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On the Allocation of Personnel in Teams

Martin J. Beckmann

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On the Allocation of Personnel in Teams*

1. This paper points out some relationships between team models which take account of the different capacities of decision makers and the personnel assignment problem. A team will be considered abstractly as a set of persons i, $i = 1, \ldots m$ employed in the performance of a set of jobs k, $k = 1, \ldots n$. The payoff to the team is assumed to be a known function of the amounts of work x_{ik} performed by persons i in job k, $f(x_{11}, \ldots x_{mn})$. These may be technological or organizational constraints limiting the amount of total employment of any person and the total labor input in any job, for instance

$$(1) \qquad \sum_k x_{ik} \leqq 1 \qquad i = 1, \ldots m$$

$$(2) \qquad \sum_k x_{ik} \leqq 1 \qquad k = 1, \ldots n;$$

alternatively the total input of labor may be limited

$$(3) \qquad \sum_{ik} x_{ik} \leqq c.$$

Always we have $x_{ik} \geqq 0$.

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The object is to determine those allocations of personnel time to jobs which will maximize the payoff to the team. If the number of persons equals that of jobs and an optimal allocation exists which is one-to-one we shall speak of a personnel assignment. One question is under what conditions the personnel allocation problem reduces to an assignment problem.

If the payoff function is smooth and the constraints are of the form (3) then the problem solution is given by the principle of marginal productivity

$$(4) \qquad \frac{\partial f}{\partial x_{ik}} = \lambda \qquad \lambda \geqq 0 \quad \text{and} = \text{if} \ \sum_{ik} x_{ik} < c$$

The necessary condition (4) is sufficient if f is a concave function of the x_{ik}. One interpretation of concavity is that in each job there are diminishing returns to the employment of any person; and that there are diminishing returns to the substitution of performance in one job for performance in any other job.

Under the constraints (1) and (2) and for a concave payoff function f the optimal allocation is similarly determined by the conditions

$$(5) \qquad f_{ik} \leqq \lambda_i + \mu_k \qquad \text{where} \qquad f_{ik} = \frac{\partial f}{\partial x_{ik}}$$

$$= \qquad\qquad \text{if} \quad x_{ik} > 0$$

(6) $\lambda_i \geqq 0$

$= \qquad$ if $\sum_k x_{ik} < 1$

(7) $\mu_k \geqq 0$

$= \qquad$ if $\sum_i x_{ik} < 1$

(1) $\sum_k x_{ik} \leqq 1$

(2) $\bar{\sum_i} x_{ik} \leqq 1$

Here λ_i and μ_k are marginal productivities attributable to persons and job respectively. Uniformity of marginal productivity has been upset by the limitations (1) and (2). Here as in the previous case the optimal allocation is not an assignment in general.

In order to get more specific results further assumptions have to be made. We shall consider the cases of <u>additivity</u> and of <u>limitationality</u> of performance in jobs.

2. The case of <u>additivity</u> may be approached as follows: Suppose that for each job there is a well defined output or score equal to the sum of outputs of the various persons employed in that job. In the case of an executive job this score may be defined as the negative of the expected loss from non-optimal decisions. Consider now an efficient set of job outputs. By this one meaning that no job score can be increased by a reallocation of personnel without decreasing some other job score.

It is well known that any efficient set of such scores can be obtained by maximizing a suitable (positively) weighted sum of the scores. It is therefore natural to consider payoff functions which are weighted sums of job scores. Once the weights have been fixed a score of each person in each job can be defined such that the payoff equals the sum of scores achieved by individuals in jobs.

(8) $$f = \sum_{ik} a_{ik} x_{ik}$$

Maximization of (8) subject to (3) is a trivial problem: all labor is put into that combination ik for which a_{ik} is maximal. The most interesting case arises, when both constraint sets (1) and (2) must be met and when $m = n$, the number of persons equals that of jobs. This case yields the well-known personnel assignment problem. The assumption that the performance of a person in a job contributes to the team payoff an amount independent of the scores achieved in other jobs and independent of the remaining personnel implies extreme independence among persons and jobs. Nevertheless, an optimal allocation of any person's time to jobs requires consideration of all the personnel available.

The conditions (5), (1), and (2) are necessary and sufficient with $\lambda_i \geqq 0$ $\mu_k \geqq 0$ $i, k = 1, \ldots n$. Moreover, as von Neumann and others have shown the optimal allocation may be chosen as an assignment

$$x_{ik} = \alpha_{ik_p}$$

where k_p represents a permutation of the integers 1, ... n. If we relabel persons in such a way that the optimal assignment is that of the i^{th} person to the i^{th} job we have

(9) $$x_{ik} = \alpha_{ik}$$

(10) $$a_{ik} \leqq \lambda_i + \mu_k$$

(11) $$a_{ii} = \lambda_i + \mu_i$$

It was shown by Koopmans and Beckmann that the λ_i may be interpreted as wages and the μ_k as rents which would sustain an optimal assignment.

From (10) and (11) certain properties of the wages follow which are of economic interest. The corresponding properties of the rents are obvious and shall not be discussed here.

2.1 <u>If a person achieves in each job a higher score than another person his wage is greater than that of the second person</u>. In symbols: $a_{1k} > a_{2k}$ for all k implies $\lambda_1 > \lambda_2$

Proof:

$\lambda_1 + \mu_2 \geqq a_{12}$ according to (10)

$a_{12} > a_{22}$ by hypothesis

$a_{22} = \lambda_2 + \mu_2$ according to (11)

$\therefore \quad \lambda_1 > \lambda_2$

Corollary: <u>If person 1 achieves a higher score in each job than any other person, then his wage is the largest.</u> If a person n has a lower score in each job than any other person, his wage is smallest. If the scores of persons in all jobs are simply ordered, so are the wages.

2.2 <u>The person whose wage is highest is the person with the maximal score for the job which he holds in the optimal assignment</u>.

Proof:

$$a_{11} = \lambda_1 + \mu_1 \qquad \text{by } (11)$$

$$a_{i1} \leqq \lambda_i + \mu_1 \qquad \text{by } (10)$$

Subtracting

$$\lambda_1 - \lambda_i \leqq a_{11} - a_{i1}$$

If now $\lambda_1 \geqq \lambda_i$ then $a_{11} \geqq a_{i1}$ for all i.

<u>Corollary</u>: There exists an ordering of jobs such that the optimal assignment is obtained by choosing the person of highest score for the first job, among the rest the person of highest score for the second job, etc.

2.3 <u>The difference between the highest and lowest wage is not greater than the largest difference of scores for any job</u>.

Proof:

$$\lambda_k - \lambda_j = \lambda_k + \mu_k - (\lambda_j + \mu_k) \leqq a_{kk} - a_{jk} \quad \text{by } (10), (11)$$

(12)
$$\underset{k,j}{\text{Max}} (\lambda_k - \lambda_j) \leqq \underset{k}{\text{Max}} (\underset{r}{\text{Max}}\ a_{rk} - \underset{r}{\text{Min}}\ a_{jk})$$

The example $(a_{ik}) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ $\lambda_1 = 2$ $\mu_1 = 0$

$\lambda_2 = 1$ $\mu_2 = 0$

shows that the "=" sign can occur in (12).

Denote the right hand side of (12) by $d(a_{ij})$.

Consider now

$$a^*_{ik} = \sum_j b_{ij}\, a_{jk}$$

where $b_{ij} \geqq 0 \quad \sum_j b_{ij} = \sum_i b_{ij} = 1.$

This means that the scores a^*_{ik} are weighted interpersonal averages of the scores a_{ik}.

Corollary: $d(a^*_{ik}) \leqq d(a_{ik})$, the wages of a more homogonous population lie between narrower bounds than those of the original set.

2.4 The payoff of an optimal assignment does not increase when the scores are replaced by weighted averages of the previous scores, provided the matrix of weights is doubly stochastic and nonsingular.

Proof: Let $((\beta_{jk}))$ be the inverse matrix of $((b_{ij}))$. It is easily shown that

$$\sum_i \beta_{ij} = \sum_j \beta_{ij} = 1$$

Now $$\sum_{ik} a_{ik} x_{ik} = \sum_{ijk} \beta_{ij} a^*_{jk} x_{ik} = \sum_{jk} a^*_{jk} y_{jk}$$

where $$y_{jk} = \sum_i \beta_{jk} x_{ik} \quad \text{or} \quad x_{ik} = \sum_j \beta_{ij} y_{jk}$$

Now $$\sum_k y_{jk} = \sum_{ij} \beta_{ij} x_{ik} = \sum_i x_{ik} = 1$$

$$\sum_k y_{jk} = \sum_{ik} \beta_{ij} x_{ik} = \sum_i \beta_{ij} = 1$$

The original problem may therefore be written

$$\underset{y_{ik}}{\text{Max}} \sum_{jk} a_{jk} y_{jk} \quad \text{subject to} \quad \sum_j y_{jk} = \sum_k y_{jk} = 1, \sum_j b_{ij} y_{jk} \geqq 0$$

$j, k = 1, \ldots n$. Since $b_{ij} \geqq 0$ the latter constraints are always satisfied if $y_{jk} \geqq$ and are therefore not more restrictive. It follows that the original problem has a maximum not below that of the problem with weighted scores. The more homogenous the population, the smaller the total score even though the average level of the scores may remain unchanged.

2.4 Raiffa has suggested to define the scores as the inner products of factor loadings of jobs and capacity vectors of persons in the manner of factor analysis

$$c_{ik} = \sum_j b_{ij} c_{jk} \text{ , say.}$$

If the number of factors is 2 or 3, the problem may be solved geometrically (by trial and error) by forming such pairs between two sets of vectors which will produce the largest sum of inner products.

3. On an assembly line where each job is <u>limitational</u>, it is the worst performance in any job that determines the payoff to the team. We are thus led to consider the payoff function

(13) $$f = \underset{k}{\text{Min}} \sum_{i} a_{ik} x_{ik}$$

and the problem of maximizing (13) subject to the constraints (1) and (2), the case of constraint (3) being trivial.

It is well known that this problem can be reduced to an ordinary linear programming problem by introducing a lower bound y, $0 \leqq y \leqq \sum_{i} a_{ik} x_{ik}$ $k = 1, \ldots n$ as an additional variable. The necessary and sufficient conditions for x_{ik} to be a solution are then

(14) $$\gamma_k a_{ik} \leqq \lambda_i + \mu_k$$

$$= \qquad \text{if} \quad x_{ik} > 0$$

(15) $$\lambda_i, \mu_k, \gamma_k \geqq 0$$

(16) $$\sum_{k} \gamma_k = 1$$

(17) $$\gamma_k = 0 \quad \text{if} \quad \sum_{i} a_{ik} x_{ik} > y$$

(18) $$\sum_i a_{ik} x_{ik} \geq y$$

(1), (2) $$\sum_k x_{ik} \leq 1 \qquad \sum_i x_{ik} \leq 1$$

It is not true that the solution will always be an assignment, e.g., not when $a_{ik} = 0$ all k for some i. (Although the problem is formally equivalent to a linear assignment problem with $\gamma_k a_{ik}$ as the scores, the argument that there must be integral solutions is vitiated by the interdependence of the γ_k and x_{ik} according to (17)). The γ_k may be interpreted as the share of the team's payoff that is attributed to job k. If the limitation (2) on labor inputs into each job is lifted, then condition (14) is replaced by the following condition

(19) $$\gamma_k a_{ik} \leq \lambda_i$$

$$= \text{ if } x_{ik} > 0$$

Suppose that for score $k = \bar{k}$ $\sum_i a_{i\bar{k}} x_{i\bar{k}} > y \geq 0$. Then $\gamma_{\bar{k}} = 0$ according to (17). From (14) it follows that then $x_{ik} = 0$ for all i in contradiction to the hypothesis. Therefore <u>the performance in all jobs are equalized</u> i.e., the marginal productivity principle applies.