**Note on Optimality Criteria for Decision-Making**

**Under Ignorance**

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This paper is written to bridge a gap in the proof of several lemmas, and so in the proof of the final result, of Hurwicz's paper "Optimality Criteria for Decision Making Under Ignorance" C.C.D.P. Statistics 370 (henceforth to be referred to as H). This paper is meant to be read in conjunction with H; in fact it uses the notations, definitions and concepts of H.

The gap in H occurs on p. 4, line 14, where an innocent little "easily seen" unfortunately becomes "too little easily seen"; in fact Properties \( A_1, \) \( A_2, \) and \( A_3 \) do not follow from property A. But, as we show here, these can be circumvented in proving the subsequent lemmas; thus the final result still carries over.

First let us see that the gap indeed gaps and in doing so, see what alternatives can be used for Property \( A_1. \) Property \( A_1 \) states: if \( p'_1 \leq p'' \) and \( d_1, d_2 \in D' \) then \( d_1 \overset{\sim}{\bowtie} d_2 [p'] \) if and only if \( d_1 \overset{\sim}{\bowtie} d_2 [p''] \). It is certainly clear that if \( d_1 \overset{\sim}{\bowtie} d_2 [p'] \) then \( d_1 \overset{\sim}{\bowtie} d_2 [p''] \) as a simple verification of the definitions involved will show. It is also true that if \( d_1 \overset{\sim}{\bowtie} d_2 [p''] \) by

virtue of the fact that $d_1 \in \hat{D}^n$, $d_2 \in \hat{D}^n$ then $d_1 \sim d_2 [p']$; this is a
trivial consequence of Property A. However, it is false, on the basis of
only Property A that if $d_1 \sim d_2 [p'']$, $d_1 \notin \hat{D}^n$, $d_2 \notin \hat{D}^n$ then $d_1 \sim d_2 [p']$.

We summarize the above into Property $A'$. If $p' \in p''$ and $d_1$, $d_2 \in \hat{D}'$ then

a) $d_1 \sim d_2 [p']$ implies $d_1 \sim d_2 [p'']$

b) $d_1 \sim d_2 [p'']$ together with $d_1 \notin \hat{D}^n$ implies $d_1 \sim d_2 [p']$.

Property $A_1$ was used in $H$ to derive Properties $A_2$ and $A_3$; however
we do not need any analogues of these here, since Property $A'$ will be enough.

The first place that Property $A_1$ was used in $H$ was in proving lemma 2.

For the sake of connectedness we reproduce the first part of Hurwicz's proof
and complete it using Property $A'$.

Lemma 2. Suppose $P$ satisfies Assumption I and let $\Psi_p$ be an optimality
criterion for which Properties A and B hold. Let $p = (p', D, E)$,
$E = (a_1, a_2, ..., a_N)$ and suppose for some $d'$, $d'' \in \hat{D}$, $u_d'$ is a permutation
of $u_d''$; then $d' \sim d'' [p]$.

Proof. Let

$$u_d' = a^0 = (a_1, a_2, ..., a_N)$$
$$u_d'' = a^{N-1} = (a_{i_1}, a_{i_2}, ..., a_{i_N})$$

where $(i_1, i_2, ..., i_N)$ is a permutation of $(1, 2, ..., N)$. The problem $P$
may be represented in matrix form as

$$
\begin{array}{c|c}
\hline
d' & a^0 \\
\hline
d'' & a^{N-1} \\
D(1) & B \\
\hline
\end{array}
$$

where $D(1) = D - (d', d'')$ may be empty.
Now define, for $j = 1, 2, ..., N-1$, $a_j$ in such a way that $a_j$ is obtained from $a_{j-1}$ by the interchange in $a_{j-1}$ of $a_j$ and $a_1$. (Clearly $a_{N-1}$ so defined is identical with that defined above.

Consider now $\bar{p} = (\bar{p}, \bar{b}, \bar{e})$ which in matrix form may be written as

\[
\begin{array}{c|c}
  d' & a^0 \\
  d_1 & a^1 \\
  d_2 & a^2 \\
  \vdots & \vdots \\
  d_{N-2} & a^{N-2} \\
  d'' & a^{N-1} \\
 \end{array}
\]

By Lemma 1, $d' \sim d_1 \ [\bar{p}], \ d_2 \sim d_{j-1} \ [\bar{p}]$ for $j = 2, 3, ..., N-2$, $d_{N-2} \sim d'' \ [\bar{p}]$. So, by the transitivity of the equivalence relation, $d' \sim d'' \ [\bar{p}]$. Suppose now that $d' \sim d'' \ [p]$ is false. Then, say, $d' \notin \hat{B}$, $d'' \notin \hat{B}$. Since $\bar{p} \supset p$, and since $d' \sim d'' \ [\bar{p}]$, by Property A', the above can happen only if $d' \notin \hat{B}$, $d'' \notin \hat{B}$. Hence $d_j \notin \hat{B}$ for all $j$ since $d_j \sim d' \ [\bar{p}]$. Since $\hat{B}$ is not null, it can thus contain only elements of $B(1)$; so $\hat{B} \cap D$ is not empty, hence $\hat{B} \cap D = \hat{D} \subset B(1)$; since $d' \notin \hat{D}$, $d'' \notin B(1)$ we have the necessary contradiction and Lemma 2 is established. The only other place Harrods used Property A or its derivatives, A2 and A3 was proving Lemma 3. We now prove Lemma 3 on the assumption of Property A (and so A') alone.

Lemma 3. Suppose $P$ satisfies assumptions I and II and let $\psi_p$ be an optimality criterion for which Properties $A$, $B$, $C$ hold. Let $p = (\bar{p}, \bar{d}, \bar{e})$, $E = (e_1, e_2, ..., e_n)$, $p \in P$. Then for $d', d'' \in D$,

$$U_d = U_{d''}$$

implies

$$d' \sim d'' \ [p].$$
Proof. \( p \) may be represented in matrix form as

\[
\begin{array}{|c|cccc|}
\hline
& 1 & 2 & \ldots & N \\
\hline
d' & v_1 & v_2 & \ldots & v_S \quad v_1 & v_2 & \ldots & v_T \\
d'' & b_1 & b_2 & \ldots & b_N \\
d(l) & c \\
\hline
\end{array}
\]

where the \( v \)'s are the \( S \) distinct elements of \( U_{d' \circ d''} \) and the columns are
so ordered that \( v_1 < v_2 < \ldots < v_S \) while \( v_1 \not\equiv v_2 \not\equiv \ldots \not\equiv v_T \); each \( w \) equals one of the \( v \)'s, and of course \( S + T = N \).

Consider now the problem \( \tilde{p} - (\tilde{d}, \tilde{u}, E) \) which may be written as

\[
\begin{array}{|c|cccc|}
\hline
& 1 & 2 & \ldots & N \\
\hline
d' & v_1 & v_2 & \ldots & v_S \quad v_1 & v_2 & \ldots & v_T \\
d'' & v_1 & v_2 & \ldots & v_S \quad z_1 & z_2 & \ldots & z_T \\
\hline
\end{array}
\]

where \((v_1, v_2, \ldots, v_S, z_1, z_2, \ldots, z_T)\) is a permutation of \((b_1, b_2, \ldots, b_N)\)
and the \( z \)'s are arranged in ascending order \( z_1 \leq z_2 \leq \ldots \leq z_T \) (again, by the assumptions each \( s \) equals same \( v \)).

Now \( d' \circ d'' (\tilde{p}) \) (this is established as Hurwitz does on p. 11).

Consider now the problem \( \bar{p} \), given in matrix form as

\[
\begin{array}{|c|cccc|}
\hline
& 1 & 2 & \ldots & N \\
\hline
d' & v_1 & v_2 & \ldots & v_S \quad v_1 & v_2 & \ldots & v_T \\
d'' & v_1 & v_2 & \ldots & v_S \quad z_1 & z_2 & \ldots & z_T \\
d & b_1 & b_2 & \ldots & b_N \\
\hline
\end{array}
\]

Since \( \bar{p} \geq \tilde{p} \), and \( d' \circ d'' (\bar{p}) \), by Property \( A' \), \( d' \circ d'' (\tilde{p}) \). Since
\( U_{d''} \) is a permutation of \( U_{d''} \), \( d'' \circ d'' (\tilde{p}) \) by Lemma 2. So \( d' \circ d'' \circ d'' (\tilde{p}) \).
and so they are all optimal in $\hat{\mathcal{D}}$, hence in $\hat{\mathcal{P}}$. If $\hat{\mathcal{P}}$ is the problem having only strategies $d', d''$, $\hat{\mathcal{P}} \subseteq \mathcal{P}$, and since $d', d'' \in \hat{\mathcal{D}}$, this forces $d' \not\sim d'' [\hat{\mathcal{P}}]$ by part (b) of Property A'. Since $p \not\in \hat{\mathcal{P}}$, and $d' \not\sim d'' [\hat{\mathcal{P}}]$ part (a) of Property A' yields $d' \not\sim d'' [\mathcal{P}]$, and so the lemma.