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Distribution of Quadratic Forms and Ratios of Quadratic Forms[1]

by

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1. Introduction.

Suppose the random variable $X = (X_1, X_2, \ldots, X_n)$ has the probability density[2]

$$p(x) = \frac{\det^{\frac{1}{2}} \Sigma}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2} x \Sigma x'}$$

where $x\Sigma x'$ is positive definite. There are various distribution problems associated with a quadratic form or a ratio of quadratic forms, which may be indicated as follows. Suppose, P, Q are arbitrary nxn matrices. The notation $(DQ)_1$ will be used to signify the problem of finding the distribution of XQX'. The notation $(DQ)_2$ represents the same problem under the restriction $XQX' \geq 0$. Also, $(DR)_1$ represents the problem of finding the distribution of $\frac{XQX'}{XPX'}$ where $XPX' \geq 0$; while $(DR)_2$ indicates the further restriction $XQX' \geq 0$. The notation $(DQ)_{11}$, $(DQ)_{21}$, $(DR)_{11}$, $(DR)_{21}$ will be reserved for the

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1. The author has had helpful discussions with E. Hille, C. Hildreth, L. Hurwicz, L.J. Savage, and M. Slater.

2. Capital letters are used to denote random variables, while corresponding small letters are used to denote particular sure values.

corresponding problems when P and Q are diagonal, and $\Sigma = I_n$, the identity matrix. It should be noted that although it is always possible, by a linear transformation, to reduce $(DQ)_i$ to $(DQ)_{i1}$, $(i = 1, 2)$, (cf. Bôcher [13]), the same is not always true for reducing $(DR)_i$ to $(DR)_{i1}$. In fact, a necessary and sufficient condition for the latter reduction is $PQ = QP$. (cf. Weyl [1].)

In a few special cases of the above problems, the distribution is available in a simple or closed form. For instance, when $Q = \Sigma$, the χ^2 distribution solves the $(DQ)_2$ problem. Simple cases of Q and P yield the F-distribution for the $(DR)_2$ problem. The exact distribution of a certain statistic given by R.L. Anderson [2] is a special case of $(DR)_{11}$ and its distribution in more general situations, given by W.G. Madow [3] is a particular case of $(DR)_1$.

In the vast majority of problems, however, the distributions are necessarily obtained by means of approximative methods, such as partial sums of infinite series. Von Neumann [4] gives such a solution for a particular case of the $(DR)_{11}$ problem. The method of partial sums is also used by Robbins [5], and by Pitman and Robbins [6], to give general solutions of $(DQ)_{21}$ and $(DR)_{21}$ respectively.

The present article gives a general solution to $(DQ)_2$ by means of Laguerrian expansions. If the semi-moments (defined below) are known, then expansions in Laguerre polynomials will afford a general solution to all problems $(DQ)_1$ and $(DR)_1$. Since the semi-moments are usually not easily found, the author proposes a new system of orthogonal polynomials, closely analogous to the Laguerre system, but which, if the expansions converge, will solve all $(DQ)_1$ and $(DR)_1$ problems. Once the convergence is rigorously established, the new system will be applicable to a much wider class of distributions than that to which Gram-Charlier series apply.

2. Reduction of the $(DR)_1$ problem.

Let $Y = (Y_1, Y_2, \ldots, Y_n)$ have the probability density

$$p(y) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2} yy'} \tag{2}$$

For any real z let

$$R_z = Q - zP \tag{3}$$

and let $\lambda_i(z)$, $i = 1, 2, \ldots, n$ be the roots of $\det (R_z - \lambda\Sigma) = 0$

It was shown previously (Gurland [7]) that[3]

$$P\left\{\frac{XQX'}{XPX'} \leq z\right\} = P\left\{\Sigma\ \lambda_i(z)\ Y_i^2 \leq 0\right\} \tag{4}$$

where Y has the probability density (2). Thus, the $(DR)_1$ problem is reduced to a $(DQ)_{11}$ problem, where only the solution at the point zero is required.

It is interesting to see how the reduction of the $(DR)_1$ problem is accomplished to yield directly the probability density of the ratio. Of course, the probability density can be obtained by differentiation of the distribution function, but this may not be feasible or advisable, depending on the type of approximation used in determining the cumulative distribution.

The following theorem will now be proved.[4]

Theorem 1.

Let X have the probability density p(x) and define

$$K = \int_{-\infty}^{\infty} xPx'\ p(x)\ dx \tag{5}$$

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3. The method by which (4) is obtained will apply to any ratio distribution problem, irrespective of quadratic forms and the normality of the original distribution.

4. The result applies generally, irrespective of quadratic forms, to any ratio (continuous random variable, positive denominator), and any distribution p(x).

(6) $$q(x) = \frac{xPx' \, p(x)}{K}$$

(7) $$G(z) = P\left\{\frac{XQX'}{XPX'} \leq z\right\}$$

Then the probability density $G'(z)$ is given by

(8) $$K \, r_z(0)$$

where $r_z(\xi)$ is the probability density of the random variable XR_zX' when the density of X is given by $q(x)$.

Proof:

From the theory of Fourier inversion formulae we have[5] (Gurland [8])

(9) $$G'(z) = \frac{1}{2\pi i} \oint \left[\frac{\partial\phi(t_1, t_2)}{t_2}\right]_{t_2 = -t_1 z} dt_1$$

where

(10) $$\phi(t_1, t_2) = E\left(e^{\,i\, t_1 XQX' + t_2 XPX'}\right)$$

Hence

(11) $$\left[\frac{\partial\phi}{\partial t_2}\right]_{t_2 = -t_1 z} = K \cdot i \int e^{it\, xR_z x'} q(x)\, dx = K\, i\, \theta_z(t)$$

where $\theta_z(t)$ is the characteristic function of XR_zX' when X has the probability density $q(x)$. Hence, (9) becomes

(12) $$G'(z) = \frac{K}{2\pi} \oint \theta_z(t)\, dt = K \cdot r_z(0)$$

by inversion of distribution functions. This completes the proof.

As the $(DR)_1$ problem may be transformed into a $(DQ)_{11}$ problem requiring

5. The notation $\oint$ signifies $\lim_{\substack{\varepsilon \to 0 \\ T \to \infty}} \left(\int_{-T}^{-\varepsilon} + \int_{\varepsilon}^{T}\right)$.

evaluation only at zero, it is advisable, in order to apply the theory of Laguerrian expansions, to consider the behavior at the tail of the distribution of a linear combination of independent χ^2 random variables, each with one degree of freedom. This will be considered after some discussion of the theory of Laguerre polynomials.

3. Laguerrian series.

By a Laguerrian series is meant an expansion of the form

$$f(x) \sim \sum_{n=0} c_n^{(\alpha)} L_n^{(\alpha)}(x) \tag{13}$$

where

$$L_n^{(\alpha)}(x) = \frac{1}{n!} e^x x^{-\alpha} \left(\frac{d}{dx}\right)^n \left(x^{n+\alpha} e^{-x}\right), \quad \alpha > -1 \tag{14}$$

The sign of equivalence in (13) signifies that the coefficients $c_n^{(\alpha)}$ are determined by

$$c_n^{(\alpha)} = \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \int_0^\infty e^{-t} t^\alpha L_n^{(\alpha)} f(t)\, dt \tag{15}$$

in view of the orthogonality relations

$$\int_0^\infty e^{-t} t^\alpha L_m^{(\alpha)}(t) L_n^{(\alpha)}(t)\, dt = \begin{cases} 0, & m \neq n \\ \dfrac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}, & m = n \end{cases} \tag{16}$$

It follows that

$$L_n^{(\alpha)}(x) = \sum_{v=0}^{n} \binom{n+\alpha}{n-v} \frac{(-x)^v}{v!} \tag{17}$$

The following theorem of Szegö [9] gives sufficient conditions for a Laguerre series to be equiconvergent with a Fourier series. (Two series

$\sum_0^\infty u_n$, $\sum_0^\infty v_n$ are called equiconvergent if the series $\sum_0^\infty (u_n - Av_n)$, $A \neq 0$, is convergent.) The notation $g(x) = o\,(h(x))$ used below means

$$\frac{g(x)}{h(x)} \to 0 \qquad \text{as} \qquad x \to \infty$$

Theorem 2 (Szëgo [9])

Let $f(x)$ be Lebesgue measurable, $0 \leq x < \infty$, and let the integrals

$$(18) \qquad \int_0^1 x^\alpha \left|f(x)\right| \, dx, \quad \int_0^1 x^{\frac{\alpha}{2} - \frac{1}{4}} \left|f(x)\right| \, dx$$

exist. If the condition

$$(19) \qquad \int_n^\infty e^{-\frac{x}{2}} \, x^{\frac{\alpha}{2} - \frac{13}{12}} \left|f(x)\right| \, dx = o\,(n^{-\frac{1}{2}})$$

is satisfied, and if $s_n(x)$ denotes the n^{th} partial sum of the Laguerre series (13), we have, for $x > 0$

$$(20) \qquad \lim_{n\to\infty} \left\{ s_n(x) - \frac{1}{\pi} \int_{x^{\frac{1}{2}}-\delta}^{x^{\frac{1}{2}}+\delta} f(\tau^2) \, \frac{\sin\left[2n^{\frac{1}{2}}\,(x^{\frac{1}{2}} - \tau)\right]}{x^{\frac{1}{2}} - \tau} \, d\tau \right\} = 0$$

where δ is a fixed positive number, $\delta < x^{\frac{1}{2}}$. This holds uniformly for every fixed positive interval

$$(21) \qquad \varepsilon \leq x \leq \omega, \qquad \delta < \varepsilon^{\frac{1}{2}}$$

The same equiconvergence theorem is valid if the integrals (18) exist and (19) is replaced by the following

$$(22) \qquad \int_1^\infty e^{-\frac{x}{2}} \, x^{\frac{\alpha}{2} - \frac{3}{4}} \left|f(x)\right| \, dx$$

is convergent and

$$\int_n^{\infty} e^{-x} x^{\alpha-2} \left| f(x) \right|^2 dx = O(n^{-\frac{3}{2}})$$

The integral occurring in (20) is essentially the partial sum of order $[n^{\frac{1}{2}}]$ of a Fourier series[6]. A sufficient condition for the validity of (20) is

(23) $$f(x) = O\left(e^{\frac{x}{2}} x^{-\frac{\alpha}{2}-\frac{1}{4}-\delta}\right), \quad \delta > 0$$

where the notation

$$f(x) = O(g(x))$$

means

$\frac{f(x)}{g(x)}$ is bounded for all x sufficiently large.

Before quoting another theorem of Szegö which ensures summability of (13) at x = 0, we shall briefly indicate what is meant by Cesaro summability. Let s_n denote the n^{th} partial sum $\sum_0^n u_\nu$. The series $\sum_0^\infty u_\nu$ is said to be (C, k) summable (cf. Zygmund [10]), $k > -1$ to the sum s if

(24) $$\lim_{n\to\infty} \frac{s_n^{(k)}}{C_n^{(k)}} = s$$

where

(25) $$C_n^{(k)} = \binom{n+k}{n} = \frac{(n+k)\,(n+k-1)\,\ldots\ldots(k+1)}{n!}$$

(26) $$s_n^{(k)} = \sum_{r=0}^{n} C_{n-r}^{(k-1)} s_r = \sum_{r=0}^{n} C_{n-r}^{(k)} u_r$$

We shall refer later, to the following theorem

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6. The notation $[n^{\frac{1}{2}}]$ means the largest integer $\leq n^{\frac{1}{2}}$.

<u>Theorem 3</u> (Szegö, [9])

Let $f(x)$ be Lebesgue measurable, $0 \leq x < \infty$, and continuous at $x = 0$. If we assume the existence of the integral

$$(27) \qquad \int_1^{\infty} e^{-x} x^{\alpha - k - 1/3} \left| f(x) \right| dx$$

the Laguerrian series (13) is summable (C,k) at $x = 0$ to the sum $f(0)$, provided

$$(28) \qquad k > \alpha + \frac{1}{2}$$

This statement is not true for $k \leq \alpha + \frac{1}{2}$.

The condition regarding (27) is satisfied if

$$(29) \qquad f(x) = 0\left(e^{\frac{x}{2}} x^{k - \alpha - \frac{2}{3} - \delta}\right), \ \delta > 0$$

It should be remarked that for the case $x = 0$, the k^{th} Cesaro mean has the simplified form

$$(30) \qquad \left\{C_n^{(k)} \Gamma(\alpha+1)\right\}^{-1} \int_0^{\infty} e^{-t} t^{\alpha} f(t) L_n^{(\alpha+k+1)}(t)\, dt$$

4. <u>Laguerrian expansions for distribution functions.</u>

Let a random variable have the distribution function

$$F(x) = \int_{-\infty}^{x} p(t)\, dt$$

By analogy with Gram-Charlier series, we may consider

$$(31) \qquad p(x) \sim e^{-x} x^{\alpha} \sum_{n=0}^{\infty} a_n^{(\alpha)} L_n^{(\alpha)}(x)$$

$$(32) \qquad F(x) - K(x) \sim e^{-x} x^{\alpha} \sum_{n=0}^{\infty} A_n^{(\alpha)} L_n^{(\alpha)}(x)$$

where

(33) $$a_n^{(\alpha)} = \int_0^\infty p(t)\, L_n^{(\alpha)}(t)\, dt$$

(34) $$A_n^{(\alpha)} = \int_0^\infty [F(t) - K(t)]\, L_n^{(\alpha)}(t)\, dt$$

and $K(x)$ is a conveniently chosen distribution function

(35) $$K(x) = \int_{-\infty}^{x} g(t)\, dt$$

Note that $a_n^{(\alpha)}$, $A_n^{(\alpha)}$ are linear functions of the "moments" taken over the interval $(0, \infty)$ and not $(-\infty, \infty)$. We shall call such "moments," semi-moments.

It is in order, at this point, to remark why Laguerrian rather than Gram-Charlier series, mentioned by Gurland [7] are being considered here for the aforementioned distribution problems. The main reason is that Cramér's condition (Cramér [11])

(36) $$\int_{-\infty}^{\infty} e^{\frac{x^2}{4}}\, dF(x) < \infty$$

is not satisfied for these problems; and theorems which guarantee Cesaro or Abel summability of Gram-Charlier series (cf. Szegö [11], Hille [12]) do not relax condition (36) very much, if at all.

In this paper, expansions of the type (32) will be considered. In order to simplify the formula (34) for $A_n^{(\alpha)}$, it is necessary to refer to the following lemma.

Lemma 1:

If all the absolute moments corresponding to $F(x)$ and $K(x)$ are finite, then

(37) $$F(x) - K(x) = o(x^{-r}) \quad , \quad r > 0, \quad x \to \pm\infty$$

Proof[7]

$$\left| F(x) - K(x) \right| = \left| 1 - K(x) - (1 - F(x) \right| \leq \int_x^{\infty} g(t)\, dt + \int_x^{\infty} p(t)\, dt$$

Let $M_r = \int_{-\infty}^{\infty} |t|^r \, p(t)\, dt$

Then

$$M_r \geq \int_x^{\infty} t^r \cdot p(t)\, dt \geq x^r \int_x^{\infty} p(t)\, dt$$

Hence $\int_x^{\infty} p(t)\, dt = 0\,(x^{-r})$, $r > 0$, $x \to \infty$, since M_r is finite for $r > 0$.

Similarly for $\int_x^{\infty} g(t)\, dt$. Also by considering, in the same manner,

$\left| F(x) - K(x) \right| \leq \int_{-\infty}^{x} p(t)\, dt + \int_{-\infty}^{x} g(t)\, dt$ the required result follows.

To apply this lemma, let $M_n^{(\alpha)}(x)$ be a polynomial such that

(38) $$\frac{d}{dx} M_n^{(\alpha)}(x) = L_n^{(\alpha)}(x)$$

and

(39) $$M_n^{(\alpha)}(0) = 0$$

Since

(40) $$\frac{d}{dx} L_{n+1}^{(\alpha-1)}(x) = - L_n^{(\alpha)}(x)$$

as can be seen from (17), we may write

(41) $$M_n^{(\alpha)}(x) = - \left[L_{n+1}^{(\alpha-1)}(x) - \binom{n+\alpha}{n+1} \right]$$

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7. This proof is due to Morton Slater.

Integrating by parts and applying lemma 1 we obtain

$$A_n^{(\alpha)} = -\int_0^{\infty} M_n^{(\alpha)}(t)\,[p(t) - g(t)]\,dt$$

(42)

$$= a_{n+1}^{(\alpha-1)} - b_{n+1}^{(\alpha-1)} - \binom{n+\alpha}{n+1}\int_0^{\infty} [p(t) - g(t)]\,dt$$

where

(43) $$b_n^{(\alpha)} = \int_0^{\infty} g(t)\, L_n^{(\alpha)}(t)\,dt$$

By choosing

(44) $$g(x) = N\, x^{\alpha}\, e^{-x}$$

where N is the normalizing constant, we get

$$b_{n+1}^{(\alpha-1)} = 0 \quad , \quad n = 0, 1, \ldots$$

If also, $p(t) = 0$ for $t \leq 0$, then (42) would become

(45) $$A_n^{(\alpha)} = a_{n+1}^{(\alpha-1)}$$

In problem $(DQ)_2$ this latter condition will be satisfied; on the other hand, in problems $(DQ)_1$, $(DR)_1$, $(DR)_2$ it will not be satisfied in general.

5. Solution to the problem $(DQ)_2$ by Laguerrian expansions.

Before proceeding to the solution it is necessary to prove the following lemma.

Lemma 2

Let Y have the probability density

(46) $$p(y) = \frac{1}{2^{\frac{n}{2}}\left(\Gamma(\frac{1}{2})\right)^n}\, e^{-\frac{1}{2}(y_1+y_2+\cdots+y_n)}(y_1\, y_2 \cdots y_n)^{-\frac{1}{2}}$$

Define

(47) $$U = \sum_{1}^{n} \gamma_i Y_i$$

where the γ_i are constants, satisfying

(48) $$0 < \gamma_i < 1.$$

Then, if $p_U(u)$ is the probability density of U,

(49) $$p_U(u) = 0\left(e^{-\frac{u}{2}(1+\varepsilon)}\, u^{\frac{n}{2}-1}\right)$$

where $\varepsilon > 0$ is defined below.

Proof:

Apply to (46) the transformation

(50)) $$\begin{aligned} u &= \gamma_1 y_1 + \gamma_2 y_2 + \cdots + \gamma_n y_n \\ v_1 &= y_1/y_n \\ &\vdots \\ v_{n-1} &= y_{n-1}/y_n \end{aligned}$$

Setting

(51) $$\Delta = \gamma_1 v_1 + \gamma_2 v_2 + \cdots + \gamma_{n-1} v_{n-1} + \gamma_n$$

this becomes

(52) $$\begin{aligned} y_1 &= \frac{uv_1}{\Delta} \\ &\vdots \\ y_{n-1} &= \frac{uv_{n-1}}{\Delta} \\ y_n &= \frac{u}{\Delta} \end{aligned}$$

and the Jacobian of the transformation is

(53) $$\frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(u, v_1, \ldots, v_{n-1})} = \frac{u^{n-1}}{\Delta^{2n-1}} \cdot \phi(v_1, v_2, \ldots, v_{n-1})$$

where[8/]

$$(54) \qquad \phi(v_1, v_2, \ldots, v_{n-1}) = \begin{vmatrix} v_1 \; {}^{(-1)}(\Delta - v_1\gamma_1) & -v_1\gamma_2 & \ldots & -v_1\gamma_{n-1} \\ v_2 & -v_2\gamma_1 \; {}^{(-1)}(\Delta - v_2\gamma_2) & \ldots & -v_2\gamma_{n-1} \\ \vdots & & & \\ v_{n-1} & -v_{n-1}\gamma_1 & -v_{n-1}\gamma_2 \ldots & {}^{(-1)}(\Delta - v_{n-1}\gamma_{n-1}) \\ 1 & -\gamma_1 & -\gamma_2 \quad \ldots & -\gamma_{n-1} \end{vmatrix}$$

Hence, the joint probability density of U, V_1, ..., V_{n-1} is given by

$$(55) \qquad p(u_1 v_1, \ldots, v_{n-1}) = \text{const.}\; e^{-\frac{u}{2\Delta}(v_1+v_2+\ldots+v_{n-1}+1)}\; u^{n-1} \left(\frac{u^n\, v_1 \ldots v_{n-1}}{\Delta^n}\right)^{-1/2} \cdot \left|\frac{\phi(v_1,\ldots,v_{n-1})}{\Delta^{2n-1}}\right|$$

Now

$$(56) \qquad \frac{v_1+v_2+\ldots+v_{n-1}+1}{\Delta} = 1 + \frac{(1-\gamma_1)v_1+\ldots+(1-\gamma_{n-1})v_{n-1}+(1-\gamma_n)}{\gamma_1 v_1+\ldots+\gamma_{n-1}v_{n-1}+\gamma_n}$$

Let

$$(57) \qquad \epsilon = \min_{v_1,\ldots,v_{n-1}} \frac{(1-\gamma_1)v_1+\ldots+(1-\gamma_{n-1})v_{n-1}+(1-\gamma_n)}{\gamma_1 v_1+\ldots+\gamma_{n-1}v_{n-1}+\gamma_n}$$

It can be shown that $\epsilon > 0$.

Integrating out the v's in (55) now yields the required result.

As mentioned earlier, the $(DQ)_2$ problem reduces to the $(DQ)_{21}$ problem by a linear transformation. Let

$$XQX' = \sum_1^n \gamma_i x_i^2$$

~~8. It can be shown that $|\phi(v_1,\ldots,v_{n-1})| = \Delta^{n-2}$.~~

$\phi(v_1, v_2) = -\Delta$

$\phi(v_1, v_2, v_3) = \Delta^2 - 2\Delta\gamma_2 v_2$

where X has the probability density of (2). There is no loss of generality in assuming condition (48) is satisfied. Setting $y_i = x_i^2$ we obtain the probability density (46); consequently the above lemma is applicable. Taking

$$(58) \qquad f(u) = e^{u}\, u^{-\alpha} p_U(u)$$

and applying (49), we get

$$(59) \qquad f(u) = O\left(e^{\frac{u}{2}(1-\epsilon)}\, u^{\frac{n}{2}-\alpha-1}\right), \quad 0 < \epsilon$$

It is evident that the conditions of Theorem 2 are satisfied. Hence the expansion (32), with $A_n^{(\alpha)}$ given by (45) will converge[9] to the distribution of XQX'. The moments of XQX' may be found conveniently from the characteristic function.

$$(60) \qquad E e^{it\, XQX'} = \frac{1}{\det^{1/2}(I_n - 2itQ)}$$

6. <u>Solution to $(DQ)_1$ and $(DR)_1$ by Laguerrian expansions if the semi-moments are known.</u>

Before applying the convergence theorems in § 3 it is necessary to extend the lemma established in § 5. We now prove

<u>Lemma 3.</u>

Suppose Y has the probability density given by (46), and define

$$(61) \qquad \begin{aligned} U_1 &= \sum_{1}^{n_1} \gamma_i Y_i \\ U_2 &= \sum_{n_1+1}^{n_1+n_2} \gamma_i Y_i \end{aligned}$$

where $0 < \gamma_i < 1$, $\quad n_1 + n_2 = n$.

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9. By theorem 2, the series will converge at each point x if the Fourier series converges there. Since F(x) - K(x) is of bounded variation, convergence is assured by Jordan's test (Titchmarsh [17]).

Let $p_V(v)$ be the probability density of

(62) $$V = U_1 - U_2$$

Then

(63) $$\int_x^{\infty} p_V(v)\,dv = O\left(e^{-\frac{x}{2}(1+\varepsilon)}\, x^{\frac{n_1}{2}}\right) \quad ,$$

where $\varepsilon > 0$ is obtained by lemma 2.

<u>Proof</u>.

By lemma 2, the probability density of U_1, U_2 has the following behavior as $x \to \infty$.

(64) $$p_{U_1,U_2}(u_1, u_2) = O\left(e^{-\frac{1}{2}(u_1+u_2)(1+\varepsilon)}\, u_1^{\frac{n_1}{2}-1}\, u_2^{\frac{n_2}{2}-1}\right)$$

Thus, for x sufficiently large

(65) $$\int_x^{\infty} p_V(v)\,dv < \text{const.} \iint_{u_1-u_2 \geq x} e^{-\frac{1}{2}(u_1+u_2)(1+\varepsilon)}\, u_1^{\frac{n_1}{2}-1}\, u_2^{\frac{n_2}{2}-1}\, du_1\,du_2$$
$$= c_1 \int_{v=x}^{\infty}\int_{u_2=0}^{\infty} e^{-\frac{1}{2}(1+\varepsilon)(2u_2+v)}\,(v+u_2)^{\frac{n_1}{2}-1}\, u_2^{\frac{n_2}{2}-1}\, du_2\,dv$$

If n_1 is an even integer the required result follows easily since

$$\int_0^{\infty} e^{-\frac{1}{2}(1+\varepsilon)u_2}\, u_2^{k}\, u_2^{\frac{n_2}{2}-1}\, du_2 = O(1)\ ,\ k = 0, 1, 2, \ldots$$

If n_1 is odd, write

$$(v + u_2)^{\frac{n_1}{2}-1} = v^{\frac{n_1}{2}-1}\left(1 + \frac{u_2}{v}\right)^{\frac{n_1}{2}-1}$$

For $0 \leq u_2 < v$, and $v > 1$ (the latter is assured when $x > 1$) it is bounded by $v^{\frac{n_1}{2}-1} (1+u_2)^{\frac{n_1}{2}-1}$

The required result follows, since

$$\int_0^\infty e^{-\frac{1}{2}(1+\varepsilon)u_2} (1+u_2)^{\frac{n_1}{2}-1} u_2^{\frac{n_2}{2}-1} du_2 = O(1).$$

In applying this result to a $(DQ)_1$ problem, we may assume XQX' is in the form (62) of lemma 3 (as can be effected by a linear transformation). The result of (63) ensures the applicability of both Theorems 2 and 3; consequently the expansion (32) will converge[10] for $x > 0$, while for $x = 0$, it will be (C, 1) summable, if α is chosen to be zero in Theorem 3. For $x < 0$, the result of Theorem 2 applies by considering the expansion of $F(-x) - K(-x)$.

From (42) and (33) it is evident that the $A_n^{(\alpha)}$ are linear functions of the semi-moments

$$\int_0^\infty v^k p_V(v)\, dv$$

which, in general, are difficult to obtain.

Lemma 3 may also be applied in solving the $(DR)_1$ problem, by using the reduction (4) of § 2, and employing the same type of argument as for the $(DQ)_1$ problem above, to show that Theorem 3 ensures the (C, 1) summability of the Laguerrian expansion at $x = 0$.

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10. To apply theorem 2, the result (63) must hold also for $x \to -\infty$. This can be proven as in lemma 3, since

$$\iint_{u_1-u_2 \leq x} p_{U_1,U_2}(u_1, u_2)\, du_1\, du_2 = \int_{u_1=0}^{\infty} du_1 \int_{u_2=u_1-x}^{\infty} p(u_1,u_2) du_2 \text{ and } x < 0.$$

7. Proposed system of polynomials for the general solution of $(DQ)_1$, $(DR)_1$.

As mentioned above, the semi-moments are, usually, difficult to obtain. The convergence properties of Laguerrian expansions are most convenient, but the main shortcoming is that the weight function is zero over $(-\infty, 0)$. What is required is a weight function over $(-\infty, \infty)$ which would generate a system of orthogonal polynomials behaving asymptotically in a manner similar to the Laguerrian system. In such a case, ordinary moments, rather than semi-moments would be used in the determination of the coefficients of the expansion, and generally these moments can be found without difficulty. A system which seems to suggest itself naturally is that generated, according to the Schmidt process (cf. Courant-Hurwitz [14]), by means of the weight function.

$$w(x) = e^{-|x|} x^{\alpha} \quad , \quad -\infty < x < \infty$$

Shohat [15] has shown that for weight functions similar to this, the resulting system of orthogonal polynomials is complete, but there appears no treatment of the convergence of such a system in the literature on orthogonal polynomials. If, as conjectured, this system behaves similarly to the Laguerrian system, then a larger class of distributions will be expansible in convergent (or summable) series than the class to which Gram-Charlier series apply.

REFERENCES

[1] H. Weyl, "The Theory of Groups and Quantum Mechanics," Methuen and Co., Ltd., London, 1931.

[2] R.L. Anderson, "Distribution of the Serial Correlation Coefficient," Annals of Mathematical Statistics, 1942.

[3] W.G. Madow, "Note on the Distribution of the Serial Correlation Coefficient," Annals of Mathematical Statistics, 1945.

[4] Von Neumann, J., "Distribution of the Ratio of the Mean Successive Difference to the Variance," Annals of Mathematical Statistics, 1941.

[5] H. Robbins, "The Distribution of a Definite Quadratic Form," Annals of Mathematical Statistics, 1948.

[6] H. Robbins and E.J.G. Pitman, "Application of the Method of Mixtures to Quadratic Forms in Normal Variates," Annals of Mathematical Statistics, 1949.

[7] J. Gurland, "Distribution of Ratios of Quadratic Forms," Cowles Commission Discussion Paper, Statistics, No. 354.

[8] J. Gurland, "Inversion Formulae for the Distribution of Ratios," Annals of Mathematical Statistics, 1948.

[9] G. Szegö, "Orthogonal Polynomials," American Mathematical Society Colloquium Publication, Vol. 23, New York, 1939.

[10] A. Zygmund, "Trigonometrical Series," Warszawa-Lwów, 1935.

[11] H. Cramér, "Mathematical Methods of Statistics," Princeton University Press, 1946.

[12] E. Hille, "A Class of Reciprocal Functions," Annals of Mathematics, 1926.

[13] M. Bôcher, "Introduction to Higher Algebra," The Macmillan Co., New York, 1924.

[14] R. Courant and D. Hilbert, "Methoden der Mathematischen Physik," Verlag von Julius Springer, Berlin, 1931.

[15] J.A. Shohat, "The Theory of Closure of Tchebycheff Polynomials for an Infinite Interval," Bulletin of the American Mathematical Society, 1924.

[16] G.E. Hardy, J.E. Littlewood, and G. Polya, "Inequalities," Cambridge University Press, 1934.

[17] E.C. Titchmarsh, "The Theory of Functions," Oxford University Press, 1939.

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1. Page 2, line 5. After the word "reduction," add the words, "when $\Sigma = I$."

2. Page 2. At the end of the third paragraph where it concludes ".... and $(DR)_{21}$ respectively." After this sentence add in parentheses "(Special case of $(DR)_{21}$).

3. Page 5. Equation (13). Write $L_n^{\alpha}(x)$ for $L_n^{(\alpha)}$.

4. Page 13. Equation (54). The elements in the first diagonal to the right of the principal diagonal should be multiplied by (-1). That is $(-1)\ (\Delta - v_1 \gamma_1)$, $(-1)\ (\Delta - v_2 \gamma_2)$, ... etc.

5. Page 13. Footnote.

Replace $\left| \phi(v_1, \dots, v_{n-1}) \right| = \Delta^{n-2}$

by $\phi(v_1, v_2) = -\Delta$

$\phi(v_1, v_2, v_3) = \Delta^2 - 2\Delta\ \gamma_2 v_2$ etc.

6. Page 15. Equation (63). $x^{\frac{n_1}{2}-1}$ should be replaced by $x^{\frac{n_1}{2}}$.

7. Page 11. Heading for § 5. Replace $(PQ)_2$ by $(DQ)_2$.