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A Simple Analysis of the Leontief System

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I have two objectives. First, I shall give an easy, straightforward proof of the more useful necessary and sufficient conditions that a Leontief system possesses a unique non-negative solution. Then I will prove the efficiency theorem of activity analysis in the appropriate form for this system.

Let B represent a square matrix and x and y column vectors. The equation $Bx = y$ represents a Leontief system when the following conditions are met:

(1) $$b_{ij} \leqq 0,\ i \neq j,\ y_i \geqq 0.$$

(2) $$\sum_{i=1}^{n} b_{ij} \geqq 0.$$

The b_{ij} for $i \neq j$ are the quantities of other goods needed to produce b_{jj} units of the j th good. x_j is the activity rate in the j th industry, and y_j is the net output of the j th good. The interest in non-negative solutions arises from the fact that negative rates of activity are not meaningful. In the latter part of the discussion where the system is treated as a linear activities model, it will prove appropriate to drop requirement (2) of "dominant"

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outputs.* This will leave the question of "workability" entirely open.**

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* See Morgenstein [5] .

** See Chipman [1] .

In dealing with the question of a non-negative solution, we will need a lemma due to Arrow [1, p. 159] . Its proof is immediate.

<u>Lemma 1. If $Bx \geqq 0$ implies $x \geqq 0$, then B is non-singular, where B is any matrix.</u>

$Bx = 0$ implies $B(-x) = 0$. Then by the hypothesis, it must be true that $x = 0$, and B is non-singular.

We may now state a Lemma which leads directly to the well-known necessary and sufficient conditions for a unique non-negative solution. Let us refer to a matrix B as positive if $Bx \geqq 0$ implies $x \geqq 0$. Let B now define a Leontief system.

<u>Lemma 2. A necessary and sufficient condition for B to be positive is that every principal minor of B have at least one column sum greater than zero.</u>

Suppose the condition is met but for some y with $y_i \geqq 0$, $Bx = y$ and $x_i < 0$ for some i. By identical rearrangements of rows and columns $Bx = y$ may be written $\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, where $x_1 < 0$ and $x_2 \geqq 0$. This gives $B_1x_1 + B_2x_2 = y_1$. Since off-diagonal elements are not positive, $B_2x_2 \leqq 0$. Therefore, $B_1x_1 \geqq 0$. By the hypothesis, $p'B_1 \geq 0$,*** where $p' = (1, \ldots, 1)$. Thus,

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*** $x \geq \bar{x}$ means $x_i \geqq \bar{x}_i$ for all i and $x_i > \bar{x}_i$ for some i.

$p'B_1x_1 < 0$, since $x_1 < 0$. But this implies $B_1x_1 \leq 0$. Since we have reached a contradiction, $x_1 < 0$ is excluded and B is positive.

On the other hand, suppose that B is positive but there is a principal minor whose column sums are all equal to zero. Without loss of generality we may suppose that this minor B_1 lies in the upper left hand corner of B. Since the column sums of B are not negative, the elements in the columns below B_1 must be zero. We may write $B = \begin{bmatrix} B_1 & B_2 \\ 0 & B_4 \end{bmatrix}$. Since $p'B_1 = 0$, B_1 is singular. Thus there is a nonzero vector x_1 with $B_1x_1 = 0$. Consider Bx, where $x = (x_1 0)$. Since $Bx = 0$, B is singular, and by lemma 1, B is not positive.

We may derive other useful necessary and sufficient conditions very easily from those of lemma 2.

Theorem. The following are equivalent:

(a) *Any Leontief system defined by* B *has a unique non-negative solution.*

(b) *Each principal minor of* B *has at least one column sum positive.*

(c) B *is non-singular.**

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* This condition may be found in Chipman [1] and Wong [8].

(d) *If* B *is completely decomposed, each indecomposable minor on the diagonal has a positive column sum.***

** More or less explicitly given by Solow [7] and Woodbury [9].

(e) <u>It is possible to perform identical permutations of rows and columns of</u> B <u>so that</u> $\sum_{i=j}^{n} b_{ij} > 0$ <u>for all</u> j.*

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* See G. B. Price [6], and in the present context Wong [8].

Lemma 2 states the equivalence of (a) and (b). (a) obviously implies (c). But (c) also implies (b) as we see from the second part of the proof of lemma 2. Therefore, (c) implies (a). (b) immediately implies (d) since the minors on the diagonal are principal minors. Conversely, if there is a principal minor whose column sums all equal zero, it may occupy the place of B_1 in the proof of lemma 2. If B_1 is then completely decomposed,** the indecomposable minor occupying the upper

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** A matrix A is decomposable if it can be put into the form $\begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}$ through identical permutations of rows and columns. It is completely decomposed if it is put by this means into the form $\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ & \ddots & \vdots \\ 0 & & A_{nn} \end{bmatrix}$ and $A_{11},\ldots,A_{nn}$ are indecomposable.

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left-hand corner will have its column sums equal to zero, since permuting columns and rows does not change the column sums. Thus (d) also implies (b).

There remains to show that (b) is equivalent to (e). This may be proved by considering what will prevent the completion of such an arrangement of the matrix when one begins with the <u>n</u>th column and proceeds backward. Suppose the $\underline{k}$th column is in order. The $\underline{k-1}$th column is available, without disturbing the arrangements already made, if one of the first $k-1$ columns as they now appear has the sum of its first $k-1$ elements greater than zero. But these elements form a principal minor, and thus (b) implies that the condition is met. This shows that (b) implies (e).

On the other hand, if (b) does not hold, that is, if there is a principal minor with all its columns summing to zero, (e) cannot hold either. For every such minor contains one column whose elements lie on and above the diagonal. Moreover, all the permutations which can be performed leave one of its columns with all its elements on and above the diagonal of B. But this column will violate (e). Thus, (e) implies (b). This completes the proof of the theorem.

The form of condition (2) is a result of the normalization of the Leontief matrix. This also explains the special role of the vector (1,..., 1) in the proof. An equivalent condition without normalization is

(2') $\quad p'B \geqq 0$ <u>for some</u> $p > 0$.

p may be interpreted as a price vector. Then for actual economies such a p is provided by the prevailing prices unless some industry realizes a loss on its current transactions. However, for a technology which is not in use, the existence of p allowing (2') to be satisfied will be an open question. If (2') can be satisfied, we may rephrase lemma 2 as

<u>Lemma 2'. A necessary and sufficient condition for B to be positive is that $p^k B_k \geq 0$ for B_k an arbitrary principal minor of B.</u>

p^k is the subvector of prices for goods appearing in B_k. The proof of lemma 2' is entirely analogous to that of lemma 2 except that p need not equal (1,..., 1). The theorem can be similarly rephrased and the proof is as before. Of course, the proofs need not be repeated anyway, since (2') is obviously equivalent to (2), and then the lemmas are equivalent. The normalization is accomplished through dividing the *i*th row of B by p_i when (2') is satisfied for p. Leontief, in addition, divides each column by its resulting diagonal element. However, he is assuming that this element is positive. This is true if (2') is met unless a trivial column of zeros is allowed in B.

If one considers the new form of condition (e) in the theorem, where we now have $\sum_{i=j}^{n} p_i b_{ij} > 0$ for all j, it becomes apparent that p can be altered so that $p'B > 0$. Suppose $\sum_{i=1}^{n} p_i b_{ij} > 0$ for $j > k$. Then $p_1, \ldots, p_k$ may be increased slightly in the same proportion, so that $\sum_{i=1}^{n} p_i b_{ij} > 0$ for $j = k$, without disturbing this inequality for $j > k$. Moreover, (2') will still hold, so that the process can be continued until $k = 1$.

Now we may note that condition (2') and lemma 2' may equally well be applied to B', the transpose of B. In particular, they imply that if we have an $x > 0$ such that $x'B' > 0$, then B' is positive. In other words, $B'p = \pi$ has a unique non-negative solution for every $\pi \geq 0$. This justifies

Theorem 2. *If B is a square matrix whose off-diagonal elements are not positive, the following are equivalent:*

(a) *There is a* $p > 0$ *such that* $p'B > 0$.

(b) B *is positive.*

(c) *There is an* $x > 0$ *such that* $Bx > 0$.

(d) B' *is positive.*

It is a corollary of Theorem 2 that if it is possible to produce something of all goods together, output can be produced in any proportions.

We may use the results now attained to prove the efficiency theorem of activity analysis for the Leontief model. In the complete statement of the open Leontief model it is recognized that each industry requires labor input. Let the columns of B represent the output of a unit of labor in the respective industries and the complementary inputs of intermediate products which are needed. Then the open Leontief model may be written, where B is a square matrix subject to condition (1),

$$Bx = y\ , \quad x \geqq 0. \qquad (3)$$

$$\Sigma x_j \leqq C, \quad C > 0.$$

C is the available labor supply. An output vector y is said to be efficient if $y + \Delta y$ where $\Delta y \geq 0$ is inconsistent with (3). It is apparent that the labor supply must be fully used. For if $\Sigma x_j = C - u$, $u > 0$, $y + \frac{u}{C-u} y$ would be possible. Also it is no lack of generality to take the price of labor to be unity. Then the efficiency theorem of activity analysis* can be written, for this model,

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* The general theorem may be found in Koopmans [3, p. 82]. An elementary proof of the theorem for the Graham model of world trade is in [4].

Theorem 3. An output y of the Leontief model (3) is efficient if and only if the labor supply is fully used and there exists a price vector $p > 0$ such that $p'B \leqq 1$ and $\sum_{i=1}^{n} p_i b_{ij} = 1$ when $x_j > 0$.

Suppose there is such a p. Then $\Delta y \geq 0$ implies $p'\Delta y = p'B\Delta x > 0$. But $p'B \leqq 1$. Therefore, $p'\Delta y \leqq \Sigma\Delta x_j$. But $\Sigma\Delta x_j \leqq 0$, since the labor supply is fully used. Thus $\Delta y \geq 0$ is impossible.

On the other hand, let y be efficient. Let $B_k x^k = y^k$ be the reduced system of industries actually in use. Then the analogue of (2') applied to B_k' and x^k is satisfied. If the analogue of the condition in lemma 2 is also satisfied B_k' is positive and $p^{k'} B_k = 1$ can be solved for $p^k > 0$. Then since every industry uses some labor, by setting components of p for industries not in use low enough, it is possible to realize $\Sigma \; p_i b_{ij} < 1$ for $x_j = 0$. For this p, the conditions of theorem 3 are met.

Suppose, however, that the conditions of the analogue of lemma 2' is not satisfied, Then for some principal minor B_h of B_k, $B_h x^h = 0$. If x^h is set equal to zero, no output is lost. However, labor is released which can be used to increase the outputs of other industries. Thus y would not be efficient.

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