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Note on Optimality Criteria for Decision-Making Under Ignorance*

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This paper is written to bridge a gap in the proof of several lemmas, and so in the proof of the final result, of Hurwicz's paper "Optimality Criteria for Decision Making Under Ignorance" C.C.D.P. Statistics 370 (henceforth to be referred to as H). This paper is meant to be read in conjunction with H; in fact it uses the notations, definitions and concepts of H.

The gap in H occurs on p. 4, line 14, where an innocent little "easily seen" unfortunately becomes "too little easily seen"; in fact Properties A_1, A_2, and A_3 __do not__ follow from property A. But, as we show here, these can be circumvented in proving the subsequent lemmas; thus the final result still carries over.

First let us see that the gap indeed gaps and in doing so, see what alternatives can be used for Property A_1. Property A_1 states: if $p' \subseteq p''$ and $d_1, d_2 \in D'$ then $d_1 \tilde{O} d_2 [p']$ if and only if $d_1 \tilde{O} d_2 [p'']$. It is certainly clear that if $d_1 \tilde{O} d_2 [p']$ then $d_1 \tilde{O} d_2 [p'']$ as a simple verification of the definitions involved will show. It is also true that if $d_1 \tilde{O} d_2 [p'']$ by

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virtue of the fact that $d_1 \in \hat{D}''$, $d_2 \in \hat{D}''$ then $d_1 \tilde{O} d_2 \left[p'\right]$; this is a trivial consequence of Property A. However, it is false, on the basis of only Property A that if $d_1 \tilde{O} d_2 \left[p''\right]$, $d_1 \notin \hat{D}''$, $d_2 \notin \hat{D}''$ then $d_1 \tilde{O} d_2 \left[p'\right]$.

We summarise the above into Property A'. If $p' \subseteq p''$ and $d_1, d_2 \in D'$ then

a) $d_1 \tilde{O} d_2 [p']$ implies $d_1 \tilde{O} d_2 [p'']$

b) $d_1 \tilde{O} d_2 [p'']$ together with $d_1 \in \hat{D}''$ implies $d_1 \tilde{O} d_2 [p']$.

Property A_1 was used in H to derive Properties A_2 and A_3; however we do not need any analogues of these here, since Property A' will be enough.

The first place that Property A_1 was used in H was in proving lemma 2. For the sake of connectedness we reproduce the first part of Hurwicz's proof and complete it using Property A'.

Lemma 2. Suppose P satisfies Assumption I and let ψ_p be an optimality criterion for which Properties A and B hold. Let $p = (\phi, D, E)$, $E = (e_1, e_2, \ldots, e_N)$ and suppose for some $d', d'' \in D$, $u_{d'}$ is a permutation of $u_{d''}$; then $d' \tilde{O} d''$ [p] .

Proof. Let

$$u_{d'} = a^o = (a_1, a_2, \ldots, a_N)$$

$$u_{d''} = a^{N-1} = (a_{i_1}, a_{i_2}, \ldots, a_{i_N})$$

where $(i_1, i_2, \ldots, i_N)$ is a permutation of $(1, 2, \ldots, N)$. The problem P may be represented in matrix form as

d'	a^o
d''	a^{N-1}
$D^{(1)}$	B

where $D^{(1)} = D - (d', d'')$ may be empty.

Now define, for $j = 1, 2, \ldots, N-1$, a^j in such a way that a^j is obtained from a^{j-1} by the interchange in a^{j-1} of a_j and a_1. (Clearly a^{N-1} so defined is identical with that defined above.

Consider now $\bar{p} = (\emptyset, \bar{D}, E)$ which in matrix form may be written as

$$\begin{array}{c|c}
d' & a^0 \\
d_1 & a^1 \\
d_2 & a^2 \\
\cdots & \cdots \\
d_{N-2} & a^{N-2} \\
d'' & a^{N-1} \\
D^{(1)} & B
\end{array}$$

By lemma 1, $d' \tilde{O} d_1 [\bar{p}]$, $d_j \tilde{O} d_{j+1} [\bar{p}]$ for $j = 2, 3, \ldots, N-2$, $d_{N-2} \tilde{O} d'' [\bar{p}]$. So, by the transitivity of the equivalence relation, $d' \tilde{O} d'' [\bar{p}]$. Suppose now that $d' \tilde{O} d'' [p]$ is false. Then, say, $d' \in \hat{D}$, $d'' \notin \hat{D}$. Since $\bar{p} \geq p$, and since $d' \tilde{O} d'' [\bar{p}]$, by Property A', the above can happen only if $d' \notin \hat{\bar{D}}$, $d'' \notin \hat{\bar{D}}$. Hence $d_j \notin \hat{\bar{D}}$ for all j since $d_j \tilde{O} d' [\bar{p}]$. Since $\hat{\bar{D}}$ is not null, it can thus contain only elements of $B^{(1)}$; so $\hat{\bar{D}} \wedge D$ is not empty, hence $\hat{\bar{D}} \wedge D = \hat{D} \subset B^{(1)}$; since $d' \in \hat{D}$, $d' \notin B^{(1)}$ we have the necessary contradiction and lemma 2 is established. The only other place Hurwicz used Property A_1 or its derivates, A_2 and A_3 was in proving lemma 3. We now prove lemma 3 on the assumption of Property A (and so A') alone.

Lemma 3. Suppose P satisfies assumptions I and II and let ψ_p be an optimality criterion for which Properties A, B, C hold. Let $p = (\emptyset, D, E)$, $E = (e_1, e_2, \ldots, e_N)$, $p \in P$. Then for $d', d'' \in D$,

$$U_{d} = U_{d''}$$

implies

$$d' \tilde{O} d'' [p].$$

Proof. p may be represented in matrix form as

	e_1	e_2	$\cdots$					e_N
d'	v_1	v_2	$\cdots$	v_S	w_1	w_2	$\cdots$	w_T
d''	b_1	b_2	$\cdots$					b_N
$D^{(1)}$					C			

where the v's are the S distinct elements of $U_{d'}$, $U_{d''}$, and the columns are so ordered that $v_1 < v_2 < \cdots\ v_S$ while $w_1 \leq w_2 \leq \cdots \leq w_T$; each w equals one of the v's, and of course $S + T = N$.

Consider now the problem $\bar{p} = (\emptyset, \bar{D}, E)$ which may be written as

	e_1	e_2	$\cdots$					e_N
d'	v_1	v_2	$\cdots$	v_S	w_1	w_2	$\cdots$	w_T
d'''	v_1	v_2	$\cdots$	v_S	z_1	z_2	$\cdots$	z_T

where $(v_1, v_2, \ldots, v_S, z_1, z_2, \ldots, z_T)$ is a permutation of $(b_1, b_2, \ldots, b_N)$ and the z's are arranged in ascending order $z_1 \leq z_2 \leq \cdots \leq z_T$ (again, by the assumptions each z equals same v).

Now $d'\ \ddot{O}\ d'''\ [\bar{p}]$ (this is established as Hurwicz does on p. 11). Consider now the problem $\ddot{p}$, given in matrix form as

	e_1	e_2	$\cdots$				e_N
d'	v_1	v_2	$\cdots$	v_S	w_1	$\cdots$	w_T
d'''	v_1	v_2	$\cdots$	v_S	z_1	$\cdots$	z_T
d''	b_1	b_2	$\cdots$				b_N

Since $\bar{\bar{p}} \geq \bar{p}$, and $d'\ \ddot{O}\ d'''\ [\bar{p}]$, by Property A', $d'\ \ddot{O}\ d'''\ [\bar{\bar{p}}]$. Since $U_{d''}$ is a permutation of $U_{d'''}$, $d''\ \tilde{O}\ d'''\ [\bar{\bar{p}}]$ by lemma 2. So $d'\ \tilde{O}\ d''\ \tilde{O}\ d'''\ [\bar{\bar{p}}]$, and so they are all optimal in $\bar{D}$, hence in $\hat{\bar{\bar{D}}}$. If $\bar{\bar{\bar{p}}}$ is the problem having only strategies d', d", $\bar{\bar{\bar{p}}} \leq \bar{\bar{p}}$, and since d', d" $\in \hat{\bar{\bar{D}}}$, this forces d' $\widetilde{O}$ d" $[\hat{\bar{\bar{\bar{p}}}}]$ by part (b) of Property A'. Since $p \gtrsim \bar{\bar{p}}$, and d' $\widetilde{O}$ d" $[\bar{\bar{p}}]$ part (a) of Property A' yields d' $\widetilde{O}$ d" [p], and so the lemma.