ADDENDUM TO
A MULTIVARIATE STOCHASTIC UNIT ROOT MODEL
WITH AN APPLICATION TO DERIVATIVE PRICING

by

Offer Lieberman and Peter C. B. Phillips

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COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

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http://cowles.yale.edu/
A Multivariate Stochastic Unit Root Model
with an Application to Derivative Pricing - An
Addendum to Lieberman and Phillips,

Offer Lieberman† and Peter C. B. Phillips‡

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In Theorem 7 of Lieberman and Phillips (Journal of Econometrics 196
(2017), 99-110), it is stated that \( \hat{\mu}_n \Rightarrow \mu_B(a, \Sigma_u) \), where

\[
B(a, \Sigma_u) = H_a^*(1) - a' \int_0^1 H_a^*(r) dB_u(r) - \frac{1}{2} a' \Sigma_u a \int_0^1 H_a^*(r) dr,
\]

and

\[
H_a^*(r) = e^{a'B_u(r)} \int_0^r e^{-a'B_u(p)} dp.
\]

In fact, stochastic differentiation of \( H_a^*(r) \) reveals that

\[
dH_a^*(r) = dr + a'H_a^*(r) dB_u(r) + \frac{1}{2} a' \Sigma_u a \int_0^1 H_a^*(r) dr,
\]

from which it follows that

\[
H_a^*(1) = \int_0^1 dH_a^*(r) = 1 + a' \int_0^1 H_a^*(r) dB_u(r) + \frac{1}{2} a' \Sigma_u a \int_0^1 H_a^*(r) dr,
\]

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†Department of Economics, Bar-Ilan University, Ramat Gan 52900, Israel. E-mail: offer.lieberman@gmail.com
‡Yale University, University of Auckland, Southampton University, and Singapore Management University.
so that $B(a, \Sigma_u) = 1$. In other words, $\hat{\mu}_n$ is consistent and as a result Remarks 9-10, which provide an alternative consistent estimator of $\mu$, are superfluous. In view of the above

\[
\left(\hat{\sigma}^\mu_{\varepsilon,n}\right)^2 := \frac{1}{n} \sum_{t=2}^{n} \left(Y_t - \hat{\mu}_n - e^{\hat{\alpha} u_t / \sqrt{n}} Y_{t-1}\right)^2 \Rightarrow \sigma^2_{\varepsilon},
\]

and consistent estimators of $\sigma^2_{\varepsilon}$ and $\Sigma_{u\varepsilon}$ can be obtained in a straightforward manner by replacing $\frac{1}{n} \sum_{t=2}^{n} Y_t$ and $\frac{1}{n} \sum_{t=2}^{n} Y_t^2$, respectively, and plugging

\[
\hat{\Sigma}^*_{\mu\varepsilon,n} := \left(1 - \left(\frac{\frac{1}{n} \sum_{t=2}^{n} Y_t}{\frac{1}{n} \sum_{t=2}^{n} Y_t^2}\right)\right)^{-1} \Rightarrow \Sigma_{u\varepsilon}
\]

into

\[
\left(\hat{\sigma}^{\mu*}_{\varepsilon,n}\right)^2 := \left(\hat{\sigma}^\mu_{\varepsilon,n}\right)^2 + \left(\frac{\frac{1}{n} \sum_{t=2}^{n} Y_t}{\frac{1}{n} \sum_{t=2}^{n} Y_t^2}\right) \hat{\Sigma}^*_{\mu\varepsilon,n} \hat{\Sigma}^*_{u\varepsilon,n} \hat{\Sigma}^*_{\mu\varepsilon,n} \Rightarrow \sigma^2_{\varepsilon}.
\]