarding the supply side we assume supplies to be uninfluenced by the forward price. This restriction can be removed at some sacrifice of simplicity. Similarly the distribution of next year's crop could be made conditional on this years crop size or this years stock by means of appropriate changes in the expression for the expected value of next year's price.

2. Demand for Consumption. The demand for consumption is assumed to depend on the (spot) price for the current period only.

$$u = u(p)$$

A linear approximation will be used sometimes.

$$u = u_o - u_1 p$$

While the total demand is assumed to be deterministic (another assumption which can be removed) the demand facing a particular manufacturer is regarded to be a random variable.

3. Demand by Manufacturers. Manufacturers demand is therefore for a certain inventory whose aggregate amount v may and typically will be in excess of annual consumption at the level determined by the spot price. We shall not consider the possibility that the proportion between inventory and consumption depends on the potential losses on inventories taken into the next period. We assume that total demand of manufacturers at a given spot price is a constant multiple k of consumption at this price.

$$v = k\, u(p).$$

4. Demand by Warehouses

We assume that warehouses (elevators) buy physical commodity stocks and hedge completely by selling it forward. Their demand is given by the level at which marginal's carrying cost $c(x)$ equals the spread $q-p$ between forward price q and spot price p.

A spread less than carrying cost implies that carryover (from which manufacturers' stocks are excluded by definition) is zero.

$$x - ku[\phi(x)] = 0 \text{ if } q < p + c(x).$$

5. The Demand by Speculators We shall use the convention of denoting by speculative demand a demand for contracts of purchase and forward sale rather than for physical stock. Thus elevator operators who speculate are regarded as having concluded fictitious forward sales contracts with

themselves by which they hedge all their stocks. By commitment we shall denote the net of forward sales over forward purchases. Manufacturers stocks which are excluded from carryover are also unhedged, as they are meant for consumption in the current period, and therefore do not contribute to commitments in our model. Aggregate commitments must then equal the carry-over

$$x - ku(p).$$

How is the demand for commitments i.e., for speculative stock determined? This calls for another consideration of expectations.

In a stationary economy not only the size distribution of crops is known, but also an induced distribution of prices after the next crop, conditional on the stocks after the present harvest. This hinges on the fact that the stock price function which we have to derive here is empirically known. As an example consider the relationship between cotton prices and stocks from 1923-1939. (figure). Expect for three years of heavy exports, 1924-1926, and the war year 1939 all observations lie very close to a straight line. The fraction of the variance explained by a linear regression* is .84.

* The regression was carried out by F. Bobkoski

Now as we shall demonstrate below this empirical stock price relation $p = \phi(x)$ permits one to obtain the distribution of next years spot price conditional on this years stocks, from the crop size distribution. This calculation need not be carried out each time. It is sufficient to assume that the conditional probability distribution of next years prices given this years stock or carryover has been estimated once and for all and is a matter of expert knowledge. Among informed persons there can then be no disagreement about the probability distribution of next years prices.

Agreement about the probability distribution of next years prices does not imply identical behavior of all speculators. They may have different risk preferences and different assets. This means that they attach different utility to the same outcomes. While as rational persons they all attempt to maximize expected utility this will result in different commitments. In order to keep things simple we shall not discuss the most general case in which everyone has possibly different utility functions. This would amount to a situation in which aggregate holding of speculative stock is a general functional of the probability distribution of next years prices. Among alternative simple hypotheses we shall adopt the venerable one of <u>normal backwardation</u> propounded by Keynes* and Hicks** among others, slightly modified

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* A treatise on Money, vol. II, pp. 142-145

** Value and Capital, pp. 138-9

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for our purposes. According to this hypothesis, speculators consider only the expected value, (the mean) of next year's price. The amount for which an individual is willing to commit himself depends on the difference between this expected value of future spot price and the current forward price, known as the backwardation

$$E\,p^1 - q \qquad \text{where } p^1 \text{ is next year's spot price.}$$

To be specific let us assume that a speculator commits himself to his limit if the backwardation exceeds a certain threshold and not at all when the backwardation is less than or equal to it. These thresholds vary among

Cotton: Total U.S. Supply and Prices, 1923-1938
(Source: Agricultural Statistics, 1942)

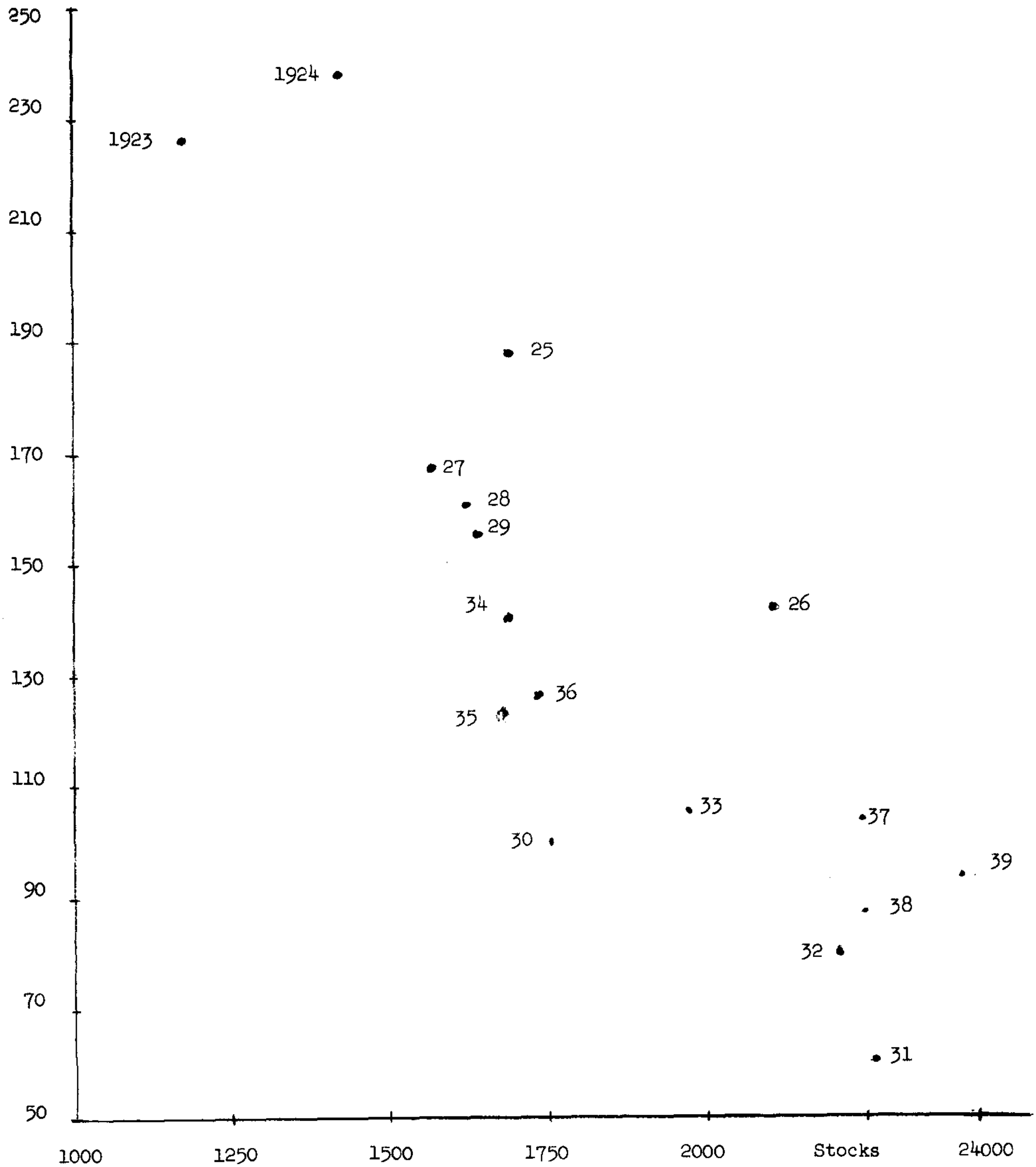

Fig. 1

individuals according to their risks preference. For instance if they are normally distributed then the total speculative commitments, i.e., the demand for speculative stock, is a sigmoid function of the backwardation.

$$s = s\ (E_p^{\ 1} - q)$$

We shall make use of a piecewise linear approximation to this demand function. Under the assumptions made, our problem becomes that of determining two prices: the spot price and the forward price for delivery at the convention of the market next year, both quoted as of this year. Since the connection between spot and forward price is a much less difficult one the following analysis will be focussed on the relation between stock and <u>spot</u> price.

3. The Stock Balance Equation

The equilibrium condition through which the unknown price function is determined is the stock balance equation. Its states that stock not absorbed by manufacturers at the prevailing stock price is held by speculators, who are motivated by the existing backwardation. Through the stock - spot price function, our unknown, a relationship is established between the stock and the expected value of the future price. Since the forward price also depends in a known manner on stock and spot price, a relationship follows between stock and backwardation. When inserted into the balance equation this determines the unknown stock price function as we shall now demonstrate.

Two cases must be distinguished. When stocks are sufficiently small

$$x \leq x^*$$

all stock is absorbed in this period.

$$(3.1) \qquad ku\,[\varphi(x)] = x$$

To be sure an amount $(k-1)u$ will still be left at the end of the period, but tied up in manufacturers' inventories. -- Of course it is part of the problem to determine the critical stock level x^*.

For stock levels above x^* the excess of stock over manufacturers' requirements

$$x - ku\,[\varphi(x)]$$

markes the supply of speculative stocks. It is a function of the spot price only, i.e., of the stock level x via $\varphi(x)$.

We consider next the demand for speculative stock and first, the backwardation. Let y be the size of next years crop and let $F(y)$ be its probability distribution. The conditional expected value of next years stock price is now given by

$$\int \varphi(x - u\,[\varphi(x)] + y)\, dF\,(y)$$

where $\varphi(\ )$ is the same function as that which related stock and price in the present period.

Since a positive amount of stock is carried over, the forward price must equal the spot price plus (marginal) carrying cost.

$$q = \varphi(x) + c(x)$$

The backwardation is therefore

$$\alpha \int \varphi(x - u\,[\varphi(x)] + y)\, dF(y) - \varphi(x) - c(x).$$

The demand for speculative stocks is a function $s(\ )$ of this expression.

Equating demand and supply we have the stock balance equation

$$(3.2) \qquad s[\alpha \int \varphi(x - u\,[\varphi(x)] + y)\, dF(y) - \varphi(x) - c(x)] =$$

$$x - ku\,[\varphi(x)] \qquad\qquad x \geqq x^* .$$

At $x = x^*$ the solutions $\varphi_1(x)$, $\varphi_2(x)$ of the two equations (3.1) and (3.2) respectively, must agree. This determines x^*.

$$(3.3) \qquad \varphi_1(x^*) = \varphi_2(x^*).$$

When carryover is zero, in other words $x \leq x^*$, the forward price q is determined by the dissappearance of speculative stocks

$$(3.4) \qquad v[\alpha \int \varphi\,(\frac{k-1}{k}\; x + y)\, dF(y) - q] = 0$$

When risk aversion is universal, then $v(0) = 0$ and then our solution is always

$$q = \alpha \int \varphi\,(\frac{k-1}{k}\; x + y)\, dF(y).$$

It is interesting to note that we have succeeded in reducing the equilibrium problem to a two period condition, although physical stocks may be carried through more than two periods.

4. Mathematical Analysis

In this section we sketch the solution of the system (3.1) and (3.2) in the mathematically simplest case. These comments are meant to be illustrative only. The first branch of the stock price function $\varphi_1(x)$ is simply determined in terms of the inverse function u^{-1} of u

$$\varphi_1(x) = u^{-1}\,(\frac{x}{k}) \qquad\qquad x \leqq x^* .$$

In particular if u is linear

$$u(x) = u_o - u_1\, x$$

then $$ku_o - ku_1\, \varphi(x) = x$$

$$(4.1) \qquad \varphi(x) = -\frac{x}{ku_1} + \frac{u_o}{u_1}$$

For $x \geqq x^*$ we have the functional equation (3.2) which in spite of its simple structure does not admit of an explicit solution, even when all the given functions are assumed to be linear. However the following device leads to an approximate solution.

Suppose that speculators employ a linear approximation, $\psi(x)$, in calculating the expected value of next years price conditional on this years stock. Then the equation system assumes the form

$$(4.2) \qquad s[\alpha\, \psi(x - u(\varphi(x)] + \bar{y}] - \varphi(x) - c(x)]$$
$$= x - ku\,[\varphi(x)] \qquad\qquad x \geqq x^*$$

where $\bar{y}$ is the expected value of the crop. Let the data functions have the following simple form

$$(4.3) \qquad s(z) = \begin{cases} s_1 z & z \leqq \frac{s^*}{s_1} \\ s^* & z \geqq \frac{s^*}{s_1} \end{cases}$$

$$(4.4) \qquad \begin{cases} \psi(x) = \psi_o - \psi_1\, x \\ u(z) = u_o - u_1\, z \\ c(x) = c_o + c_1\, x \end{cases}$$

If

$$(4.5) \qquad x - ku_o + ku_1\, \varphi(x) \geqq z^*$$

then (4.2) reduces to

$$s^* = x - ku_o + ku_1\, \varphi(x)$$

$$(4.6) \qquad \varphi(x) = \frac{s^* + ku_o - x}{ku_1}$$

When (4.5) does not hold we have instead an equation

$$(4.7) \qquad s_1\, (\alpha\, \psi_o - \alpha\, \psi_1\, [x - u_o + u_1\, \varphi(x) + \bar{y}] - \varphi(x) - c_o - cx)$$

$$= x - ku_o + ku_1\, \varphi(x)$$

whose solution is

$$\varphi(x) = - \varphi_1\, x + \varphi_o$$

$$(4.8) \qquad = - x\, \frac{1 + c_1\, s_1 + \alpha\, \psi_1\, s_1}{s_1 + ku_1 + \alpha\, u_1\, \psi_1\, s_1}$$

$$+ \frac{ku_o + s_1\, \alpha\, \psi_o + s_1\, \alpha\, \psi_1\, u_o + s_1\, \alpha\, \psi_1\, \bar{y} - s_1\, c_o}{s_1 + ku_1 + s_1\, u_1\, \alpha\, \psi_1}$$

It can be shown that the slope of this line segment is less in absolute value than that of either one of the adjacent lines, provided c_1 is of small order, and $k-1$ is small, as one may expect.

To summarize, the solution has the shape of a broken line with an upward kink at x^* and a downward kink at s^*, the slope of the last segment being the same as that of the first segment. (Fig. 2).

The points x^* and s^* may be determined by solving two linear equations equating the respective expressions for $\varphi(x)$. The formulae are somewhat cumbersome and have been suppressed here.

The parameters ψ_o and ψ_1 entering into this solution are in turn determined by fitting a straight line to the broken line that represents φ.

For instance we may put

Fig. 2

$$\psi_o = \beta\, \varphi_o + (1-\beta) .\ \frac{u_o}{u_1}$$

$$\psi_1 = \beta\, \varphi_1 + (1-\beta) .\ \frac{1}{ku_1}$$

where $0 < \beta < 1$ is a weight whose precise value still depends on x^* and s^*, but may be regarded as given here. For the slope of the middle section we have then an equation

$$(4.9) \qquad \varphi_1 = \frac{1 + c_1\, s_1 + \alpha\, s_1 \cdot [\beta\, \varphi_1 + (1-\beta) \frac{1}{ku_1}]}{s_1 + ku_1 + \alpha\, u_1\, s_1\, [\beta\, \varphi_1 + (1-\beta) \frac{1}{ku_1}]}$$

which is of the second degree in φ_1. It is interesting to note that with certain simplifying assumptions

$c_1 = 0$

$\alpha = 1$

$u_1 \gg s$

i.e., inelastic demand for consumption we obtain again $\varphi_1 = \frac{1}{ku_1}$.

5. Conclusions

The purpose of this paper has been to give a theoretical account of the principles by which the stock level after harvest determines spot and forward prices in a forward market. Although the stocks inherited from the past were itself determined by prices, after the random event of a harvest it is clearly stocks that are the determining and prices that are the determined variables. The essential assumptions were that there are favorable conditions for rational behavior (the stable environment of a stationary economy) and that the participants do in fact act rationally. What hypothesis follow if this model is granted?

<u>5.1. Do forward prices predict the future spot prices?</u> They understate the future spot price on the average by the amount of the backwardation. This backwardation, and hence the bias in average prediction, will be larger, the greater the stock level.

<u>5.2. Does a forward price express anything that is not indicated by the spot price?</u>

While the difference between the two prices is fixed by the stock level as long as speculative stocks are positive, the forward price is determined (almost) independently of the spot price when speculative stocks are zero. In that case the forward price mainly serves to regulate bets among speculators as to the future spot price. But since there is no criterion for the absence of speculative stocks other than the failure of spot and forward price to differ vy less than the marginal carrying cost, the forward price serves as an indicator of the fact whether speculative stocks are carried over or not.

5.3. Is speculation economically harmful?

No, under the conditions assumed here, because the expectations of all the speculators are right. Therefore interannual price fluctuations are reduced by speculation. -- It is also economically desirable that farmers should plant crops in anticipation of the probability distribution of the next period's prices.

5.4. In forward markets as described by this model, is the amount of commodity stored less than is economically desirable?

Yes, because storage is reduced to a level where speculators receive a risk premium which is higher, because of capital rationing, than the premium that is necessary to reimburse an agency operating without a capital limit, but possibly at an increasing schedule of interest expressing the social cost of tying down these funds. To put it differently, the risk preference of individuals is more conservative than the risk perference of a storage pool would be that can afford to maximize long run profit rather than having to forego opportunities at lower margins of profitability in order to avoid brankruptcy. But with all storing being done by one agency, competition and hence all the advantages of a forward market would cease to be. Thus the higher risk premium, and lower insurance against severe shortage is the price of competition.

The purpose of this model has been primarily to point out which aspects of forward trading are easy to explain in principle and which require a more complex theory -- one that would take into account the fact that grade and location are not specified by a futures contract, and that would not steer clear of the interdependence of several futures markets.