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Optimality Criteria for Decision Making Under Ignorance

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1. The main result of this paper is a theorem stating that if certain properties are assumed for the optimality criterion [in particular the property of "independence of irrelevant alternatives," as well as certain invariance properties], then an action must be evaluated merely in terms of the best and worst it may accomplish. Criteria of this type were considered in the writer's CCDP 355 where the use of a linear function was suggested.

The properties required in the theorem are closely related to the first ten axioms in Chernoff's CCDP 326A. The chief difference consists in the fact that the requirement of independence of irrelevant alternatives as formulated by the writer is essentially equivalent to that of Arrow's in "Social Choice and Individual Values," i.e., stronger than Chernoff's.

It will be seen that the requirements imposed are so strong that they at times conflict with the admissibility axiom. That is, it can so happen that two decisions are optimal though one of them is never inferior and

1. The main result below corresponds to 1.12 in John Milnor's "Games Against Nature" (RAND RM-679) which came to the author's attention during the proof-reading of the present paper. Detailed comparison of the two formulations has not as yet been made.

sometimes superior to the other one. It is possible to alter the axiom system so as to remove the conflict with admissibility while making the "best-worst" criterion modified by admissibility as the only one permissible.[2] This point will be treated in a separate paper.

The writer feels strongly indebted to Chernoff and Arrow who have developed the type of approach, both with regard to formulation and method of proof, found useful in the present paper.

2. The decision maker's problem is to select an "optimal" subset $\hat{D}$ from his strategic domain D; he is assumed to be totally ignorant[3] as to which state of nature within the given set E prevails.[4] The payoff, in the nature of utility, is given by the function

$$(2.1) \qquad u = \phi(d,e) \qquad d \in D,\ e \in E$$

where <u>u</u> is a (finite) real number. (If we wish to permit "mixed" strategies, we think of some of the elements of D or E as probability distributions on the respective "pure" strategy domains A and B. In that case it is convenient to assume that utility is "measurable," i.e., satisfies the von Neumann-Morgenstern or Marschak postulates.)

3. By a <u>decision problem</u> <u>p</u> is meant the ordered triple (ϕ, D, E), i.e., we define

$$(3.1) \qquad p = (\phi, D, E).$$

A <u>class</u> of problems <u>p</u> is denoted by P. It is assumed that P is nonempty.

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2. It was the admissibility-modified "best-worst" criterion that was considered in CCDP 355 .

3. No element of E is "more plausible" than any other.

4. The sets D and E are nonempty. (The interesting case arises when each of these sets has more than one element.)

4. A criterion of optimality Ψ_p for a given class P of problems is a function assigning to each problem $(\emptyset, D, E) = p \in P$ a nonempty subset $0 < \hat{D} \subseteq D$. $\hat{D}$ is called the solution of p and the elements of $\hat{D}$ are said to be optimal. We have

$$(4.1) \qquad \hat{D} = \Psi_p(p), \quad 0 < \hat{D} \subseteq D, \; p = (\emptyset, D, E), \; p \in P.$$

Assumption: Every $p \notin P$ has a solution. (This assumption is implicit in the remainder of the paper.)

Two elements d', d", D are said to be optimally equivalent in p if, for a given p, either both d' and d" are in $\hat{D}$ or both d' and d" are in $D - \hat{D}$. Clearly the relation "optimally equivalent in p" is reflexive, symmetric and transitive, hence it is a proper equivalence relation. When the relation holds we write [5]

$$(4.2) \qquad d' \tilde{o} \; d''[p].$$

5. Independence of irrelevant alternatives. This property, formulated in somewhat different ways, appears in the axiomatic system of Chernoff's CCDP 326A (as well as his remarks on the "regret principle" in CCDP 326) and in Arrow's "Social Choice and Individual Values" (Condition 3). The recent work by Marschak and Radner (CCDP 2018) is concerned, inter alia, with examples of failure of the independence property.

In what follows we shall adopt a formulation which is equivalent to Arrow's and is somewhat stronger than Chernoff's. It is found convenient to follow Chernoff's technique and start by defining the concept of strategic inclusion for two decision problems. Given two problems $p' = (\emptyset', D', E')$ and

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5. The notation in (4.2) could be made more explicit by including a reference to the optimality criterion used, e.g. writing $d' \tilde{o} \; d''[p, \Psi_p]$.

$p'' = (\emptyset'', D'', E'')$, we say that p' is <u>strategically</u> <u>included</u> in p'' if

(5.1) $$\begin{cases} D' \subseteq D'' \\ E' = E'' = E \\ \emptyset'(d,e) = \emptyset''(d,e) \text{ for all } d \in D'' \text{ and } e \in E. \end{cases}$$

When (5.1) holds, we write

(5.2) $$p' \subseteq p''.$$

PROPERTY A. (Independence of Irrelevant Alternatives.)

<u>If</u>

(5.3) $$p' \subseteq p''$$

<u>then</u>

(5.4) $$\hat{D}'' \cap D' \neq 0$$

<u>implies</u>

(5.5) $$\hat{D}'' \cap D' = \hat{D}'.$$

It is easily seen that Property A implies Properties A_1, A_2, A_3 below.

PROPERTY A_1. <u>If</u>

(5.6) $$p' \subseteq p'' \text{ and } d_1, d_2 \in D',$$

<u>then</u>

(5.7) $$d_1 \tilde{o} d_2[p'] \text{ if and only if } d_1 \tilde{o} d_2[p''].$$

PROPERTY A_2. <u>Let</u> $p_i = (\emptyset_i, D_i, E)$, $i = 1, 2$ <u>where</u>

(5.8) $$\emptyset_1(d,e) = \emptyset_2(d,e) \text{ for all } d \in D_1 \cap D_2 \text{ and } e \in E$$

<u>and let</u> d', d'' <u>be elements of</u> $D_1 \cap D_2$; <u>then</u>

(5.9) $$d' \tilde{o} d''[p_1] \text{ if and only if } d' \tilde{o} d''[p_2].$$

<u>Proof.</u> Suppose $d' \tilde{o} d'' [p_1]$. Consider $p_3 = (\emptyset_3, D_3, E)$ where D_3 $D_1 \cup D_2$ and

$$\emptyset_3(d,e) = \emptyset_i(d,e) \text{ for } d \in D_i, \qquad i = 1, 2.$$

Clearly $p_1 \subseteq p_3$; hence, in virtue of Property A_1, $d' \tilde{o} d'' [p_3]$. But, since $p_2 \subseteq p_3$, $d' \tilde{o} d'' [p_2]$ also follows from Property A_1. The converse is proved in the same manner.

PROPERTY A_3. Let $p_i = (\phi_i, D_i, E)$, $i = 1, 2, 3$, where

(5.10) $$\phi_{i_1}(d,e) = \phi_{i_2}(d,e) \text{ for all } d \in D_{i_1} \cap D_{i_2},\ i_1, i_2 = 1, 2, 3;$$

then

(5.11) $$d' \tilde{o} d'' [p_1],\ d'' \tilde{o} d''' [p_2],\ d', d''' \in D_3$$

implies

(5.12) $$d' \tilde{o} d''' [p_3].$$

Proof. Consider $p = (\phi, D, E)$ where $D = D_1 \cup D_2 \cup D_3$ and

$$\phi(d,e) = \phi_i(d,e) \text{ for all } d \in D_i, \quad i = 1, 2, 3.$$

Since

$$p_i \subseteq p \qquad \text{for } i = 1, 2, 3,$$

it follows from (5.11) that $d' \tilde{o} d'' [p]$, $d'' \tilde{o} d''' [p]$; hence $d' \tilde{o} d''' [p]$ and, in virtue of Property A_1, (5.12) follows.

6. Invariance under transformations of strategic domains. This invariance property is, except for a minor qualification equivalent of Chernoff's Consequence 1 (obtained from his Axioms 9 and 10).

We again follow Chernoff and start by defining a certain equivalence concept. (Chernoff's "isomorphism" in his Definition 14.)

p' is said to be equivalent to p'', in symbols

(6.1) $$p' \sim p'',$$

if there exists a one to one transformation f from D' onto D'' and a one to one transformation g from E' onto E'' with the property that, for each $d' \in D'$ and $e' \in E'$, we have

(6.2) $\phi'(d',e') = \phi''(d'',e'')$

where

(6.3) $d'' = f(d')$

$e'' = g(e')$.

PROPERTY B. (Invariance under Transformations of Domains.)

<u>If</u>

(6.4) $p' \sim p''$

<u>then</u>

(6.5) $\hat{D}'' = f(\hat{D}')$

<u>where</u> $f(\hat{D}')$ <u>denotes the image of</u> $\hat{D}'$ <u>in</u> D'' <u>under the transformation</u> f.[6]

7. <u>Lemma 1</u>. Let ψ_p be an optimality criterion for which Properties A and B hold. Let $p = (\phi, D, E)$ and suppose that for some $d', d'' \in D$ and some $e', e'' \in E$ the following relations hold:

(7.1) $\phi(d',e') = \phi(d'',e'')$

$\phi(d'',e') = \phi(d',e'')$

while

(7.2) $\phi(d',e^*) = \phi(d'',e^*)$ for all $e^* \quad E = E - (e',e'')$;

then

(7.3) $d' \tilde{o}\, d''\ [p]$.

<u>Proof.</u> Let $\bar{\phi}(d',e') = \alpha$, $\bar{\phi}(d',e'') = \beta$. Without loss of generality[7] <u>p</u> may be

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6. For terminology concerning the transformations, cf., Halmos, <u>Measure Theory</u>, p. 161.

7. It is important to note that a problem (and hence its solution) is unchanged when rows and/or columns of the payoff matrix (together with their "headings"!) are interchanged. Thus

	e_1	e_2
d_1	a_{11}	a_{12}
d_2	a_{21}	a_{22}

	e_2	e_1
d_2	a_{22}	a_{21}
d_1	a_{12}	a_{11}

are merely two ways of representing $p = (\phi, D, E)$ where $D = (d_1, d_2)$, $E = (e_1, e_2)$, and $\phi(d_i, e_j) = a_{ij}$.

represented in its matrix form as

(7.4)

	e'	e''	E_1
d'	α	β	a
d''	β	α	a
D_1	b	c	A_{11}

where $E_1 = E - (e',e'')$, $D_1 = D - (d',d'')$ and both may conceivably be empty; a, b, c, and A_{11} are sub-matrices of the payoff matrix;[8] α and β are scalars.

Now p is equivalent to itself in two ways: first, obviously, by the identity transformation, and second by the one-to-one transformations

(7.5)
$$\begin{aligned} d' &= f(d'') \\ d'' &= f(d') \\ D_1 &= f(D_1) \\ e' &= g(e'') \\ e'' &= g(e') \\ E_1 &= g(E_1) \ . \end{aligned}$$

Hence, by Property B, if (say) d' is optimal then so is d" (as the image of d' according to f). Similarly d" would not be optimal if d' were not. Hence (7.3) follows.

8. Consider a problem $p = (\phi, D, E)$ where $E = (e_1, e_2, \ldots, e_N)$. Denote by u_d the N-dimensional vector

$$u_d = [\phi(d,e_1), \phi(d,e_2), \ldots, \phi(d,e_N)],$$

so that u_d is the row of the payoff matrix corresponding to d. An N-dimensional vector differing from u_d at most with regard to the order of components is called a permutation of u_d.

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8. We find it convenient to use the matrix language even though E might be a continuum and $\phi(d,e)$, for a given d, a continuous curve (when the "natural" ordering on E is preserved).

<u>Assumption I.</u> Let $p \in P$. Then there exists a $p^* \in P$ such that $p \subseteq p^*$, the payoff matrix of p^* contains all the permutations of u_d for all $d \in D$, and $p^{**} \in P$ if $p^{**} \subseteq p^*$.

9. <u>Lemma 2.</u> Suppose P satisfies Assumption I and let ψ_P be an optimality criterion for which Properties A and B hold. Let $p = (\emptyset, D, E)$, $E = (e_1, e_2, \ldots, e_N)$, and suppose for some $d', d'' \in D$ $u_{d'}$ is a permutation of $u_{d''}$; then $d' \tilde{o}\, d''$ [p].

<u>Proof</u>. Let

$$u_{d'} = a^0 = (a_1, a_2, \ldots, a_N)$$

$$u_{d''} = a^{N-1} = (a_{i_1}, a_{i_2}, \ldots, a_{i_N})$$

where $(i_1, i_2, \ldots, i_N)$ is a permutation of $(1, 2, \ldots, N)$. The problem p may be represented in matrix form as

d'	a^0
d''	a^{N-1}
$D^{(1)}$	B

where $D^{(1)} = D - (d', d'')$ may be empty.

Now define, for $j = 1, 2, \ldots, N-1$, a^j in such a way that a^j is obtained from a^{j-1} by the interchange in a^{j-1} of a_j and a_{i_j}. (Clearly, a^{N-1} so defined is identical with that defined above.)

Consider now $\bar{p} = (\bar{\emptyset}, \bar{D}, E)$ which in matrix form may be written as

d'	a^0
d_1	a^1
d_2	a^2
. . . . .	. . .
d_{N-2}	a^{N-2}
d''	a^{N-1}
$D^{(1)}$	B

By Property A_1 the Lemma is proved if it is shown that $d' \tilde{o} d'' [\bar{p}]$, since $\bar{p} \geqq p$. But, by Lemma 1, $d' \tilde{o} d_1 [\bar{p}]$, $d_j \tilde{o} d_{j+1} [\bar{p}]$, for $j = 2, 3, \ldots, N-2$, $d_{N-2} \tilde{o} d'' [\bar{p}]$. Therefore, by transitivity of the equivalence relation, $d' \tilde{o} d'' [\bar{p}]$ which completes the proof. (Problems $\bar{p}$ and others implicitly used in the above proof are elements of P by Assumption I.)

10. <u>Invariance under deletion of repetitious columns</u>. This property is postulated by Chernoff in Axiom 8.

We shall say that the problem p" is <u>derived</u> from p' (through deletion of repetitious columns), and write

(10.1) $$p' \Longrightarrow p'',$$

if

(10.2) $$D' = D''$$
$$E' \supset E''$$

and

(10.3) $$\phi'(d,e) = \phi''(d,e) \qquad \text{for all } e \in E''$$

while for each $e^* \in (E' - E'')$ there is an $e'' \in E''$ such that

(10.4) $$\phi'(d,e^*) = \phi''(d,e'') \qquad \text{for all } d \in D.$$

PROPERTY C. (Invariance under Deletion of Repetitious Columns.)

<u>If</u>

(10.5) $$p' \Longrightarrow p''$$

<u>then</u>

(10.6) $$\hat{D}' = \hat{D}''.$$

11. <u>Assumption II.</u> If $p \in P$ and $p \Longrightarrow p^*$ then $p^* \in P$.

11. bis. We shall denote by U_d the (unordered) set of distinct elements of u_d.

[U_d is the image of <u>d</u> in the utility space by $\phi(d,e)$.]

12. Lemma 3. Suppose P satisfies Assumptions I and II and let ψ_P be an optimality criterion for which Properties A, B, and C hold. Let $p = (\emptyset, D, E)$, $E = (e_1, e_2, \ldots, e_N)$, $p \in P$. Then, for $d', d'' \in D$,

(12.1) $$U_{d'} = U_{d''}$$

implies

(12.2) $$d' \tilde{o}\, d'' \; [p].$$

Proof. p may be represented in matrix form as

$$\begin{array}{c||ccccccc}
 & e_1 & e_2 & & \cdots & & & e_N \\ \hline\hline
d' & v_1 & v_2 \cdots & v_S & w_1 & w_2 & \cdots & w_T \\
d'' & b_1 & b_2 \cdots & & \cdots & & \cdots & b_N \\
D^{(1)} & & & & C & & &
\end{array}$$

where the v's are the S distinct elements of $U_{d'}$, $U_{d''}$, and the columns are so ordered that $v_1 < v_2 < \cdots < v_S$ while $w_1 \leq w_2 \leq \cdots \leq w_T$; each w equals one of the v's, and, of course, $S + T = N$.

Consider now the problem $\bar{p} = (\bar{\emptyset}, \bar{D}, E)$ which may be written as

$$\begin{array}{c||ccccccc}
 & e_1 & e_2 & & \cdots & & & e_N \\ \hline\hline
d' & v_1 & v_2 \cdots & v_S & w_1 & w_2 & \cdots & w_T \\
d''' & v_1 & v_2 \cdots & v_S & z_1 & z_2 & \cdots & z_T
\end{array}$$

where $(v_1, v_2, \ldots, v_S, z_1, z_2, \ldots, z_T)$ is a permutation of $(b_1, b_2, \ldots, b_N)$ and the z's are arranged in ascending order $z_1 \leq z_2 \leq \cdots \leq z_T$. (Again, by (12.1), each z equals some v.)

By Properties A_2 and A_3 (with Lemma 2) it will suffice to show that $d' \tilde{o}\, d''' \; [\bar{p}]$.

Now suppose that, for some $\underline{t}$, $w_t = z_t$. Clearly, by Property C, the deletion of the column containing these two entries would not affect the optimality properties of d' and d'' (since it is a repetition of one of the first $\underline{s}$ columns). Hence we may assume that all such columns have been deleted, i.e., $w_t \neq z_t$ for all $\underline{t}$.

For some fixed t_o, let $w_{t_o} = u_{k'}$ while $z_{t_o} = u_{k''}$, $k' \neq k''$, say $k' < k''$. Denote by $p^* = (\phi^*, D^*, E)$ the problem represented by

$$\begin{array}{c||cccccccccccccccc}
 & e_1 & e_2 & & & & \cdots & & & & & & & & & & e_N \\
\hline\hline
d' & v_1 & v_2 & \cdots & v_{k'} & \cdots & v_{k''} & \cdots & v_S & w_1 & w_2 & \cdots & w_{t_o} & \cdots & w_T \\
d_1 & v_1 & v_2 & \cdots & v_{k''} & \cdots & v_{k'} & \cdots & v_S & z_1 & z_2 & \cdots & z_{t_o} & \cdots & z_T
\end{array}$$

where the second row in p^* is obtained by interchanging $v_{k'}$ and $v_{k''}$ in the second row of $\bar{p}$ while the first row is the same as in $\bar{p}$. By Lemma 2 it will suffice to prove the optimal equivalence of d' and d_1 in p^*. But in p^* the column containing w_{t_o} above z_{t_o} repeats the column with $v_{k'}$ above $v_{k''}$. Hence, by Property C, the former may be deleted. In this manner we get rid of all the columns containing w's and z's, and are left with two identical rows (consisting of the S v's) which, by Lemma 1, are optimally equivalent. Hence the conclusion of Lemma 2 follows. (Problems $\bar{p}$, p^*, etc., are elements of P by Assumptions I and II.)

13. Let $p = (\phi, D, E)$ and let d', d'' be elements of D.

PROPERTY D_1.

$\underline{\text{If}}$

(13.1) $\qquad \phi(d',e) \doteq \phi(d'',e)$ for all $e \in E$

$\underline{\text{and}}$

(13.2) $\qquad d' \in \hat{D}$,

<u>then</u>

(13.3) $\qquad d'' \in \hat{D}.$

PROPERTY D_2.

<u>If</u>

(13.4) $\qquad \phi(d',e) \doteq \phi(d'',e)$ for all $e \in E$

<u>and</u>

(13.5) $\qquad d'' \notin \hat{D},$

<u>then</u>

(13.6) $\qquad d' \notin \hat{D}.$

14. Let $p \in P$, $p = (\phi, D, E)$. For any $d \in D$, write

$$m_d = \max U_d$$
$$m_d = \min U_d$$

whenever the right-hand members exist.

14. bis. <u>Assumption III</u>. Let $p \in P$, $d' \in D$, $M_{d'} = M_{d''} = M$ and $m_{d'} = m_{d''} = m$. Then there is a $p* = (\phi*, D*, E)$ such that $p \subseteq p*$ and there are two elements d_1, $d_2 \in$ in $D*$ for which

$$\phi*(d_1,e') = m \text{ for } e' \in E_1 \text{ and } \phi*(d_1,e) = M \text{ for } e \notin E_1$$
$$\phi*(d_2,e'') = M \text{ for } e'' \in E_2 \text{ and } \phi*(d_2,e) = m \text{ for } e \notin E_2$$

where E_1, E_2 are chosen at will.

15. <u>Lemma 4.</u> Let the conditions of Lemma 3 be satisfied; assume that ψ_p also satisfies the Properties D_1, D_2 and let P be such that Assumption III holds. Define

$$M_d = \max U_d$$
$$m_d = \min U_d.$$

Then, for $p = (\phi, D, E)$, d', $d'' \in D$, the relations

(15.1) $$M_{d'} = M_{d''}$$

$$m_{d'} = m_{d''}$$

hold if and only if

(15.2) $$d' \tilde{o} d'' [p].$$

In virtue of the preceding results, we may assume that $D = (d',d'')$ and write the entries of each row of the payoff matrix in arbitrary order. Thus $\underline{p}$ is represented by

(15.3)

	e_1	e_2	...	e_{N-1}	e_N
d'	a_1	a_2	...	a_{N-1}	a_N
d''	b_1	b_2	...	b_{N-1}	b_N

where each row is arranged in ascending order, so that $a_i \leq a_{i+1}$ and $b_i \leq b_{i+1}$ for $i = 1,2, \ldots, N-1$. Let the common value $M_{d'} = M_{d''}$ be denoted by M and the common value $m_{d'} = m_{d''}$ by $\underline{m}$, so that

$$a_1 = b_1 = m$$

$$a_N = b_N = M.$$

Now consider[9] the problem $\bar{p} \in P$ such that $\bar{p} \geqq p$ and $\bar{p}$ is represented by

(15.4)

	e_1	e_2	...	e_{N-1}	e_N
d_1	m	M	...	M	M
d'	a_1	a_2	...	a_{N-1}	a_N
d''	b_1	b_2	...	b_{N-1}	b_N
d_2	m	m	...	m	M

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9. $\bar{p} \in P$ by Assumption III.

where every entry (except the first one) in the first row is M and every entry in the last row (except the last one) is $\underline{m}$. Now d_1 and d_2 are optimally equivalent in p* in virtue of Lemma 3 (since they have the same image $U_{d_1} = U_{d_2} = \{m, M\}$). If they are both optimal, the optimality of d_2 implies that of d' and d" (by Property D_1). If neither d_1 nor d_2 is optimal, then the nonoptimality of d_1 implies that of d' and d" (by Property D_2). (The latter possibility is excluded by the definition of Ψ_p implying that every $p \in P$ has a solution.)

16. In Section 7 we defined a permutation of a row of the payoff matrix for the case of a finite E, We now generalize this to the case where E may be infinite. As before, let

$$u_d = [\phi(d,e),\ e \in E].$$

We say that $u_{d'}$ is a <u>permutation</u> of $u_{d''}$ if there exists an automorphism $\underline{f}$ on E such that

$$\phi(d',e) = \phi[d'',f(e)]$$

with

$$f(e) = e \qquad \text{for all } e \in E_1 \subseteq E,$$

where $E - E_1$ is not infinite.

17. <u>Lemma 2*</u>. With the term "permutation" defined as in Section 16, Lemma 2 holds without the restriction that E be finite, since the proof of Lemma 2 only uses the finiteness of $E - E_1$ rather than that of E itself.

18. <u>Lemma 3*</u>. Suppose Assumptions I-III are satisfied and Properties A, B, and C hold. Let $p = (\phi, D, E)$, $D = (d', d'')$, E possibly infinite, and, for some $e', e'' \in E$,

$$\begin{aligned}
&\phi(d',e') = m \\
&\phi(d',e) = M \qquad \text{for all } e \in [E - (e')] \\
(18.1) \qquad &\phi(d'',e'') = M \\
&\phi(d'',e) = M \qquad \text{for all } e \in [E - (e'')].
\end{aligned}$$

Then

(18.2) $\qquad d' \tilde{o} d'' [p]$.

Proof. This case may be represented by

	e'	e''	E_1
d'	m	M	(M)
d''	m	M	(m)

where (m) and (M) denote respectively row vectors (possibly infinite) of m's and M's. Now by Property C this may be reduced to the problem p* with

	e'	e''	e'''
d'	m	M	M
d''	m	M	m

where $e''' \in E_1$. But p* has a finite $E^* = (e', e'', e''')$, hence Lemma 3 applies and (18.2) follows.

19. Let $p = (\phi, D, E)$ where E may be infinite.

Assumption IV. For every $d \in D$, M_d and m_d exist and are finite.

Assumption V.1. For every $d \in D$, $\phi(d,e) = M_d$ for a finite number of e's only.[10/]

Assumption V.2. For every $d \in D$, $\phi(d,e) = m_d$ for a finite number of e's only.[10/]

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1. Assumptions V.1 and V.2 can be relaxed to a considerable extent.

20. Lemma 4*. Let Properties A, B, C, D_1, D_2 and Assumptions I-V hold. Then Lemma 4 holds without the restriction that E be finite.

Proof. By Assumptions IV and V and Lemma 2* it is possible to perform the reordering analogous to (15.3). Consequently the analogue of (15.4) may also be constructed. It then follows from Lemma 3* that d_1 and d_2 are optimally equivalent and the remainder of the proof is exactly as in Lemma 4.

21. Theorem 1. Suppose P satisfies Assumptions I-V. Let $v_d^{\chi} = \chi(M_d, m_d)$ be a scalar-valued function and let a class Υ^{o} of Ψ's be given by the requirement that, for some χ, $d \in \hat{D}$ if and only if $v_d^{\chi} \geq v_{d'}^{\chi}$ for all $d' \in D$. Then $\Psi \in \Upsilon^{o}$ if and only if the Properties A, B, C, D_1, D_2 hold.

It is easily verified that the Properties A, B, C, D_1, D_2 are satisfied for $\Psi \in \Upsilon^{o}$. The converse follows from Lemma 4*.