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Demand Theory Without a Utility Index

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DEMAND THEORY WITHOUT A UTILITY INDEX[1]

The modern revolution in the theory of demand has been to replace utility as a measurable quantity with a utility index which is arbitrary up to a strictly

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1. This paper was first written independently, but it has since benefited from the very similar approach in Professor Hicks' new book [3]. The chief virtues of the presentation here are perhaps the mathematical form, and brevity.

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monotonic transformation. It is my purpose here to describe an approach to the theory of demand which dispenses with the utility index entirely. This use of Occam's razor does not, however, complicate the derivation of the major propositions of demand theory, but rather, in at least the case of the Slutsky equation, leads to an important simplification. My approach can be related to three developments by other authors, the revealed preference analysis of Samuelson [7], the use of the indirect utility function[2] by Houthakker [5], and the use of Convex

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2. The indirect utility function represents the level of utility as a function of prices and income, rather than as a function of quantities of goods consumed.

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set methods by Arrow and Debreu [1].

The Slutsky equation asserts that, for compensated price changes, the rate at which the consumer varies the consumption of the i-th good per unit change of the j-th price equals the rate at which the consumer varies the consumption of the j-th good per unit change of the i-th price. By compensated price changes is meant that income changes are made at the same time of the proper magnitude to keep the consumer on the same indifference locus [7, pp. 103-104]. In other words, if $f_i(p,m)$ is the demand function for the i-th good, where p is the vector of prices and M is money income,

$$(1)\ \frac{\partial f_i}{\partial p_j} + k_j\frac{\partial f_i}{\partial M} = \frac{\partial f_j}{\partial p_i} + k_i\frac{\partial f_j}{\partial M},$$

where k_i is the appropriate compensation per unit change of p_i.

Let $\frac{\delta f_i}{\delta p_j} = \frac{\partial f_i}{\partial p_j} + k_j \frac{\partial f_i}{\partial M}$. Then $\frac{\delta f_i}{\delta p_j}$ is the effect of a compensated price change. It is written by Hicks $\underline{x}_{ji}$ [4, p. 309] and by Samuelson $k_{ij} = (\frac{\partial x_j}{\partial p_i})$ U = constant [7, p. 103]. Hicks calls $\underline{x}_{ji}$ the substitution term.

Define $M_x(p)$ as the minimum income[3] which allows the consumer to attain a

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3. $M_x(p)$ is called a concave function of p. A function g(p) is said to be concave if $g(tp + (1-t)\,p') \geq tg(p) + (1-t)\,g(p')$ when $0 \leq t \leq 1$. That is, every chord of the surface defined by the function lies on or below the surface. Write $M_x(p) = M(p)$. That M(p) is concave is clear, for suppose $p'' = tp + (1-t)p'$ and $p''\cdot x = \min p''\cdot x'$ for x' in C_x. Then $tp\cdot x + (1-t)p'\cdot x = p''\cdot x = M(p'')$. But $p\cdot x \geq M(p)$ and $p'\cdot x \geq M(p')$. Thus $tM(p) + (1-t)\,M(p') \leq M(p'')$. The partial derivatives of the second order exist for a concave function (technically speaking) "almost everywhere" [2, pp. 87, 102, 142].

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commodity combination at least as good as x when the price vector is p. If C_x is the set of commodity combinations at least as good as x, then

(2) $M_x(p) = \min p\cdot x'$ for x' in C_x.[4]

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4. $p\cdot x$ equals $\sum_i p_i x_i$, the inner product of p and x, which are vectors in an n-dimensional vector space, n being the number of goods.

I shall show that $\frac{\delta x_i}{\delta p_j}$ and $\frac{\delta x_j}{\delta p_i}$ are cross partial derivatives of the function $M_x(p)$, differing only in the order of derivation, and therefore equal whenever they exist.

Let S be the set of combinations x of goods which the consumer is capable of consuming. S is often taken to be all x such that $x_i \geq 0$ for each i and, if the i-th good is indivisible, x_i is an integer. I assume that the set S is completely ordered by a preference ordering, which I will write with inequality signs enclosed in circles. The preference ordering of S is assumed to be closed. By closure, I mean

(3) If $\bar{x}^1, \ldots, \bar{x}^i, \ldots \to \bar{x}$ and $x^1, \ldots, x^i, \ldots \to x$, and $\bar{x}^i \; (>) \; x^i$, then $\bar{x} \; (\geq) \; x$.

The arrow symbol should be read "converges to."

As a consequence of (3), the set C_x of combinations of goods indifferent with or preferred to x will be closed if S is closed. This means that the minimum in (2) may be expected to exist for any non-negative p. The consumer is presumed to choose a combination of goods which is at least as good as any combination available to him when he is restrained by the budget condition

(4) $p \cdot x \leqq M$.

In order to carry through our demonstrations we must provide for two things. First, we must ensure that the consumer has to spend all his income to achieve a best combination of goods subject to the budget restraint. Then the chosen combination x will be a least cost combination in C_x. Second, we must require that the nearby combination to which the consumer is led to move when a compensated price change is made is actually indifferent to the initial combination. In other words, compensation in the Slutsky sense must be possible for the price change in question. Notice, however, that neither of these requirements represents a limitation on the analysis, for they are needed to make the Slutsky equation meaningful in any case.

The first requirement is met if the consumer is not satiated in some good which is divisible, at points of S which lie within the budget limitation. That is to say, for a combination x in S and within the budget limitation, there is a j-th good and a quantity x^*_j of it such that any combination x' lies in S and is preferred to x if $x'_i = x_i$ for $i \neq j$ and $x^*_j > x'_j > x_j$. Indeed, it is enough for this condition to hold at the chosen point.[5]

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5. The strategic role of having a divisible good was made clear by Professor Hicks [3, p. 387]. He suggested that savings might play this role.

This assumption implies that any chosen point x involves the expenditure of all income, that is, $p \cdot x = M$. For if $p \cdot x < M$, there would be, by the assumption, an $x' > x$ and $p \cdot x' < M$, where $x'_i = x_i$ for $i \neq j$ and $x'_j > x_j$. Thus x could not be a chosen point. Moreover, we must have $p \cdot x = M_x(p)$. Necessarily, $p \cdot x \geqq M_x(p)$, since x does lie in C_x. But if $p \cdot x > M_x(p)$, by the definition of $M_x(p)$ it would not be necessary to spend as much as $M = p \cdot x$ to reach a point in C_x, in contradiction to the proposition just proved.

The second requirement can be met if we assume the existence of a point $\tilde{x}$ in S cheaper than the chosen point x and such that the points also lie in S, which are intermediate between $\tilde{x}$ and any point x' of S in a small neighborhood of x. Let x' lie in the small neighborhood of x, and suppose x' is chosen at p' and $M_x(p')$. Then $x' \geqq x$. Now consider a sequence of points x^i which lie on the line segment from $\tilde{x}$ to x' and which converge to x' in the limit as $i \rightarrow \infty$. Since x^i is in S but does not lie in C_x, $x > x^i$. Therefore, by the closure of the preference relation, $x \geqq x'$. Thus $x = x'$. A basis for the proof has now been prepared.[6]

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6. The basis for the proof is illustrated in Figure 1.

Let $f_x(p) = f\,(p, M_x(p))$, that is, $f_x(p)$ is the combination chosen at p and a value of M equal to $M_x(p)$. Consider $\dfrac{\partial M_x(p)}{\partial p_i}$, or equivalently $\dfrac{\partial p \cdot f_x(p)}{\partial p_i}$.

Assume that the requirements of non-satiation in a divisible good and of the existen e of the cheaper combination in S are met at x. Write $f(p)$ for $f_x(p)$. Then, by the argument just made, points $x' = f(p')$ are indifferent with x if they lie close enough to x. If the minimum cost combination in C_x is unique in the neighborhood of p, then $f(p)$ is well defined and continuous. It is also differentiable. Performing the derivation,

$$\frac{\partial p \cdot f(p)}{\partial p_i} = f_i + \sum_k p_k \frac{\partial f_k(p)}{\partial p_i}$$

But the derivatives $\partial f_k(p) / \partial p_i$ are defined by values of $x' = f(p')$ which lie arbitrarily near $x = f(p)$, and which are consequently indifferent with x. Thus

$$\frac{\partial p \cdot f(p)}{\partial p_i} = x_i + p \cdot \frac{\delta x}{\delta p_i},$$

where the symbol δ has its earlier meaning.

We may now show that $p \cdot \frac{\delta x}{\delta p_i} = 0$. Choose Δp_i small enough so that $p \cdot \Delta x$ has the sign of the first differential $p \cdot \frac{\delta x}{\delta p_i} \Delta p_i$. Choose the sign of Δp_i so that the first differential is negative. Then $p \cdot (x + \Delta x) < p \cdot x$ in contradiction to the requirement that $p \cdot x = \min p \cdot x'$ for $x' \in C_x$. Therefore, $p \cdot \frac{\delta x}{\delta p_i} = 0$ and $\frac{\partial p \cdot f(p)}{\partial p_i} = x_i$.

Write $M(p)$ for $M_x(p)$. If $\frac{\partial M(p)}{\partial p_i}$ exists, it is equal to x_i. Therefore, if $\frac{\partial M(p)}{\partial p_i \partial p_j}$ exists, it must be $\frac{\delta x_i}{\delta p_j}$. Similarly, $\frac{\delta x_j}{\delta p_i} = \frac{\partial M(p)}{\partial p_j \partial p_i}$. Therefore, at these points, $\frac{\partial M(p)}{\partial p_i \partial p_j} = \frac{\partial M(p)}{\partial p_j \partial p_i}$ and $\frac{\delta x_i}{\delta p_j} = \frac{\delta x_j}{\delta p_i}$, while elsewhere the Slutsky equation is not defined.

As we have seen, $M = M_x(p)$ for movements along the indifference locus of x. But $\frac{\partial M_x(p)}{\partial p_i} = x_i$. This shows that k_i in (1) is equal to x_i, or the compensation needed per unit change of p_i is x_i. Then the Slutsky equation appears in its traditional form

$$(5) \quad \frac{\partial f_i}{\partial p_j} + x_j \frac{\partial f_i}{\partial M} = \frac{\partial f_j}{\partial p_i} + x_i \frac{\partial f_j}{\partial M},$$

where f_i is written for $f_i(p,M)$.

The other relations satisfied by the substitution terms $\frac{\delta x_j}{\delta p_i}$ can be proved through the manipulation of the inequality $p \cdot x \leqq p \cdot x'$ for x' in C_x. This proof corresponds so closely to that used by Samuelson in his revealed preference approach that it does not require repeating. Therefore, I will merely indicate the implication from properties of $M_x(p)$.

As we have noted, $M_x(p)$ is a concave function. It is a fundamental property of concave functions that the second differential is non-positive [2, p. 88] where it exists. Thus, $\sum_i \sum_j \frac{\partial^2 M_x(p)}{\partial p_i \partial p_j} dp_i \, dp_j = 0$. Therefore, by our results,

$$(6) \quad \sum_i \sum_j \frac{\delta x}{\delta p_i} dp_i dp_j = 0.$$

In other words, (6) is a negative semi-definite quadratic form. In particular, we may set $dp_k > 0$ and $dp_i = 0$ for $i \neq k$. This shows that $\frac{\delta x_k}{\delta p_k} \leq 0$.

The strict inequality cannot hold in (6), for, as appeared earlier, $p \cdot \frac{\delta x}{\delta p_i} = 0$ for every i. However, it may happen that the strict inequality holds whenever dp is not proportionate to p. This means that the indifference surface does not have a corner at x in any direction. Then $\frac{\delta x_k}{\delta p_k} < 0$ for each k.

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