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Market Equilibrium*

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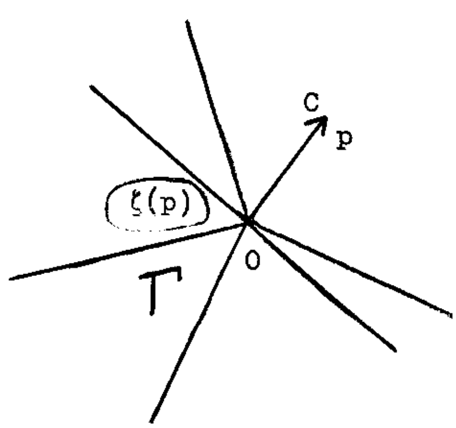
Market Equilibrium

Let there be ℓ commodities in the economy. When the price system is $p \in R^{\ell}$, the excess of demand over supply is $z \in R^{\ell}$. Generally, p does not uniquely determine z, it determines a set $\zeta(p)$ of which z can be any element. The problem of market equilibrium has the natural formulation: is there a p compatible with $z = 0$, i.e., is there a p such that $0 \in \zeta(p)$?

In usual contexts, two price systems derived from each other by multiplication by a positive number are equivalent, and all prices do not vanish simultaneously. Thus the domain of p is a cone C with vertex 0, but with 0 excluded.

Since no agent spends more than he receives, the value of total demand does not exceed the value of total supply, hence $p \cdot z \leqq 0$ for every z in $\zeta(p)$. This can also be written $p \cdot \zeta(p) \leqq 0$, i.e., the set $\zeta(p)$ is below (with possibly points in) the hyperplane through 0 orthogonal to p.



It is intuitive that, under proper regularity assumptions, there is in C a p, different from 0, for which $\zeta(p)$ intersects Γ, the polar of C (whose definition is recalled in the appendix). The theorem gives a precise statement of this result. Its interest lies in that, for a wide class of economies, $\Gamma \cap \zeta(p) \neq \emptyset$ implies $0 \in \zeta(p)$. (See ref. 4.)

It is convenient to normalize p by restricting it to the unit sphere $S = \left\{ p \in R^{\ell} \;\middle|\; |p| = 1 \right\}$.

<u>Theorem</u>. <u>Let</u> C <u>be a closed, convex cone with vertex</u> 0 <u>in</u> R^{ℓ}, <u>different from</u> $\left\{ 0 \right\}$ <u>and</u> R^{ℓ}; <u>let</u> Γ <u>be its polar. If the multi-valued function</u> ζ <u>from</u> $C \cap S$ <u>to</u> R^{ℓ} <u>is upper semi-continuous and bounded; if for every</u> p <u>in</u> $C \cap S$ <u>the set</u> $\zeta(p)$ <u>is non-empty, convex and satisfies</u> $p \cdot \zeta(p) \leqq 0$; <u>then there is a</u> p <u>in</u> $C \cap S$ <u>such that</u> $\Gamma \cap \zeta(p) \neq \emptyset$.

<u>Proof</u>: Throughout, Z denotes a compact, convex subset of R^{ℓ} in which ζ takes its value; such a subset exists since ζ is bounded.

1.) The theorem is first proved in the case where Γ has an interior point z^{o}. It is more convenient here to normalize p by restricting it to the set $P = \left\{ p \in C \mid p.z^{o} = -1 \right\}$, which is easily seen to be non-empty, compact, convex.

Given z in Z, let $\pi(z)$ be the set of maximizers of $p.z$ in P. The set $\pi(z)$ is clearly non-empty, convex, and the multi-valued function π from Z to P is easily seen to be upper semi-continuous.

Consider then the set $P \times Z$ and the multi-valued transformation φ of this set into itself defined by $\varphi(p,z) = \pi(z) \times \zeta(p)$. Since φ satisfies the conditions of the Kakutani[7] fixed point theorem, there is a pair (p^*, z^*) which belongs to $\varphi(p^*, z^*)$, i.e., $p^* \in \pi(z^*)$ and $z^* \in \zeta(p^*)$.

The first relation implies that $p.z^* \leqq p^* . z^*$ for every p in P; the second implies $p^* . z^* \leqq 0$; therefore $p.z^* \leqq 0$ for every p in P; hence $z^* \in \Gamma$. This, with $z^* \in \zeta(p^*)$, proves that $\Gamma \cap \zeta(p^*) \neq \emptyset$.

2.) In the general case, Γ is considered as the limit of an infinite sequence of cones Γ^m with vertex 0, having <u>non-empty</u> interiors. These cones are constrained to be closed, convex, different from R^ℓ, and to contain Γ.

Let C^m be the polar of Γ^m; it is contained in C, which is the polar of Γ. Apply then the result of 1) to the pair C^m, Γ^m: there is a pair (p^m, z^m) such that $p^m \in C^m \cap S$, $z^m \in \Gamma^m$, and $z^m \in \zeta(p^m)$.

Since $S \times Z$ is compact, one can extract from the sequence (p^m, z^m) a sub-sequence converging to (p^*, z^*). Clearly, $p^* \in C \cap S$, $z^* \in \Gamma$, and $z^* \in \zeta(p^*)$ (the last relation by upper semi-continuity of ζ).

<u>Remarks</u>: The central idea of the proof is taken from Arrow-Debreu[1] (see also ref. 3). It consists, given an excess z of demand over supply, in choosing p so as to maximize $p.z$. It has a simple economic interpretation: in order to reduce the excess demand, the weight of the price system is brought to bear on those commodities for which the excess demand is the largest.

Since the convexity assumptions in Kakutani's theorem can be weakened (see Eilenberg-Montgomery[5] and Begle[2]), the assumption that $\zeta(p)$ is convex is inessential.

Gale[6] and Debreu[4] have, independently, stated the theorem in the particular case where C is the set of points in R^{ℓ} all of whose coordinates are non-negative. The underlying economic assumption is that commodities can be freely disposed of. As McKenzie[8] emphasizes, it is very desirable to relax that assumption. The purpose of this note was to give a general market equilibrium theorem with a simple and economically meaningful proof.

I thank Shizuo Kakutani for the valuable discussion of this question I had with him.

<u>Appendix</u>:

Let C be a cone with vertex 0 in R^{ℓ}; its polar Γ is the set $\left\{ z \in R^{\ell} \mid p \cdot z \leqq 0 \text{ for every } p \text{ in } C \right\}$. This set is a closed, convex cone with vertex 0. It can also be described as the intersection of the closed half-spaces below the hyperplanes through 0 with normals p in C.

It is immediate that «C^1 contains C^2» implies «Γ^2 contains Γ^1». One can prove that if C is closed, and convex, then C is the polar of Γ, i.e., the relation "is the polar of" becomes symmetric.

Let ψ be a multi-valued function from a subset E of R^n to R^n; it is said to be upper semi-continuous if «$x^q \rightarrow x^o$, $y^q \in \psi(x^q)$, $y^q \rightarrow y^o$» implies «$y^o \in \psi(x^o)$».

References

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