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A Theory of the Movement and Solution of Problems within an Organization*

Roy Radner

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A Theory of the Movement and Solution of Problems within an Organization

SUMMARY

Introduction

The purpose of this note is to describe a theory of the movement and solution of problems within an organization, as they are related to the formal structure of authority, the informal channels of communication, and the rates at which problems arise at different points within the organization. First a general framework for such a theory is outlined, and then two particular alternative schemes are proposed. The simpler of the two is analyzed in some detail, with the goal of determining the backlog of problems that each person has when the system is in equilibrium, the rates of flow of problems between persons, and the average time it takes for a problem to get solved by the organization. This theory was suggested by an organization model developed by R. E. Gaskell, G. Brigham, and others.

General Framework

Consider a set of individuals organized according to a given authority structure. At any given time each individual is originating problems at a certain rate; he is also solving problems, passing some to other members of the organization, and accumulating some in his backlog, each of these activities going on at certain rates. For each problem there is a corresponding expert (who is one of the members of the organization). A problem may get solved by the individual possessing the problem or by means of the individual delivering the problem to the appropriate expert; it is assumed that an individual solves all the problems he receives for which he is the expert.

More precisely, the variables to be considered are:

Originations $Y_{ei}(t)$ - the rate at which problems for which e is the expert (or briefly, of type e) are being originated by individual i at time t .

Flows $Z_{eij}(t)$ - the rate at which problems of type e are being sent by individual i to individual j , at time t .

Completions $Q_{ei}(t)$ - rate at which problems of type e are solved by i at time t .

Backlogs $X_{ei}(t)$ - the backlog, or stock, of problems of type e held by i at time t .

These variables must satisfy the following "accounting identity," which essentially states that, for problems of any type, and for any individual, the rate at which the backlog is increasing equals the difference between the rate of total originations and inflow, and the rate of total outflow and completions, of problems of that type.

$$(1) \quad \frac{dX_{ei}(t)}{dt} = [Y_{ei}(t) + \sum_j Z_{eji}(t)] - [Q_{ei}(t) + \sum_j Z_{eij}(t)] \; .$$

The originations $Y_{ei}(t)$ are functions of time that are given from outside the system. The completions $Q_{ei}(t)$ and flows $Z_{eij}(t)$ will depend upon the stocks $X_{ei}(t)$ according to some functional relationships, symbolized, say, by

$$(2) \quad \begin{aligned} Q_{ei}(t) &= f_{ei}(X(t), t), \\ Z_{eij}(t) &= g_{eij}(X(t), t), \end{aligned}$$

where $X(t)$ denotes the vector of all the variables $X_{ei}(t)$; $e = 1, \ldots, N$; $i = 1, \ldots, N$.

The next two sections deal with two different assumptions about the functional relations f_{ei} and g_{eij}. In the first case it is assumed that, for any given type of problem, the completions at i and the flows from i are all proportional to the backlog of problems of that type currently held by i. In the second case, this last assumption is modified by assuming that each individual has a definite limitation on the total rate at which he can handle problems of all types. Furthermore, in both cases it is assumed that the flow of problems is restricted in a certain way that is related to the hierarchy of authority within the organization.

These two cases only represent examples of the kinds of assumptions that would be made about the functional relationships (2). Other assumptions might be more appropriate in certain situations; these examples have been chosen as possessing both simplicity and a certain degree of general realism.

A Linear System without Feedback

In the model simulating problem flow and solution, developed by Gaskell, Brigham and others (see memo by G. Brigham and R. E. Gaskell, April 29, 1955), an individual could

1. solve a problem himself,
2. pass it on to the expert,
3. pass it to the expert's supervisor,
4. pass it either to his own supervisor or to a subordinate, depending upon whether he is not, or is, above the expert in the chain of command,
5. accumulate the problem in his backlog.

These different events happened with various probabilities, depending upon the busyness and ability of the individual, upon the urgency and difficulty of the problem, and upon the readyness with which the individual communicated with the expert and the expert's supervisor. This suggests a linear form for the function relationships (2), as follows:

$$(3) \quad \begin{cases} Q_{ei}(t) = q_{ei}X_{ei}(t) \\ Z_{eij}(t) = z_{eij}X_{ei}(t), \end{cases}$$

where the coefficients q_{ei} , z_{eij} represent the average values of the probabilities of a problem of type e , held by i , being completed, and passed to j , respectively, in the next "instant of time."

The above list of the five things that can happen to a problem implies that the present system has no feedback, i.e., that there are no loops in the flow of any given type of problem. From this it will follow that, if the originations $Y_{ei}(t)$ are constant over time, then as t gets large, the system approaches a unique steady state that can be computed fairly easily. The steady state backlogs X_{ei} are determined by the following system of linear "steady state" equations.

$$X_{ei} = \frac{Y_{ei} + \sum_j z_{eji} X_{ej}}{k_{ei}} , \quad i = 1,\ldots,N. \tag{4}$$

where Y_{ei} is the (constant) rate of originations of problems of type e originated by i, and

$$k_{ei} = q_{ei} + \sum_j z_{eij} \tag{5}$$

is the ratio of i's rate of "processing" -- that is, completion or passing to others -- to his backlog, for problems of type e.

Another quantity of interest is the mean time T_{ei} before a problem of type e is solved, given that it has just arrived (or originated) at i. In the steady state these mean times are determined by

$$T_{ei} = \frac{1 + \sum_j z_{eij} T_{ej}}{k_{ei}} , \quad i = 1,\ldots,N. \tag{6}$$

Since there is no feedback, the equations of both systems, (4) and (6), can be arranged in such an order that the solution of any one equation requires only the values of the variables determined by the previous equations.

A System with Limitations on the Rate of Handling Problems

One feature of the system described in the previous section that would be particularly unrealistic in many situations is the fact that there is no absolute upper limit on the rate at which an individual can process problems. Thus, no matter how large the individual's backlog is, he processes (solves or passes on to others) a constant proportion of that backlog per unit time. Indeed, if the backlog starts at zero, then it will build up until the problems are being processed at the same rate as they are coming in.

The present section will describe one possible way of introducing limitations on the rate at which individuals can process problems. This involves specifying the functional relationships (2) in a somewhat different way than in equations (3) of the previous section.

Let $S_i(t)$ be the total backlog of problems of all types held by i at time t; that is,

$$(7) \qquad S_i(t) = \sum_e X_{ei}(t).$$

Assume now that i processes a constant proportion of his backlog only if it remains less than some given level C_i, which will be called the

<u>capacity of i</u>; otherwise i processes only a certain proportion of C_i. More precisely, replace equation (3) of the last section by:

$$(8) \qquad Z_{eij}(t) = \begin{cases} z_{eij} X_{ei}(t) & \text{, if } S_i(t) \leq C_i \\ z_{eij} \left(\dfrac{X_{ei}(t)}{S_i(t)} \right) C_i & \text{, if } S_i(t) > C_i \end{cases}$$

$$Q_{ei}(t) = \begin{cases} q_{ei} X_{ei}(t) & \text{, if } S_i(t) = C_i \\ q_{ei} \dfrac{X_{ei}(t)}{S_i(t)} C_i & \text{, if } S_i(t) \geq C_i \end{cases}$$

Notice that when i is working at capacity, it is assumed that he distributes his effort among the different types of problems in the same proportions as they are present in the total backlog.

The effect of placing capacity limitations as in equations (8) is to introduce both feedback and non-linearity into the system. The analysis is not as simple as that for the previous model, and will not be attempted in this note. One thing however is clear; if an individual's total backlog $S_i(t)$ has increased from below C_i up to C_i, then as t increases, $S_i(t)$ will continue to increase <u>without limit</u>. In a sense, the model just presented is not complete, for no mechanism for taking care of such a situation has been incorporated into it, although such mechanisms do exist in real organizations. More study of such mechanisms (both the empirical and theoretical aspects) is needed.

It seems very likely that a system constructed along these lines may have more than one limiting (steady) state, depending upon the initial conditions; i.e., the state in which the system will settle down may depend upon which individuals reach their capacity limitations first.